The path spaces of a graph

Graph algebras workshop, BIRS

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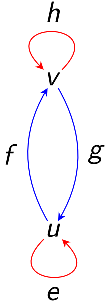
MODEL / ANALYSE / FORMULATE / ILLUMINATE
CONNECT:IMIA

Directed graphs

A *directed graph* E consists of a set E^0 of vertices and a set E^1 of directed edges, with direction determined by range and source maps $r, s : E^1 \to E^0$. A *k-coloured graph* is a directed graph with a map $c : E^1 \to \{c_1, \ldots, c_k\}$.



$$E^0 = \{v, u\}$$
$$E^1 = \{e, f, g, h\}$$
$$v = s(g) = s(h) = r(h) = r(f)$$
$$u = s(f) = s(e) = r(e) = r(g)$$
$$c(f) = c(g) = c_1 \ (= \text{blue})$$
$$c(e) = c(h) = c_2 \ (= \text{red})$$

Paths: Raeburn Mk II / Australian

- A sequence $\mu_1\mu_2\mu_3\dots$ of edges is a *path* if $s(\mu_i) = r(\mu_{i+1})$ for all i.

$$r(\mu) \xleftarrow{\ \mu_1\ } \text{- - - - - - - - - -} \xleftarrow{\ \mu_n\ } s(\mu)$$

- $E^n = \{\mu : \mu \text{ is a path with } n \text{ (possibly } = \infty) \text{ edges}\}$
- $E^* = \{\mu : \mu \text{ has finitely many edges}\}$.

Higher-rank graphs

- A *higher-rank* graph, or k-graph, is a small category Λ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorisation property: if $\lambda \in \mathrm{Mor}(\Lambda)$ has $d(\lambda) = m + n$, then there exists unique $\mu, \nu \in \mathrm{Mor}(\Lambda)$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.
- Call d the *degree* functor.

Examples

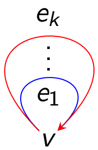
1. Suppose E is a directed graph. The path category $\mathcal{P}(E)$ of E has $\mathrm{Obj}(\mathcal{P}(E)) = E^0$, $\mathrm{Mor}(\mathcal{P}(E)) = E^*$, range, source and composition inherited from E. With $d(\lambda) := |\lambda|$, $\mathcal{P}(E)$ is a 1-graph. Moreover, every 1-graph occurs as the path category of a directed graph
2. Let T_k be the category with a single object and morphisms $\mathbb{N}^k$. With $d = \mathrm{id}_{\mathbb{N}^k}$, T_k is a k-graph.

Skeletons

We may visualise a k-graph Λ by its *skeleton*: the k-coloured directed graph E_Λ with $E_\Lambda^0 = \mathrm{Obj}(\Lambda)$, $E_\Lambda^1 = \bigcup_{i \leq k} d^{-1}(e_i)$, range and source as in Λ, and colouring $c^{-1}(c_i) = d^{-1}(e_i)$.

Examples

1. $\mathcal{P}(E)$ has skeleton isomorphic to E.
2. The skeleton of T_k has single vertex, and a different coloured loop for each generator of $\mathbb{N}^k$:



Skeletons

Examples

3. For each $m \in (\mathbb{N} \cup \{\infty\})^k$ there is a k-graph $\Omega_{k,m}$ with objects $\{p \in \mathbb{N}^k : p \le m\}$, morphisms $\{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \le q \le m\}$, $r(p,q) = p$, $s(p,q) = q$, $(p,q)(q,t) = (p,t)$, and $d(p,q) = q - p$. The skeleton of $\Omega_{k,m}$ is denoted $E_{k,m}$. The following 2-coloured graph is $E_{2,(3,2)}$

$$
\begin{array}{ccccccc}
(0,2) & \xleftarrow{} & (1,2) & \xleftarrow{} & (2,2) & \xleftarrow{} & (3,2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(0,1) & \xleftarrow{} & (1,1) & \xleftarrow{} & (2,1) & \xleftarrow{} & (3,1) \\
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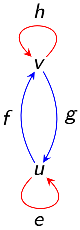
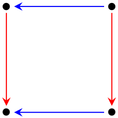
$$

k-coloured graphs

A *coloured-graph morphism* is a range, source and colour preserving map between two coloured graphs.





