A Multi-scale Jacobi Method for Anderson Localization

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October 31, 2013

Overview

Before working on proving many-body localization in some system, we need a systematic procedure for constructing eigenfunctions of a random Hamiltonian.

Traditional approaches to Anderson localization using Green's functions are problematic for many-body problems because the energy is extensive.

Instead, we will perform a sequence of similarity transformations on the Hamiltonian to drive down off-diagonal terms. We will need to prove exponential localization. In the many-body problem, exponential decay of probabilities will be required to counter the geometric growth of the number of configurations with the volume.

Joint work with Tom Spencer

Anderson model

$$H = H_0 + J$$

$$H_{ij} = v_i \delta_{ij} + J_{ij}$$

Here

$$J_{ij} = \begin{cases} J_0 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and v_i are iid random variables representing unperturbed energies.

Results

Our goal is to prove exponential decay of the eigenfunctions of $H^{(\Lambda)}$ uniformly in Λ for small J_0.

Theorem

There is a $p > 0$ such that if J is sufficiently small (depending only on D and ρ_0), the eigenfunction correlator obeys the bound

$$\int d\lambda(v) \sum_{\alpha} |\psi_\alpha(i)\psi_\alpha(j)| \ \leq J_0^{p|i-j|}$$

As a consequence, if $\tilde{J} > 0, \tilde{\epsilon} > 0$, with $\tilde{J}\tilde{\epsilon} = J_0^p$, then

$$\sum_{\alpha} |\psi_\alpha(i)\psi_\alpha(j)| \leq \tilde{J}^{|i-j|} \text{ with probability } 1 - \tilde{\epsilon}^{|i-j|}.$$

See http://pi.math.virginia.edu/ ji2k/banff/jacobi.pdf for a draft.

Resonances

Perturbation theory works if there are gaps between eigenvalues. This is a problem because the eigenvalues become dense as $\Lambda \longrightarrow Z^d$. Luckily, J couples nearest neighbors only, so we only need to worry about nearest neighbor resonances (for the moment):

$$\langle i, j \rangle \text{ resonant if } |v_i - v_j| < \epsilon \equiv J_0^{1/20}.$$

Resonant bonds form a dilute set of regions where perturbation theory breaks down.

Perturbation Theory

Let

$$J = J^{\mathrm{res}} + J^{\mathrm{per}}$$

where J^{res} contains links internal to resonant blocks. Then put

$$A_{ij} = \frac{J^{\mathrm{per}}_{ij}}{E_i - E_j}.$$

First order perturbation theory:

$$\begin{aligned} \psi_i^{(1)} &= \psi_i + \sum_j \frac{J^{\mathrm{per}}_{ij}}{E_i - E_j}\, \psi_j \\ &= \sum_j (I + A)_{ij} \psi_j. \end{aligned}$$

Here $\psi_i(j) = \delta_{ij}$ are the unperturbed eigenvectors.

Effective Hamiltonian

Instead, use $\Omega = e^{-A}$ for the basis change (preserves norm).

$$\psi_i^{(1)} = \sum_j \Omega_{ij}^{\mathrm{tr}} \psi_j.$$

Renormalized Hamiltonian:

$$H^{(1)} = \Omega^{\mathrm{tr}} H \Omega$$

Observe that $[A, H_0] = -J^{\mathrm{per}}$:

$$[A, H_0]_{ij} = \frac{J_{ij}^{\mathrm{per}} E_j - E_i J_{ij}^{\mathrm{per}}}{E_i - E_j} = -J_{ij}^{\mathrm{per}}.$$

Then, using $H = H_0 + J$, we have $[A, H] = -J^{\mathrm{per}} + [A, J]$, and so

$$\begin{aligned}
H^{(1)} &= e^A H e^{-A} = H + [A, H] + \frac{[A, [A, H]]}{2!} + \dots \\
&= H_0 + J^{\mathrm{res}} + J^{\mathrm{per}} - J^{\mathrm{per}} + [A, J] + \frac{[A, -J^{\mathrm{per}} + [A, J]]}{2!} + \dots \\
&= H_0 + J^{\mathrm{res}} + \frac{1}{2}[A, J^{\mathrm{per}}] + [A, J^{\mathrm{res}}] + \dots \\
&= H_0 + J^{\mathrm{res}} + J^{(1)}.
\end{aligned}$$

Properties of new Hamiltonian:

After the change of basis:

J^{per} is gone

J^{res} is still there

$J^{(1)}$ is quadratic and higher order in J_0, containing terms of the form AJ, JA, AJA, AAJ, etc.

