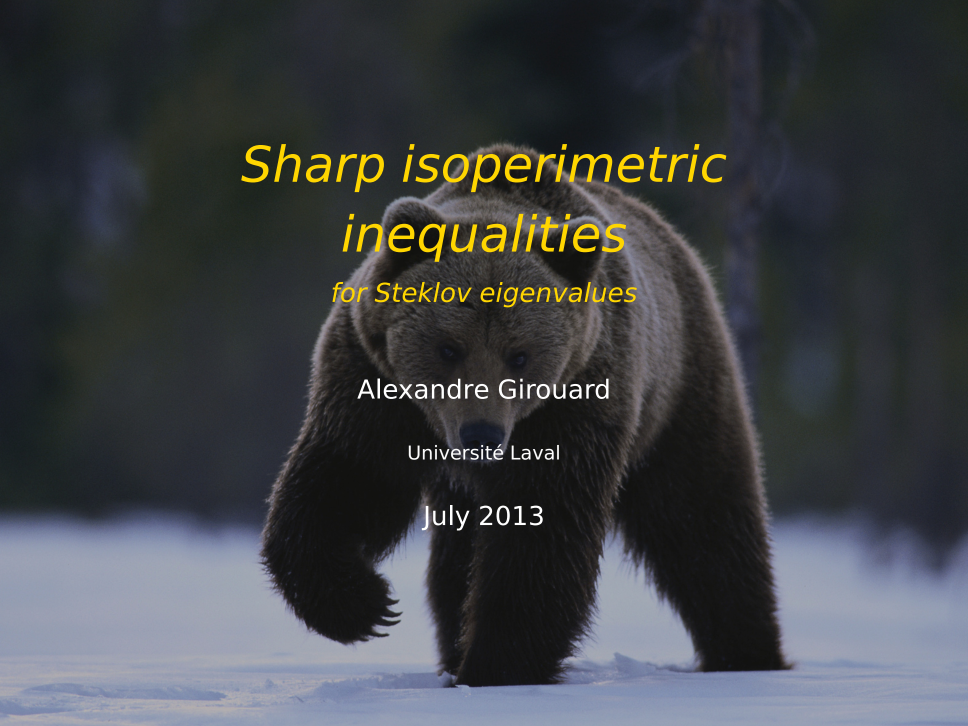


Sharp isoperimetric inequalities

for Steklov eigenvalues

Alexandre Girouard

Université Laval

July 2013

The *Steklov spectral problem* on a bounded domain $\Omega \subset \mathbb{R}^d$ is

$$\Delta u = 0 \text{ in } \Omega, \qquad \partial_n u = \sigma u \text{ on } \partial\Omega.$$

$$\boxed{0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty}$$

The *Steklov spectral problem* on a bounded domain $\Omega \subset \mathbb{R}^d$ is

$$\Delta u = 0 \text{ in } \Omega, \qquad \partial_n u = \sigma u \text{ on } \partial\Omega.$$

$$\boxed{0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty}$$

The Steklov eigenvalues are the eigenvalues of the *Dirichlet–to–Neumann operator* $\Lambda : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$, defined by

$$\Lambda f = \partial_n u$$

where $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$.

The *Steklov spectral problem* on a bounded domain $\Omega \subset \mathbb{R}^d$ is

$$\Delta u = 0 \text{ in } \Omega, \qquad \partial_n u = \sigma u \text{ on } \partial\Omega.$$

$$\boxed{0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty}$$

The Steklov eigenvalues are the eigenvalues of the *Dirichlet–to–Neumann operator* $\Lambda : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$, defined by

$$\Lambda f = \partial_n u$$

where $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$.

The operator Λ is an elliptic self-adjoint Ψdo of order 1, with principal symbol $|\xi|$. It follows that

$$\sigma_k \sim C(d) \left(\frac{k}{|\partial\Omega|} \right)^{1/(d-1)} \quad \text{as} \quad k \nearrow \infty.$$

Isoperimetric problems for Steklov eigenvalues

Problem

Maximize $\sigma_k(\Omega)$ among domains $\Omega \subset \mathbb{R}^d$ with $|\partial\Omega| = 1$.

Isoperimetric problems for Steklov eigenvalues

Problem

Maximize $\sigma_k(\Omega)$ among domains $\Omega \subset \mathbb{R}^d$ with $|\partial\Omega| = 1$.

Given $c > 0$, it is clear that

$$\sigma_k(c\Omega) = \frac{1}{c}\sigma_k(\Omega).$$

Therefore, the functional

$$\Omega \mapsto \tilde{\sigma}_k(\Omega) := \sigma_k(\Omega)|\partial\Omega|^{1/d-1}$$

is scaling invariant.

