Recent advances in isoperimetric inequalities for eigenvalues

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Outline

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Notations

DIRICHLET: $\lambda_k(\Omega)$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

NEUMANN: $\mu_k(\Omega)$ ($\mu_0 = 0$)

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

ROBIN: $\sigma_k(\Omega, \alpha)$

$$\begin{cases} -\Delta u = \sigma u & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha u = 0 & \text{on } \partial\Omega \end{cases}$$

STEKLOV: $p_k(\Omega)$ ($p_0 = 0$)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = pu & \text{on } \partial\Omega \end{cases}$$

Isoperimetric inequalities

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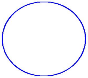
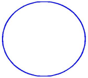
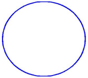
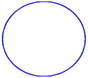
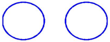
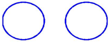
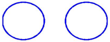
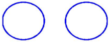
By homogeneity, it is equivalent to consider problems like $\min\{|\Omega|^{2/N}\lambda_k(\Omega); \Omega \in \mathbb{R}^N\}$, $\min\{P(\Omega)^{2/(N-1)}\lambda_k(\Omega); \Omega \in \mathbb{R}^N\}$, etc...

The two lowest eigenvalues (volume constraint)

	Dirichlet	**Neumann**	**Robin**	**Steklov**
	min	**max**	**min**	**max**
1st eigenvalue	 Faber 1923 Krahn 1924	 Szegö 1954 Weinberger 1956	 Bossel 1986 Daners 2006	 Weinstock 1954 Brock 2001
2nd eigenvalue	 Krahn 1926 Hong 1950	 Girouard-Nadirashvili-Polterovitch 2009	 Kennedy 2009	 Girouard-Polterovitch 2009

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A general existence result

Theorem (Bucur; Mazzoleni-Pratelli 2011)

The problem $\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^N, |\Omega| = c\}$ *has a solution. This one is an open set which is bounded and has finite perimeter.*

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More generally, the problem $\min\{F(\lambda_1(\Omega), \ldots, \lambda_p(\Omega)), \Omega \subset \mathbb{R}^N, |\Omega| = c\}$, *where* $F : \mathbb{R}^p \to \mathbb{R}$ *is increasing in each variable and lower-semicontinuous, has a solution.*

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Open problem*: If Ω_k^* denotes a minimizer for λ_k, $k \geq 2$, prove that λ_k is a multiple eigenvalue, $\lambda_{k-1}(\Omega_k^*) = \lambda_k(\Omega_k^*)$.

Techniques of proof

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Bucur introduces the notion of *local shape sub-solution for the energy* (which are bounded), proves that minimizers for the eigenvalues satisfy this definition and conclude by induction thanks to a concentration-compactness argument.

The third eigenvalue λ_3

Dimension 2:
Open problem*** Prove that the disk is the minimizer!
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Dimension ≥ 4: Open problem** Prove that the union of three identical balls is the minimizer.

Numerical results for λ_k

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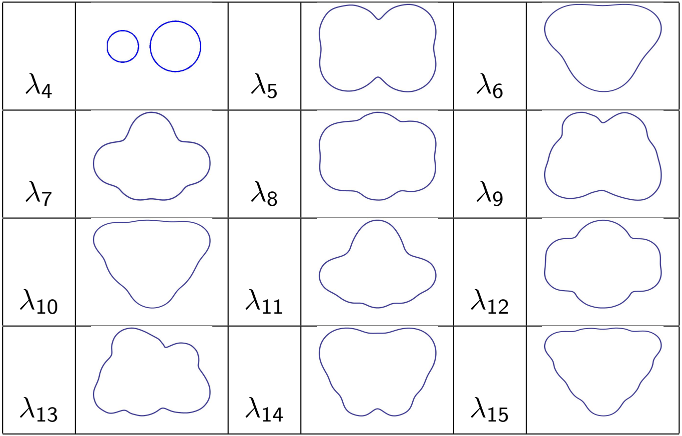
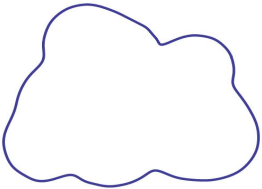


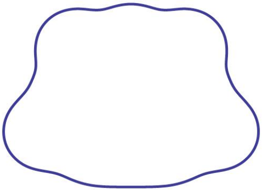
Table: Minimizers of $\lambda_k(\Omega)$, $k = 4 \ldots 15$ in the plane, by courtesy of P. Antunes and P. Freitas

Symmetry?