$J^{(1)}$ is now long-range: it has a random walk expansion exhibiting exponential decay away from the resonant blocks.

We also perform exact rotations O in "small" resonant blocks to diagonalize the Hamiltonian there.

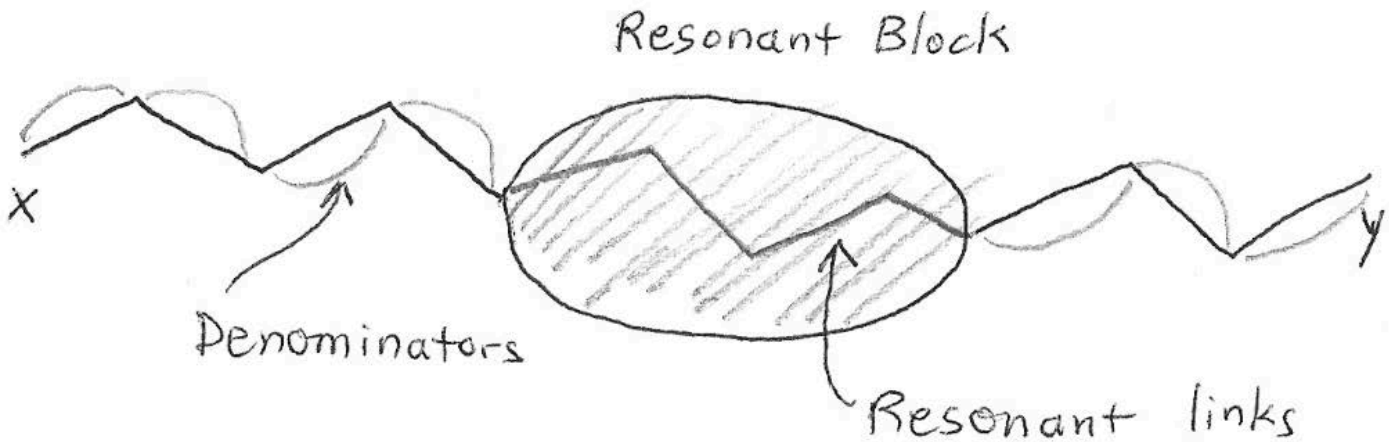
Average over the potential v

Note that A_{ij} is bounded by J_0/ϵ, which is small.

After averaging we get exponential decay either from the small probability of resonant links or from the random walk expansion in $\Omega = e^{-A}$:

$$\int d\lambda(v) \sum_{\alpha} |(\Omega O)_{x\alpha} (\Omega O)^{\mathrm{tr}}_{\alpha y}| \leq (c_D \epsilon)^{|x-y|}.$$

If we could prove this for the complete diagonalization of H, that would be exponential localization.



Continue the process on a sequence of length scales $L_k = (15/8)^k$

$J^{(k)}$ is a sum of graphs $J^{(k)}_{xy}(g)$

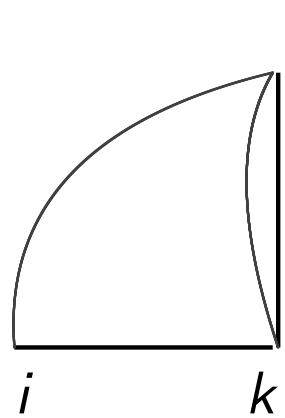
g is resonant if $A^{(k)}_{xy}(g) \equiv \frac{J^{(k)}_{xy}(g)}{E^{(k)}_x - E^{(k)}_y}$ is larger than $(J/\epsilon)^{|g|}$.

Energies $E^{(k)}$ are the diagonal elements of $H^{(k)}$: the potential v is renormalized by interactions up to the k^{th} scale.

Multi-denominator estimates

Example: two step graph, two denominator lines. $|i-j|=2$, $|k-j|=1$.