Isoperimetric problems for Steklov eigenvalues

Problem

Maximize $\sigma_k(\Omega)$ among domains $\Omega \subset \mathbb{R}^d$ with $|\partial\Omega| = 1$.

Given $c > 0$, it is clear that

$$\sigma_k(c\Omega) = \frac{1}{c}\sigma_k(\Omega).$$

Therefore, the functional

$$\Omega \mapsto \tilde{\sigma}_k(\Omega) := \sigma_k(\Omega)|\partial\Omega|^{1/d-1}$$

is scaling invariant.

Equivalent problem

Maximize $\tilde{\sigma}_k(\Omega)$ among all regular domains $\Omega \subset \mathbb{R}^d$.

Variational characterization of σ_k

The starting point of many strategies to obtain isoperimetric results is to use a variational characterization...

Let

$$\mathcal{H}_k = \{V \subset H^1(\Omega) \, : \, \dim V = k\}.$$

$$\boxed{\sigma_k = \min_{V \in \mathcal{H}_k} \max_{f \in V \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial\Omega} f^2 \, dS}}$$

Variational characterization of σ_k

The starting point of many strategies to obtain isoperimetric results is to use a variational characterization...

Let

$$\mathcal{H}_k = \{V \subset H^1(\Omega) : \dim V = k\}.$$

$$\boxed{\sigma_k = \min_{V \in \mathcal{H}_k} \max_{f \in V \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial\Omega} f^2 \, dS}}$$

Observation

The *infimum* of $\sigma_k(\Omega)$ among domains with $|\partial\Omega| = 1$ is *zero*.

Variational characterization of σ_k

The starting point of many strategies to obtain isoperimetric results is to use a variational characterization...

Let

$$\mathcal{H}_k = \{V \subset H^1(\Omega) \, : \, \dim V = k\}.$$

$$\boxed{\sigma_k = \min_{V \in \mathcal{H}_k} \max_{f \in V \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial\Omega} f^2 \, dS}}$$

Observation

The *infimum* of $\sigma_k(\Omega)$ among domains with $|\partial\Omega| = 1$ is *zero*.

This is related to loss of compactness for the trace map

$$H^1(\Omega) \to L^2(\partial\Omega)$$

Channels, cusps,...

Physical interpretation in two dimension

The *non homogeneous Neumann* spectral problem with density $0 < \rho \in C^\infty(\overline{\Omega})$ is

$$-\Delta u = \mu \rho u \text{ in } \Omega, \qquad \partial_n u = 0 \text{ on } \partial\Omega.$$

$$\boxed{0 = \mu_0 < \mu_1(\rho) \leq \mu_2(\rho) \leq \cdots \nearrow \infty}$$

Physical interpretation in two dimension

The *non homogeneous Neumann* spectral problem with density $0 < \rho \in C^\infty(\overline{\Omega})$ is

$$-\Delta u = \mu \rho u \text{ in } \Omega, \qquad \partial_n u = 0 \text{ on } \partial\Omega.$$

$$\boxed{0 = \mu_0 < \mu_1(\rho) \leq \mu_2(\rho) \leq \cdots \nearrow \infty}$$

These are characterized using the Rayleigh quotient

$$\frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \, \rho \, dx}$$

Physical interpretation in two dimension

The *non homogeneous Neumann* spectral problem with density $0 < \rho \in C^\infty(\overline{\Omega})$ is

$$-\Delta u = \mu \rho u \text{ in } \Omega, \qquad \partial_n u = 0 \text{ on } \partial\Omega.$$

$$\boxed{0 = \mu_0 < \mu_1(\rho) \le \mu_2(\rho) \le \cdots \nearrow \infty}$$

These are characterized using the Rayleigh quotient

$$\frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \, \rho \, dx}$$

If $\rho_n dx \overset{n\to\infty}{\longrightarrow} dS$, then for $f \in H^1(\Omega)$

$$\lim_{n\to\infty} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \, \rho_n \, dx} = \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial\Omega} f^2 \, dS}$$

Physical interpretation in two dimension

The *non homogeneous Neumann* spectral problem with density $0 < \rho \in C^\infty(\overline{\Omega})$ is

$$-\Delta u = \mu \rho u \text{ in } \Omega, \qquad \partial_n u = 0 \text{ on } \partial\Omega.$$

$$\boxed{0 = \mu_0 < \mu_1(\rho) \le \mu_2(\rho) \le \cdots \nearrow \infty}$$

These are characterized using the Rayleigh quotient

$$\frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \, \rho \, dx}$$

If $\rho_n dx \xrightarrow{n\to\infty} dS$, then for $f \in H^1(\Omega)$

$$\lim_{n\to\infty} \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_\Omega f^2 \, \rho_n \, dx} = \frac{\int_\Omega |\nabla f|^2 \, dx}{\int_{\partial\Omega} f^2 \, dS}$$

One can think of the Steklov problem as a free membrane with its mass uniformly distributed along its boundary.