A *square* is a coloured-graph morphism from the coloured graph on the right into E. We think of this as a labelling of the picture on the right with elements of our graph.

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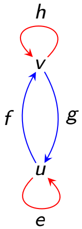

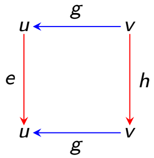

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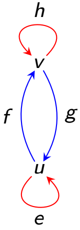
Squares

Given a k-coloured graph E, we say a collection of squares $\mathcal{C}$ is *complete* if for each $c_i c_j$-coloured path $x \in E^2$, there exists a unique square in $\mathcal{C}$ of which x is a subpath.



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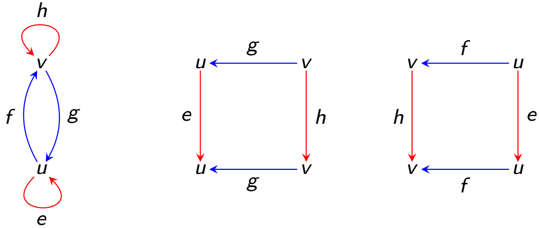
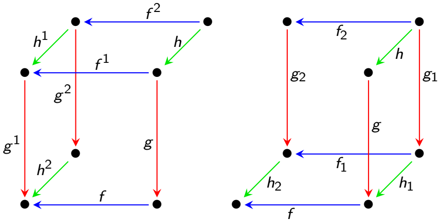


For example: these two squares are a complete collection for E. Such a collection is not typically unique.

Associativity of $\mathcal{C}$

Let E be a k-coloured graph and $\mathcal{C}$ be a complete collection of squares. Given a 3-coloured path $fgh \in E^3$, the squares in $\mathcal{C}$ give $f_i, g_i, h_i, f^i, g^i, h^i \in E^1$ as shown in the following diagram.



We say that $\mathcal{C}$ *associative* if $f^2 = f_2$, $g^2 = g_2$ and $h^2 = h_2$.

k-coloured graphs and k-graphs

- Suppose that E is a k-coloured graph and $\mathcal{C}$ complete collection of squares which is associative.
- For each Λ, $\{\lambda \in \Lambda : d(\lambda) = e_i + e_j, i \neq j\}$ determines a complete collection of squares $\mathcal{C}_\Lambda$ for E_Λ which is associative.

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Theorem (Hazlewood-Raeburn-Sims-W)

There is a k-graph $\Lambda_{E,\mathcal{C}}$ and an isomorphism $\psi : E_{\Lambda_{E,\mathcal{C}}} \cong E$ such that $\psi \circ \phi \in \mathcal{C}_{\Lambda_{E,\mathcal{C}}}$ for each $\phi \in \mathcal{C}$ (i.e. ψ preserves squares).

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Theorem (Hazlewood-Raeburn-Sims-W)

Let $\sim$ be the equivalence relation on $\mathcal{P}(E)$ generated by $\mathcal{C}$. Then $\mathcal{P}(E)/\sim$ is a k-graph which is isomorphic to $\Lambda_{E,\mathcal{C}}$.

Notation and Nomenclature

- A *k-graph morphism* is a degree preserving functor between two k-graphs.
- Each $\lambda \in \text{Mor}(\Lambda)$ may be uniquely identified with a k-graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$: for $m \leq n \leq d(\lambda)$ the factorisation property gives us a unique $x_\lambda(m,n) \in d^{-1}(m-n)$ satisfying $\lambda = \lambda' x_\lambda(m,n)\lambda''$. Then $x_\lambda(0, d(\lambda)) = \lambda$.