Naive bound: $\frac{J_0^2}{\epsilon\cdot\epsilon^2}$ not adequate



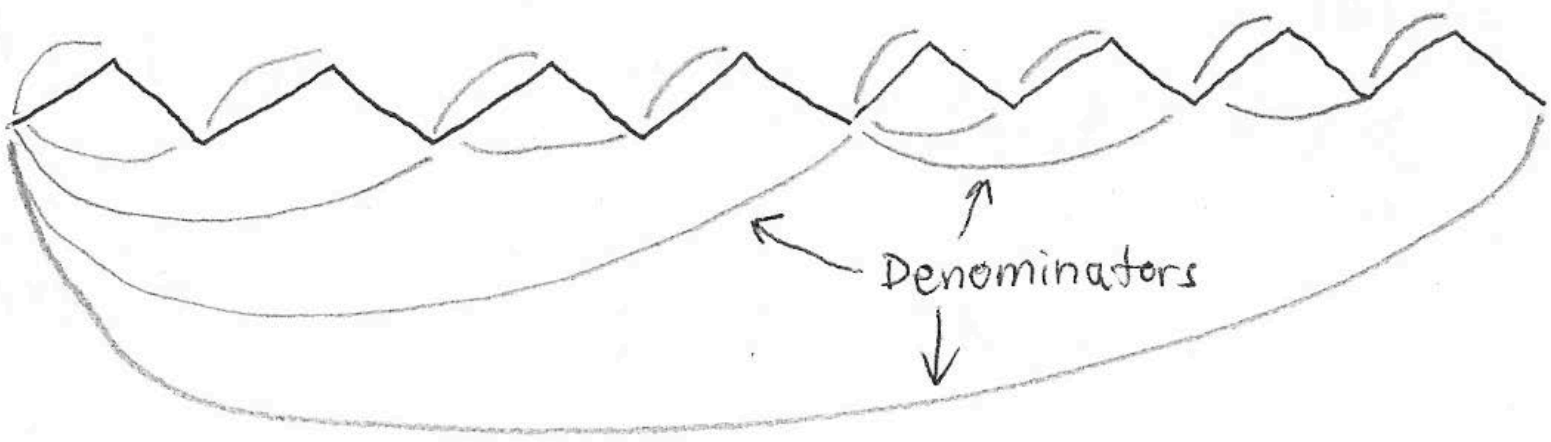
Markov inequality: $E(A(g)^s) \sim J_0^{2s}$ with $s = 4/5$.

$P(|A(g)| > (J_0/\epsilon)^2) \leq \epsilon^{2s}$

Retains exponential decay in $|g|$ for $A(g)$ when non-resonant.

The probability that g is resonant also decays exponentially in $|g|$.

Diagrams on the k^{th} scale



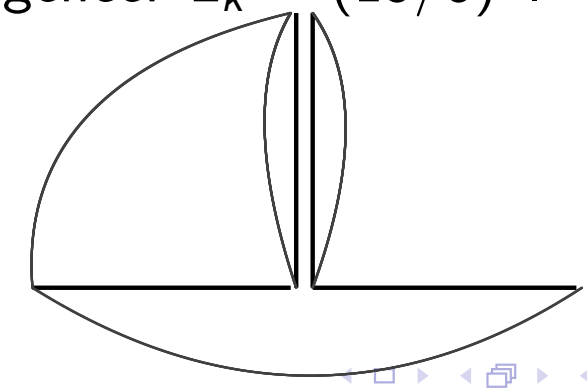
Loops

If g has loops, then the denominator graph has loops, too, so there are duplicated or non-independent denominators. We have integrals like $\int_{|v|\geq\epsilon} \frac{1}{v^2} d\lambda(v) \leq \epsilon^{-1}$ which lead to negative powers of ϵ. This weakens the bound on the probability that the graph is resonant.

If a graph is not too much longer than the distance traveled, it will have at least $1/4$ of its length free of looping problems, and then

$$P\left(|A(g)| \geq (J_0/\epsilon)^{|g|}\right) \leq \epsilon^{s|g|/4}.$$

Graphs that are quite a bit longer than the distance traveled need to be resummed, but the extra links ensure exponential decay in the distance (probabilistic bounds not needed). Resummation leads to slower than quadratic convergence: $L_k = (15/8)^k$.



