SDE limits for transfer matrices with hyperbolic channels and limiting eigenvalue processes

Christian Sadel[1]
joint work with Bálint Virág[2]

[1] University of British Columbia, Vancouver
[2] University of Toronto

work in progress

Banff - 31 Oct 2013

Random Schrödinger operators on strips

- Consider the random Schrödinger operator

$$(H_\lambda \psi)(n) = \psi(n+1) + \psi(n-1) + A\psi(n) + \lambda V(n)\psi(n)$$

  where $\psi = (\psi(n))_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^d$, $\lambda \in \mathbb{R}$ is a small coupling constant, A, $V(n) \in \mathrm{Her}(d)$.
- The $V(n)$ are i.i.d. random matrices with $\mathbb{E}(V(n)) = \mathbf{0}$, such that $\mathbb{E}(\|V(n)\|^{6+\epsilon}) < \infty$.
- If A is the adjacency matrix of a finite graph $\mathbb{G}$ and the $V(n)$ are diagonal matrices with i.i.d. entries along the diagonal, then this corresponds to the Anderson model on the product graph $\mathbb{Z} \times \mathbb{G}$.

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elliptic and hyperbolic channels

- Solving $H_\lambda \psi = E\psi$ leads to $\binom{\psi_{n+1}}{\psi_n} = \mathcal{T}^E_{\lambda,n}\binom{\psi_n}{\psi_{n-1}}$ where

$$\mathcal{T}^E_{\lambda,n} = \begin{pmatrix} E\,\mathbf{1} - A - \lambda V(n) & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

  are called transfer matrices. We call $\mathcal{T}^E_0 = \mathcal{T}^E_{0,n}$ the unperturbed or free transfer matrix.
- Let E be an energy such that $E\mathbf{1} - A$ has no eigenvalue ± 2.
- Let $(\varphi_j)_{j=1}^d$ be an orthonormal set of eigenvectors of A. We call φ_j an elliptic channel at energy E if it corresponds to an eigenvalue of $E\mathbf{1} - A$ with absolute values < 2, and we call it an hyperbolic channel if it corresponds to an eigenvaue of $E\mathbf{1} - A$ of absolute value > 2.

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- Each elliptic channel φ_j gives a pair of eigenvalues $e^{\pm ik}$, $k \in (0, \pi)$ of $\mathcal{T}_0^E$ with eigenvectors $\binom{e^{\pm ik}\varphi_j}{\varphi_j}$ corresponding to one left and one right moving wave (extended eigenstate) of H_0.
- Each hyperbolic channel gives a pair of eigenvalues $\gamma^{\pm 1}$, $|\gamma| > 1$, for $\mathcal{T}_0^E$.
- If at an energy E one has d_e elliptic and d_h hyperbolic channels, $d_e + d_h = d$, then the multiplicity of the spectrum of H_0 at E is $2d_e$ (there are d_e overlapping bands at E for each of which one has one right and one left moving extended eigenstate)
- We want to describe the Markov process of the transfer matrix from 1 to n,

$$\mathcal{T}_{\lambda,[1,n]}^E = \mathcal{T}_{\lambda,n}^E \mathcal{T}_{\lambda,n-1}^E \cdots \mathcal{T}_{\lambda,2}^E \mathcal{T}_{\lambda,1}^E$$

in a critical scaling limit $n \sim \lambda^{-2}$, $\lambda \to 0$, $n \to \infty$.

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SDE limit for $\mathcal{T}_0^E$ having only elliptic channels

Recall:

$$\mathcal{T}^E_{\lambda,[1,n]} = \mathcal{T}^E_{\lambda,n}\mathcal{T}^E_{\lambda,n-1}\cdots\mathcal{T}^E_{\lambda,2}\mathcal{T}^E_{\lambda,1}$$

Theorem (Valko, Virag; Bachmann, de Roeck)

Let A and E be such that there are only elliptic channels (i.e. $\mathcal{T}_0^E$ is conjugated to a unitary matrix), and consider the process $X_{\lambda,n} = \left(\mathcal{T}_0^E\right)^{-n}\mathcal{T}^E_{\lambda,[1,n]}$. Then, in distribution for $n \to \infty$

$$X_{\frac{1}{\sqrt{n}},\lfloor tn\rfloor} \implies X_t$$

where X_t satisfies a SDE (stochastic differential equation) of the form

$$dX_t \;=\; d\mathcal{B}_t\, X_t$$

where $d\mathcal{B}_t$ is a matrix-Brownian motion with certain variances and covariances of its entries.

Applications in the pure one-dimensional model:

- Kritchevski, Valko and Virag studied the eigenvalue statistics in this critical scaling.
- Rifkind and Virag studied distribution of shape of eigenfunctions in this scaling limit

Application on strip models:

- Valko and Virag obtained GOE limiting statistics for certain sequences of modified Anderson models on long boxes
- Bachmann and De Roeck discussed relations from Random Matrix Theory to the Anderson model and the DMPK equation
- In both papers the Anderson model on a strip is slsightly modified by scaling down the vertical Laplacian to ensure that one has an energy interval around 0 such that for all these energies there are only has elliptic channels.
- **If you want to treat an energy interval for the honest Anderson model in a limit to infinite width or if you want to treat all energies in the spectrum of H_0 in this critical limit, then one has to deal with hyperbolic channels.**

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Heuristics

$X_{\lambda,n} = \left(\mathcal{T}_0^E\right)^{-n}\mathcal{T}^E_{\lambda,[1,n]}$ follows an evolution of the form:

$$X_{\lambda,n+1} = \Big(\mathbf{1} + \lambda\left(\mathcal{T}_0^E\right)^{-n-1}\mathcal{V}_n\left(\mathcal{T}_0^E\right)^n\Big)X_{\lambda,n}$$