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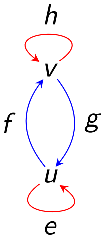
- Given a subset $F \subset \Lambda$ and a vertex $v \in \Lambda^0$, define $vF := r^{-1}(v) \cap F$ and $Fv := s^{-1}(v) \cap F$.

Example



- Let λ be the path of degree $(3, 2)$ with range v in the k-graph Λ represented on the left.
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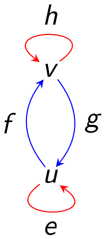

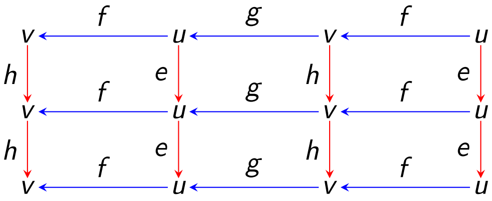

- Let λ be the path of degree $(3, 2)$ with range v in the k-graph Λ represented on the left.
- Unique factorisation forces $\lambda = fgfee = feghf = hhfgf = \ldots$
- λ is represented by the k-graph morphism $\Omega_{2,(3,2)} \to \Lambda$ encoded by the labelling of $\Omega_{2,(3,2)}$ on the right.
- The path $\lambda((2, 1), (3, 2)) = fe = hf$, the square on the top right.

The path space

- Given a k-graph Λ, We call $W_\Lambda := \bigcup_{m\in(\mathbb{N}\cup\{\infty\})^k} \Lambda^m$ the *path space* of Λ.
- We endow W_Λ with the cylinder set topology (or initial topology) given by the indicator function $\chi : W_\Lambda \to \{0,1\}^\Lambda$, where $\chi_x(\lambda) = 1$ if $x(0, d(\lambda)) = \lambda$ and 0 otherwise [PW].

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- A base for this topology on W_Λ consists of the sets

$$\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \bigcup_{\nu \in G} \mathcal{Z}(\mu\nu),$$

  where $\mathcal{Z}(\mu) := \{\lambda \in W_\Lambda : \lambda(0, d(\mu)) = \mu\}$, $\mu \in \Lambda$, and $G \subset \Lambda$. We may insist that $G \subset \bigcup_{i \leq k} \Lambda^{e_i}$. [W]
- With this topology W_Λ is a locally compact, Hausdorff space [W, PW].

Minimal common extensions

Given $\mu, \nu \in \Lambda$, we say that λ is a *minimal common extension of* μ *and* ν if $\lambda \in \mathcal{Z}(\mu) \cap \mathcal{Z}(\nu)$ and $d(\lambda) = d(\mu) \vee d(\nu)$. We denote the set of all such λ by $\mathrm{MCE}(\mu, \nu)$.

Example (1)

Given a directed graph E, and two paths $\mu, \nu \in E^*$, then

$$\mathrm{MCE}(\mu, \nu) = \begin{cases} \{\mu\} & \text{if } \mu \in \mathcal{Z}(\nu) \\ \{\nu\} & \text{if } \nu \in \mathcal{Z}(\mu) \\ \emptyset & \text{otherwise.} \end{cases}$$

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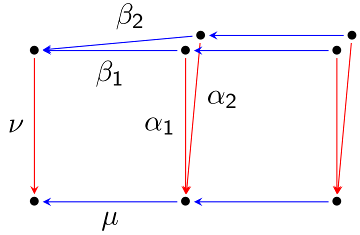

$$\mathrm{MCE}(\mu, \nu) = \{\mu\alpha_1, \mu\alpha_2\} = \{\nu\beta_1, \nu\beta_2\}$$

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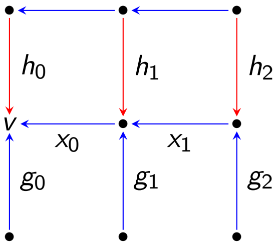

$\mathrm{MCE}(g_0, h_0) = \emptyset$
$\mathrm{MCE}(x_0 x_1, h_0) = \{x_0 x_1 h_2\}$
$\mathrm{MCE}(x_0, x_0 g_1) = \{x_0 g_1\}$.