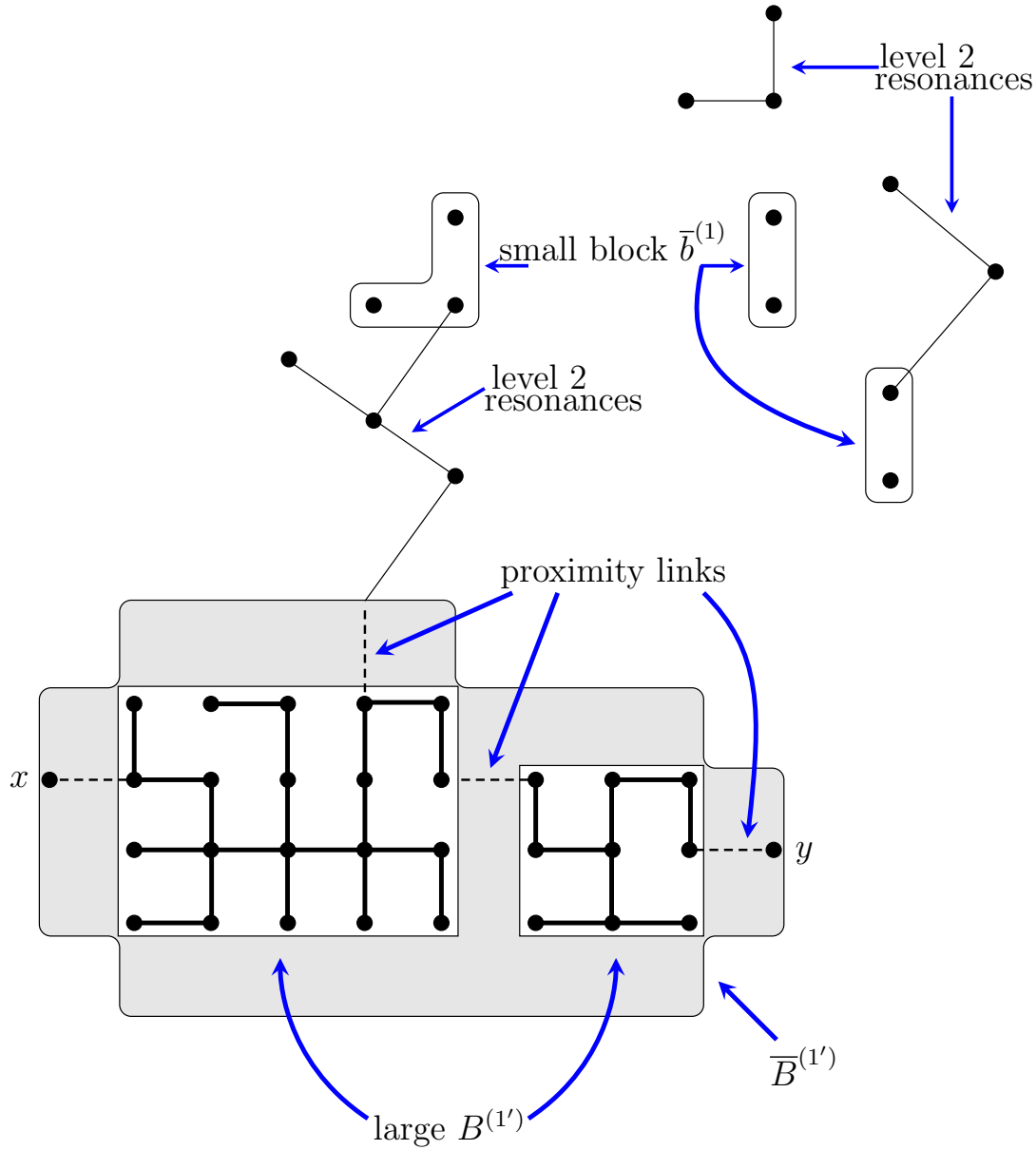
Walks move through landscape of resonant blocks

Resonant blocks can be treated on scale k if they have volume $\leq exp(ML_k^{2/3})$.