- The $\mathcal{V}_n$ are i.i.d. random matrices, giving a diffusive term of order λ^2 (variance)
- Since $\mathcal{T}_0^E$ is conjugated to a unitary, i.e. generates a compact group, the conjugations with $\left(\mathcal{T}_0^E\right)^n$ lead to an average over the compact group generated by $\mathcal{T}_0^E$.
- In the scaling limit $n \sim \lambda^{-2} \to \infty$ the total diffussion after n steps is of order 1 and one obtains a limiting process as in the central limit theorem.
- If $\mathcal{T}_0^E$ has eigenvalues of different size, then conjugates lead to exponential growing terms in n, preventing the existence of a limit process.
- One needs to project on the elliptic channels in a good way.

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Case with hyperbolic channels

- Let us assume we have $d_e > 0$ elliptic and $d_h > 0$ hyperbolic channels at energy E, $d_e + d_h = d$.
- Then with an adequate basis change we find that $\mathcal{T}_{\lambda,n} = \mathcal{C}\mathcal{T}^E_{\lambda,n}\mathcal{C}^{-1}$ is of the form

$$\mathcal{T}_{\lambda,n} = \begin{pmatrix} \Upsilon & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Upsilon^{-1} \end{pmatrix} + \lambda \mathcal{V}(n)$$

  where $U \in \mathrm{U}(2d_e)$ is unitary and $\Upsilon \in \mathrm{GL}(d_h)$ satisfies $\|\Upsilon\| < 1$, $\mathcal{V}(n)$ are i.i.d. random matrices.
- Let

$$\mathcal{X}_{\lambda,n} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & U^{-n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \mathcal{T}_{\lambda,n}\mathcal{T}_{\lambda,n-1}\cdots\mathcal{T}_{\lambda,2}\mathcal{T}_{\lambda,1}\,.$$

  We will eliminate the exponential growing part of $\mathcal{X}_{\lambda,n}$ by taking a Schur complement.

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where $A_{\lambda,n} \in \mathrm{Mat}(d_h + 2d_e, \mathbb{C})$ and $D_{\lambda,n} \in \mathrm{Mat}(d_h, \mathbb{C})$.

- Equivalence relation: Let $\mathcal{X}_1 \sim \mathcal{X}_2$ if $\mathcal{X}_1 = \mathcal{X}_2 \begin{pmatrix} \mathbf{1}_{2d_e+d_h} & \mathbf{0} \\ C & D \end{pmatrix}$.
- Since

$$\begin{pmatrix} A_{\lambda,n} & B_{\lambda,n} \\ C_{\lambda,n} & D_{\lambda,n} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -D_{\lambda,n}^{-1} C_{\lambda,n} & D_{\lambda,n}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} A_{\lambda,n} - B_{\lambda,n} D_{\lambda,n}^{-1} C_{\lambda,n} & B_{\lambda,n} D_{\lambda,n}^{-1} \\ \mathbf{0} & \mathbf{1}_{d_h} \end{pmatrix}$$

the equivalence class of $\mathcal{X}_{\lambda,n}$ is determined by

$$X_{\lambda,n} := A_{\lambda,n} - B_{\lambda,n} D_{\lambda,n}^{-1} C_{\lambda,n}, \qquad B_{\lambda,n} D_{\lambda,n}^{-1}$$

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Theorem (Virag, S.)

In distribution, for $n \to \infty$ and any $t > 0$,

$$B_{\frac{1}{\sqrt{n}},\lfloor tn \rfloor}\, D^{-1}_{\frac{1}{\sqrt{n}},\lfloor tn \rfloor} \Longrightarrow \mathbf{0}\ , \quad X_{\frac{1}{\sqrt{n}},\lfloor tn \rfloor} \Longrightarrow X_t = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \Lambda_t X_{21} & \Lambda_t X_{22} \end{pmatrix}$$

where Λ_t is a $2d_e \times 2d_e$ matrix process satisfying a SDE

$$d\Lambda_t = V\,\Lambda_t\,dt \;+\; d\mathcal{B}_t\,\Lambda_t\ , \quad \Lambda_0 = \mathbf{1}$$

Doing the same procedure for the transfer matrices $\mathcal{T}_{\lambda,n}^{E+\lambda^2\varepsilon}$ we obtain the following:

Theorem (Virag, S.)

Let $H_{\lambda,n}$ be the restriction of H_λ to $\ell^2(\{1,\ldots,n\})\otimes\mathbb{C}^d$ with Dirichlet boundary conditions, let $\mathcal{E}_{\lambda,n}$ be the eigenvalue process of $H_{\lambda,n}-E$, then for subsequences $n_k\to\infty$

$$n_k\,\mathcal{E}_{\frac{1}{\sqrt{n_k}},n_k} \implies \mathrm{zeros}_\varepsilon \det(M_0^*\Lambda_1^\varepsilon M_1)$$

where $M_0, M_1\in\mathbb{C}^{2d_e\times d_e}$, Λ_t^ε is a $2d_e\times 2d_e$ matrix process that for fixed ε satisfies a SDE of the form

$$d\Lambda_t^\varepsilon = (V+\varepsilon W)\Lambda_t^\varepsilon\,dt + d\mathcal{B}_t\,\Lambda_t^\varepsilon\,, \quad \Lambda_t^\varepsilon = \mathbf{1}$$

THANK YOU!!