P. Antunes and P. Freitas got, as a possible solution for the minimizer of λ_{13} the following domain



To confirm this non-symmetry result, they tried to look for the best symmetric domain (by imposing an axis of symmetry) and they got the following domain



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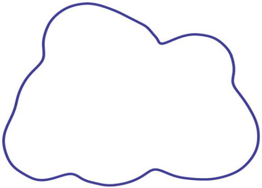

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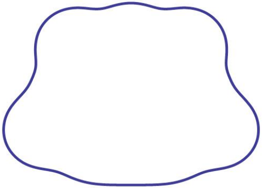


But the second one has a worse 13-th eigenvalue than the first one. Thus it may appear that the minimizers are not necessarily symmetric.

Some three-dimensional results

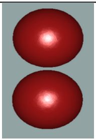
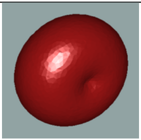
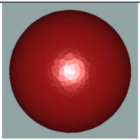
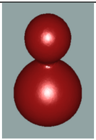
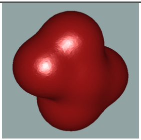
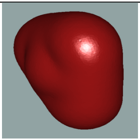
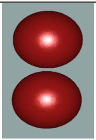
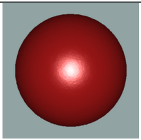
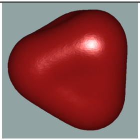
λ_2		λ_3		λ_4	
λ_5		λ_6		λ_7	
λ_8		λ_9		λ_{10}	

Table: Minimizers of $\lambda_k(\Omega)$, $k = 2 \dots 10$ in the 3D space, by courtesy of A. Berger and E. Oudet

Perimeter constraint

Theorem (Bucur-Buttazzo-H. 2009; De Philippis-Velichkov 2013)

The problem $\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^N, P(\Omega) = c\}$ *has a solution. This one is bounded, connected. Its boundary is* $C^{1,\alpha}$ *outside a closed set of Hausdorff dimension at most* $N - 8$*. It is analytic in dimension* $N = 2$*.*

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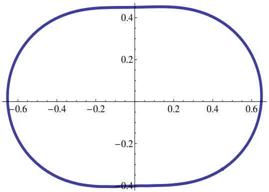
The minimizer for λ_2 in the plane is a regular strictly convex domain with a curvature vanishing at exactly two points.



Diameter constraint

First motivation: the gap conjecture (which is now the gap theorem by B. Andrews and J. Clutterbuck!). We wanted to prove (see AIM Palo-Alto meeting Low Eigenvalues of Laplace and Schrödinger Operators in 2006) that the problem

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Is it possible that the ball is the minimizer for any λ_k with a diameter constraint?

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are $\mu_0 = 0 \leq \mu_1 \leq \mu_2 \ldots$

Open problem**: Prove a general existence result for

$$\max\{\mu_k(\Omega); \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz }, |\Omega| = c\}.$$

The second Neumann eigenvalue

Theorem (A. Girouard-N. Nadirashvili-I. Polterovitch 2009)

The union of two disjoint balls solves the problem

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It remains to prove:
Open problem**: Extend the theorem to non simply-connected domains and to higher dimensions.

Numerical results

Numerical results for the maximization of $\mu_k(\Omega), k = 4 \dots 15$ have been obtained in the plane e.g. by P. Antunes and P. Freitas (2012), A. Berger and E. Oudet (2013)

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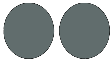
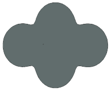
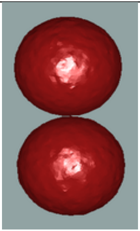
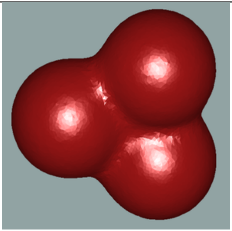
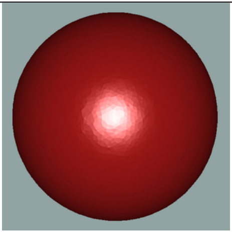
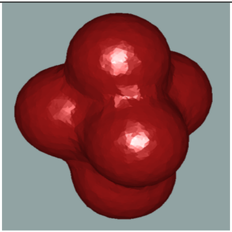
μ_2		μ_3		μ_4	
μ_5		μ_6		μ_7	
μ_8		μ_9		μ_{10}	