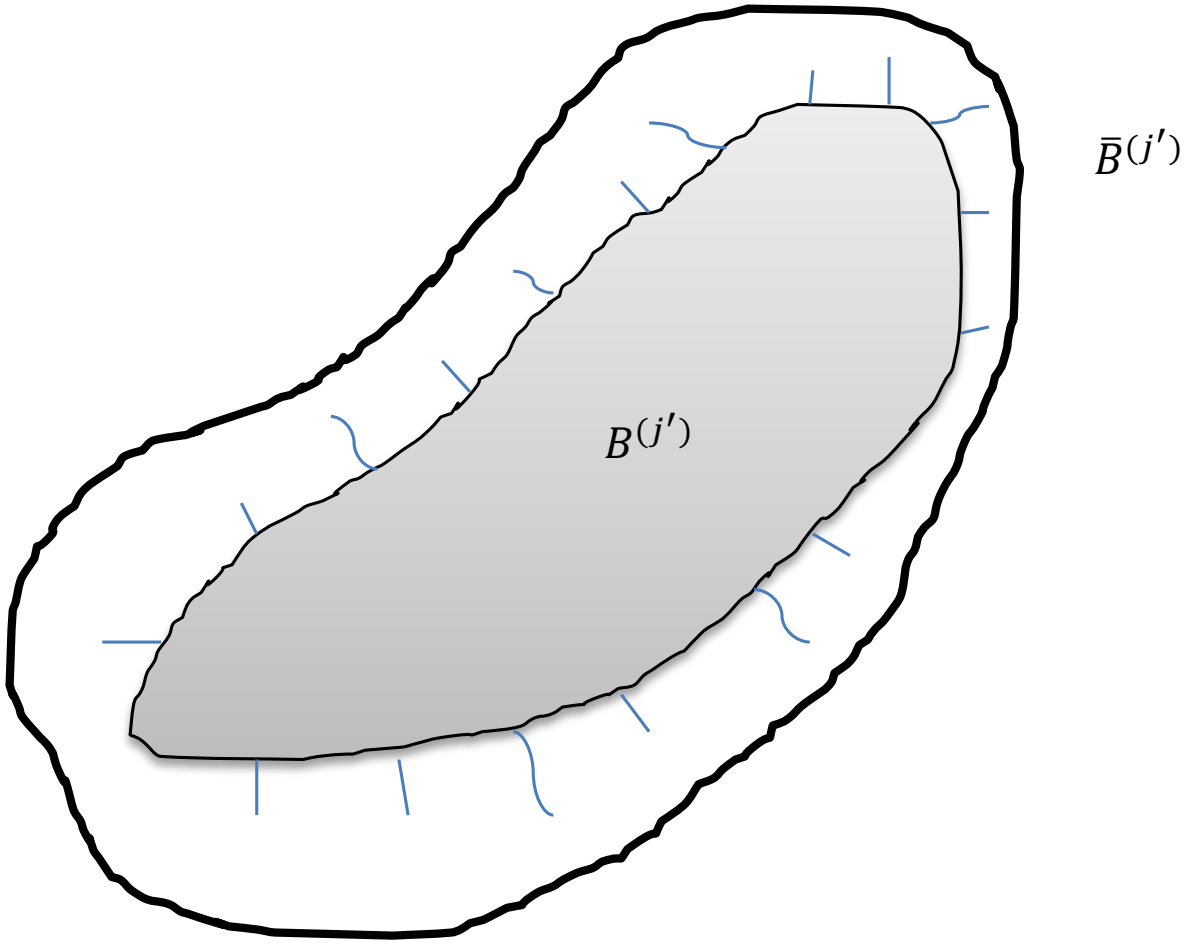
Such blocks effectively have a connectivity constant $exp(ML_k^{2/3})$ from the sum over states in the block and the sum over sites adjacent to the block.

Not a problem, because couplings are $O(J_0/\epsilon)^{L_k}$ and resonant graphs have probability $O(\epsilon^{sL_k/4})$

Walks move through landscape of resonant blocks



Large blocks



Large blocks need a collar (width logarithmic in the volume) so that they interact only through diagrams of sufficiently high order.

Many-body Hamiltonian

What are the prospects for proving many-body localization with this method?

Start with a simple 2 by 2 Hamiltonian on each site:

$$H_0 = \sum_{i \in \Lambda} \begin{pmatrix} h_i & J \\ J & -h_i \end{pmatrix} = \sum_{i \in \Lambda} (h_i \sigma_i^z + J \sigma_i^x)$$

where h_i is a random field. Then let

$$H = H_0 + \sum_{\langle i,j \rangle} J_0 \sigma_i^z \sigma_j^z$$

This is a quantum Ising model with a random magnetic field.

Many-body localization

We would like to diagonalize this Hamiltonian and understand the nature of the eigenfunctions for small J and/or small J_0. What does localization mean in this context?

In the Anderson model, localization means that the eigenfunctions closely resemble the $J = 0$ eigenfunctions, which are δ-functions at the sites in Λ.

In the many-body context, it should mean that the spin variables should (with rare exceptions) resemble those of the $J = 0$ states. That is, a prescribed set of 1's and -1's in the lattice. Let's call the state-labeling spin configuration σ_i^{label}.

The many-body wave function should be "concentrated" on configurations "close to" σ_i^{label}. This means in particular that each wave function is approximately a product state, with very little entanglement.