Isoperimetric inequalities for planar domains.

Weinstock, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\sigma_1(\Omega)|\partial\Omega| \leq \sigma_1(\mathbb{D})|\partial\mathbb{D}| = 2\pi.$$

Szegő, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\mu_1(\Omega)|\Omega| \leq \mu_1(\mathbb{D})|\mathbb{D}|.$$

Isoperimetric inequalities for planar domains.

Weinstock, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\sigma_1(\Omega)|\partial\Omega| \leq \sigma_1(\mathbb{D})|\partial\mathbb{D}| = 2\pi.$$

Weinberger, 1956.

If $\Omega \subset \mathbb{R}^2$ ~~is *simply connected*~~,

$$\mu_1(\Omega)|\Omega| \leq \mu_1(\mathbb{D})|\mathbb{D}|.$$

Isoperimetric inequalities for planar domains.

Weinstock, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\sigma_1(\Omega)|\partial\Omega| \leq \sigma_1(\mathbb{D})|\partial\mathbb{D}| = 2\pi.$$

Weinberger, 1956.

If $\Omega \subset \mathbb{R}^2$ ~~is *simply connected*~~,

$$\mu_1(\Omega)|\Omega| \leq \mu_1(\mathbb{D})|\mathbb{D}|.$$

Observation

Let $A_\epsilon = \mathbb{D} \setminus B(0, \epsilon)$. Then for small $\epsilon > 0$ one has

$$\sigma_1(A_\epsilon)|\partial A_\epsilon| > 2\pi$$

Isoperimetric inequalities for planar domains.

Weinstock, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\sigma_1(\Omega)|\partial\Omega| \leq \sigma_1(\mathbb{D})|\partial\mathbb{D}| = 2\pi.$$

Weinberger, 1956.

If $\Omega \subset \mathbb{R}^2$ ~~is *simply connected*~~,

$$\mu_1(\Omega)|\Omega| \leq \mu_1(\mathbb{D})|\mathbb{D}|.$$

Observation

Let $A_\epsilon = \mathbb{D} \setminus B(0,\epsilon)$. Then for small $\epsilon > 0$ one has

$$\sigma_1(A_\epsilon)|\partial A_\epsilon| > 2\pi$$

Simple-connectedness is not merely a technical assumption!

Isoperimetric inequalities for planar domains.

Weinstock, 1954

If $\Omega \subset \mathbb{R}^2$ is *simply connected*,

$$\sigma_1(\Omega)|\partial\Omega| \leq \sigma_1(\mathbb{D})|\partial\mathbb{D}| = 2\pi.$$

Weinberger, 1956.

If $\Omega \subset \mathbb{R}^2$ ~~is *simply connected*~~,

$$\mu_1(\Omega)|\Omega| \leq \mu_1(\mathbb{D})|\mathbb{D}|.$$

Observation

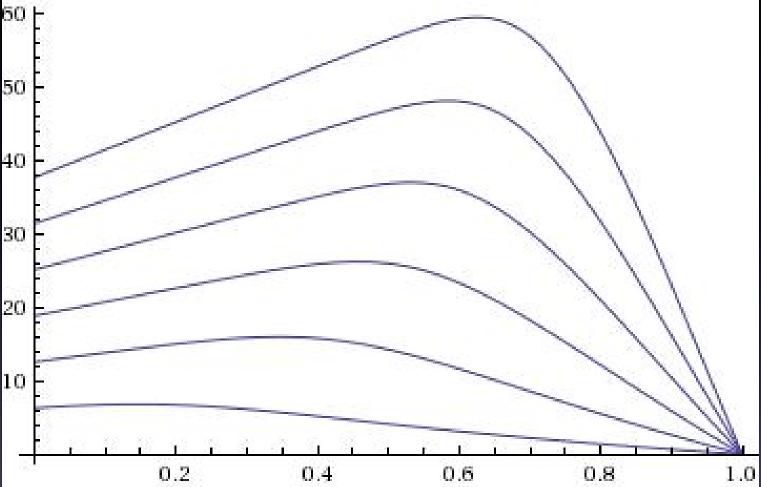
Let $A_\epsilon = \mathbb{D} \setminus B(0,\epsilon)$. Then for small $\epsilon > 0$ one has

$$\sigma_1(A_\epsilon)|\partial A_\epsilon| > 2\pi$$

Simple-connectedness is not merely a technical assumption!

What can we say for multiply connected domains?