Table: Maximizers of $\mu_k(\Omega), k = 2 \dots 10$ in the plane, by courtesy of A. Berger and E. Oudet

Numerical results - 3D case

μ_2		μ_3	
μ_4		μ_5	

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Theorem (J. Kennedy 2009)

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Idea of the proof: similar to the Dirichlet case. Some supplementary work to deal with the possible non regularity of the nodal surface.

The Robin eigenvalues

No general existence result available. The case of $\sigma_1(\Omega, \alpha)$ has been solved in 1986 by M.H. Bossel in the plane and more recently in 2006 by D. Daners in any dimension, following the same strategy.

The case of $\sigma_2(\Omega, \alpha)$ has been solved recently:

Theorem (J. Kennedy 2009)

The union of two disjoint balls solves the problem $\min\{\sigma_2(\Omega, \alpha), \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz }, |\Omega| = c\}$ *for any* $\alpha > 0$.

Idea of the proof: similar to the Dirichlet case. Some supplementary work to deal with the possible non regularity of the nodal surface.

The case $\alpha < 0$ seems completely open even for σ_1.
Open problem**: prove that the ball maximizes $\sigma_1(\Omega, \alpha)$ for $\alpha < 0$ among bounded Lipschitz domains.

Outline

Introduction

The eigenvalues of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = p\, u & \text{on } \partial\Omega \end{cases}$$

are $p_0 = 0 \leq p_1 \leq p_2 \ldots$.

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are $p_0 = 0 \leq p_1 \leq p_2 \ldots$. We can look at problems like

$$\max\{p_k(\Omega); \Omega \subset \mathbb{R}^N, \Omega \text{ bounded and Lipschitz } \}$$

either with an area constraint $|\Omega| = c$ or a perimeter constraint $P(\Omega) = c$.

Theorem

- *Weinstock 1954: if $N = 2$, the disk maximizes $p_1(\Omega)$ among sets of given perimeter.*
- *Brock 2001: if $N \geq 2$, the ball maximizes $p_1(\Omega)$ among sets of given volume.*

Open problem**: Extend Brock's result to the perimeter constraint.

The second Steklov eigenvalue

Theorem (A. Girouard-I. Polterovitch 2009)

The union of two disjoint disks solves the problem

$$\max\{p_2(\Omega), \Omega \subset \mathbb{R}^2, \Omega \textit{ regular and simply connected }, P(\Omega) = c\}.$$

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It remains to prove:
Open problem**: Extend the Theorem to non simply-connected domains and to higher dimensions.

Some other inequalities

Let Ω be any (regular) domain and denote by Ω^* the ball with same volume. F. Brock in 2001 proved the inequality:

$$\sum_{i=2}^{N+1} \frac{1}{p_i(\Omega)} \geq \sum_{i=2}^{N+1} \frac{1}{p_i(\Omega^*)}$$

which clearly implies $p_2(\Omega) \leq p_2(\Omega^*)$.

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Theorem (H.-Philippin-Safoui 2008)

For any convex domain in $\mathbb{R}^N$, we have

$$\Pi_{k=2}^{N+1} p_k(\Omega) \leq \Pi_{k=2}^{N+1} p_k(\Omega^*).$$

Open problem*: remove the convexity assumption in the previous inequality

The trace operator

Let Ω be a Lipschitz domain and let us consider the norm of the trace operator $\tau : H^1(\Omega) \longrightarrow L^2(\partial\Omega)$. Computation of its norm leads to consider the eigenvalue

$$\frac{1}{\|\tau\|} = \lambda(\Omega) := \min \left\{ \frac{\int_\Omega |\nabla u|^2 + u^2 \, dx}{\int_{\partial\Omega} u^2 \, d\sigma} \; ; u \in H^1(\Omega) \right\}$$

which corresponds to the eigenvalue problem of Steklov type

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Open problem**: Prove that the ball maximizes $\lambda(\Omega)$ among sets of given volume.

Known: (J. Rossi 2008) the ball is a critical point.