Background on many-body localization: A special kind of quantum phase transition

Basko, Aleiner & Altschuler: Annals of Physics 321, 1126 (2006)
Pal & Huse: Phys. Rev. B 82, 174411 (2010), arxiv:1010:1992

Weak Randomness:	Strong randomness:
ergodic	localized
long-range entanglement	short range entanglement
thermalized	non-thermalized

Thermalized means an eigenstate at energy E would look locally like a statistical ensemble of states in thermal equilibrium.
Localized means one bare state σ^{label} (plus rare localized variations of it) predominates.

Desired Result

We would like to diagonalize H with a set of eigenfunctions given by graphical expansions with explicit bounds, including bounds on probabilities of rare events. Then, because of the smallness or rarity of deviations from σ^{label}, one should be able to show that

$$\text{Av}_h \ \text{Av}_{\sigma^{\text{label}}} \ |\langle \sigma_0^z \rangle_{\psi_{\sigma^{\text{label}}}}| \text{ is close to 1,}$$

which shows that for most h's and for most states the state follows the label. In the thermalized case, this would be a mixture of many (unperturbed) states and hence presumably 0.

This is analogous to the situation for the mixed-state classical Ising model at low temperature:

$$\langle \cdot \rangle_{\text{mixed}} = \frac{1}{2}\langle \cdot \rangle_+ + \frac{1}{2}\langle \cdot \rangle_-$$

with

$$\langle \sigma_0 \rangle_{\text{mixed}} = 0, \qquad |\langle \sigma_0 \rangle_\pm| \text{ close to 1.}$$

A more general framework

It is useful to consider a broader class of Hamiltonians:

$$H = H_0 + \sum_{X \in \Lambda} J(X),$$

where

X is a connected subset of the lattice $\Lambda \subset Z^d$, and $J(X)$ is an off-diagonal matrix operating on $\otimes_{i \in X} C^2$, the Hilbert space for spins in X.

H_0 is diagonal, may also contain nonlocal terms, but basically looks like $\sum_i h_i \sigma_i^z$.

$J(X)$ and nonlocal terms in H_0 are small, with exponential decay in $|X|$.

Initially $J(X)$ will couple nearest neighbor spins only. But as we proceed with our KAM/RG scheme, more general interactions $J(X)$ will be generated.

Resonances

Perturbation theory works if there are gaps between eigenvalues.

More precisely, if off-diagonal terms of H connect states whose energies are separated by gaps.

The many-body Hamiltonian has only local interactions, so we need only be concerned about resonances from local transitions. Initially, we would say that

$$\langle i, j \rangle \text{ resonant if } |\Delta E| < \epsilon$$

for some transition involving the spins at nearest-neighbor sites i, j.

- Resonant bonds form a dilute set of regions where perturbation theory breaks down.

Perturbation Theory

As in the discussion for the Anderson model, let

$$J = J^{\mathrm{res}} + J^{\mathrm{per}}$$

where J^{res} contains terms internal to resonant blocks. Then put

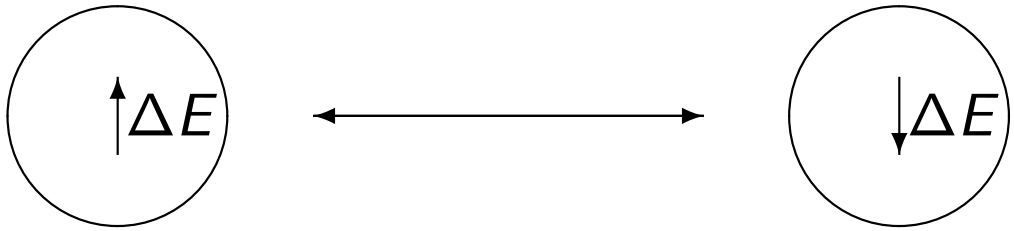
$$A(X) = \frac{J^{\mathrm{per}}(X)}{\Delta E}$$

which now generates local moves in spin space, instead of real space. This means that $\Omega = e^{-A} = \exp\left(-\sum_X A(X)\right)$ generates disconnected graphs when we define the effective Hamiltonian $H^{(1)} = \Omega^{\mathrm{tr}} H \Omega$ (unlike the situation with the Anderson model).

However, rotations of local observables $\mathcal{O} \to \Omega^{\mathrm{tr}} \mathcal{O} \Omega$ do produce connected graphs, so in the end the procedure is not dissimilar from the Anderson model case.

The most important difference is the exponential growth of the number of states—this is why we insisted on exponential bounds on resonance probabilities.

Block-Block Resonances



How to control the probability of resonance between blocks?

Can one show that energy differences within a block vary with the randomness?

This is still an open problem. Only an issue for length scales exponental in a power of $1/J_0$.