Finite exhaustive sets

Given $v \in \Lambda^0$, a subset $E \subset v\Lambda$ is *exhaustive at* v if for each $\mu \in v\Lambda$, there exists $\nu \in E$ such that $\mathrm{MCE}(\mu, \nu) \neq \emptyset$. We denote the set of all finite exhaustive sets at v by $v\mathcal{FE}(\Lambda)$.

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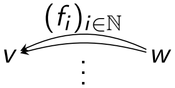

We have $v \in E$ for every $E \in v\mathcal{FE}(\Lambda)$, and $w\mathcal{FE}(\Lambda) = \{\{w\}\}$.

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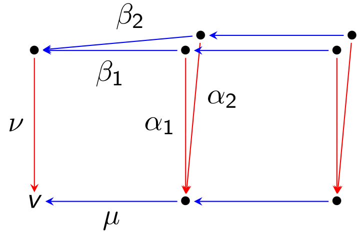

$\{v\}, \{\nu\}, \{\mu\}, \{\nu, \mu\}, \{\mu\alpha_1, \mu\alpha_2\} \in v\mathcal{FE}(\Lambda)$
$\{\mu\alpha_1\}, \{\nu\beta_2\} \notin v\mathcal{FE}(\Lambda)$

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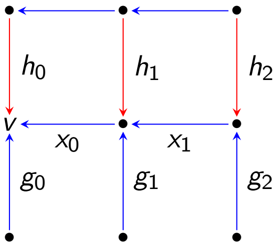

$\{h_0, x_0, g_0\}, \{g_0, x_0\}, \{g_0, x_0 g_1, x_0 x_1\}, \{h_0, g_0, x_0 g_1, x_0 x_1 g_2\} \in v\mathcal{FE}(\Lambda)$
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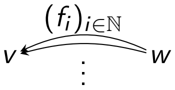
Boundary paths

A path $x \in W_\Lambda$ is a *boundary path* if for each $n \in \mathbb{N}^k$ with $n \leq d(x)$ and $E \in x(n)\mathcal{F}\mathcal{E}(\Lambda)$, there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. Denote the set of all boundary paths by $\partial\Lambda$.

Examples (1)

$\Lambda^\infty = \{x : \Omega_{k,(\infty)^k} \to \Lambda : x \text{ is a } k\text{-graph morphism}\} \subset \partial\Lambda$.
$\partial\Lambda = \Lambda^\infty$ if $0 < |v\Lambda^m| < \infty$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.
If k=1, then $\partial\Lambda = \Lambda^\infty \cup \{x \in \Lambda : |s(x)\Lambda^1| = 0 \text{ or } \infty\}$. E.g. if Λ is the 1-graph



then $\partial\Lambda = \{v, w\} \cup \{f_i : i \in \mathbb{N}\}$

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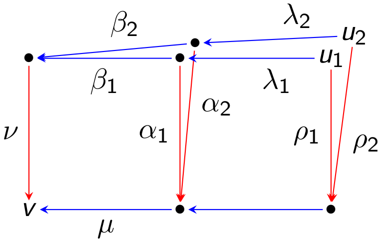

$\partial\Lambda = \Lambda u_1 \cup \Lambda u_2$, where $\Lambda v := s^{-1}(v)$.