Normalized eigenvalues of A_ϵ



Higher eigenvalues for simply connected domains

Hersch–Payne–Schifer, 1974.

If $\Omega \subset \mathbb{R}^2$ is *simply connected*, then for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\partial\Omega| \leq k\sigma_1(\mathbb{D})|\partial\mathbb{D} = 2k\pi.$$

Higher eigenvalues for simply connected domains

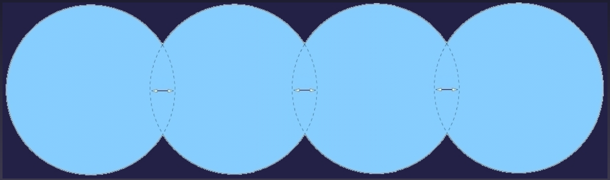
Hersch–Payne–Schifer, 1974.

If $\Omega \subset \mathbb{R}^2$ is *simply connected*, then for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\partial\Omega| \leq k\sigma_1(\mathbb{D})|\partial\mathbb{D} = 2k\pi.$$

G.–Polterovich, 2010.

This inequality is *sharp*, and attained in the limit by a family of domains Ω_ϵ degenerating to k disjoint identical disks.



$k = 4$

This contrasts with Neumann eigenvalues. . .

Upper bounds for surfaces

Fraser–Schoen, 2011.

If Ω is a smooth compact surface of genus γ with l boundary components, then

$$\sigma_1(\Omega)|\partial\Omega| \leq 2(\gamma + l)\pi.$$

Upper bounds for surfaces

G.–Polterovich, 2012

If Ω is a smooth compact surface of genus γ with l boundary components, then for each $k \in \mathbb{N}$

$$\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k.$$

Upper bounds for surfaces

G.–Polterovich, 2012

If Ω is a smooth compact surface of genus γ with l boundary components, then for each $k \in \mathbb{N}$

$$\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k.$$

- Weinstock: $\gamma = 0$, $l = 1$, $k = 1$.
- Hersch–Payne–Schiffer: $\gamma = 0$, $l = 1$, arbitrary $k \in \mathbb{N}$.
- Fraser–Schoen, 2011: $k = 1$, arbitrary γ and l.

Upper bounds for surfaces

G.–Polterovich, 2012

If Ω is a smooth compact surface of genus γ with l boundary components, then for each $k \in \mathbb{N}$

$$\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k.$$

- Weinstock: $\gamma = 0$, $l = 1$, $k = 1$.
- Hersch–Payne–Schiffer: $\gamma = 0$, $l = 1$, arbitrary $k \in \mathbb{N}$.
- Fraser–Schoen, 2011: $k = 1$, arbitrary γ and l.

These inequality are in general *not sharp*. For instance,

Fraser–Schoen, 2011

For $l = 2$ and $\gamma = 0$, the maximum of $\sigma_1(\Omega)|\partial\Omega|$ is attained at the *critical catenoid*. (max $\approx 4\pi/1.2$)

Upper bounds for surfaces

G.–Polterovich, 2012

If Ω is a smooth compact surface of genus γ with l boundary components, then for each $k \in \mathbb{N}$

$$\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k.$$

- Weinstock: $\gamma = 0$, $l = 1$, $k = 1$.
- Hersch–Payne–Schiffer: $\gamma = 0$, $l = 1$, arbitrary $k \in \mathbb{N}$.
- Fraser–Schoen, 2011: $k = 1$, arbitrary γ and l.

These inequality are in general *not sharp*. For instance,

Fraser–Schoen, 2011

For $l = 2$ and $\gamma = 0$, the maximum of $\sigma_1(\Omega)|\partial\Omega|$ is attained at the *critical catenoid*. (max $\approx 4\pi/1.2$)

Also, not sharp for large l...

Open problems/projects

Let Ω is a smooth compact surface of genus γ with l boundary components.

$$\boxed{\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k}$$

Open problems/projects

Let Ω is a smooth compact surface of genus γ with l boundary components.

$$\boxed{\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k}$$

Problem

Find sharp upper bounds in the general case

Open problems/projects

Let Ω is a smooth compact surface of genus γ with l boundary components.

$$\boxed{\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k}$$

Problem

Find sharp upper bounds in the general case (*good luck!*)

Open problems/projects

Let Ω is a smooth compact surface of genus γ with l boundary components.

$$\boxed{\sigma_k(\Omega)|\partial\Omega| \leq 2\pi(\gamma + l)k}$$

Problem

Find sharp upper bounds in the general case (*good luck!*)

Ongoing project with Bruno Colbois

There exists a sequence Ω_n of surfaces such that

$$\sigma_1(\Omega_n)|\partial\Omega_n| \nearrow \infty.$$

(In this situation, the genus will have to diverge.)

Thank you for your attention!