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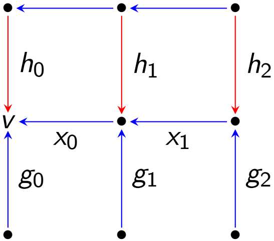
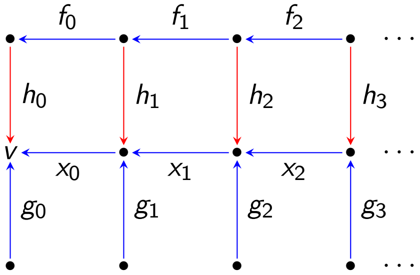


$$v\partial\Lambda = \{x_0 x_1 h_2,\ g_0, x_0 g_1,\ x_0 x_1 g_2\}$$

Boundary paths

A path $x \in W_\Lambda$ is a *boundary path* if for each $n \in \mathbb{N}^k$ with $n \leq d(x)$ and $E \in x(n)\mathcal{FE}(\Lambda)$, there exists $m \in \mathbb{N}^k$ such that $x(n, m) \in E$. Denote the set of all boundary paths by $\partial\Lambda$.

Example (4)



$$v\partial\Lambda = \{x_0 \ldots x_{i-1} g_i : i \in \mathbb{N}\} \cup \{h_0 f_0 \ldots, \; x_0 x_1 \ldots\}$$

Boundary paths

- Let σ be the shift action of $\mathbb{N}^k$ partially defined by $\sigma_n(\lambda)(p,q) = \lambda(n+p, n+q)$ for $d(\lambda) \geq n$.
- $\sigma_n(x) \in \partial\Lambda$ for each $n \in \mathbb{N}^k$ and $x \in \partial\Lambda$ with $d(x) \geq n$.
- $\lambda x \in \partial\Lambda$ for every $\lambda \in \Lambda$ and $x \in s(\lambda)\partial\Lambda$.
- $v\partial\Lambda \neq \emptyset$ for all $v \in \Lambda^0$
- Notice that

$$W_\Lambda \setminus \partial\Lambda = \bigcup_{\lambda\in\Lambda}\Big(\bigcup_{E\in s(\lambda)\mathcal{FE}(\Lambda)} \mathcal{Z}(\lambda \setminus E)\Big),$$

- so $\partial\Lambda$ is closed in W_Λ, and hence a locally compact Hausdorff space.

Filters [Exel]

- Give W_Λ a partial order $\leq$ defined by $\mu \leq \lambda \iff \lambda \in \mathcal{Z}(\mu)$.
- A *filter* in W_Λ is a subset $U \subset W_\Lambda$ such that
  1. if $\lambda \in U$ and $\mu \leq \lambda$, then $\mu \in U$, and
  2. if $\mu, \nu \in U$, then there exists $\lambda \in U$ with $\mu, \nu \leq \lambda$.

  Denote the set of all filters by $\widehat{\Lambda}$. Say U is an *ultrafilter* if U is a maximal filter. Denote the set of ultrafilters by $\widehat{\Lambda}_\infty$.

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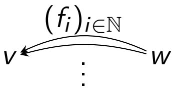
- $\widehat{\Lambda}$ has similar looking topology, replacing $\mathcal{Z}(\mu)$ with $\widehat{\mathcal{Z}}(\mu) := \{U \in \widehat{\Lambda} : \mu \in U\}$.
- $\widehat{\Lambda} \cong W_\Lambda$.

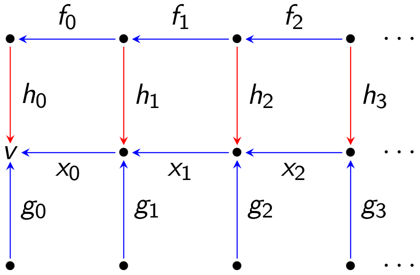
Filters

Example (1)



- $\widehat{\Lambda}_\infty = \{\{w\}\} \cup \{U_{f_i} : i \in \mathbb{N}\}$
- $f_i \to v$ in W_Λ.
- $\widehat{\Lambda}_\infty$ is not closed!
- Don't need infinite receivers to see this.

Example (2)



- $U_{g_0}, \; U_{h_0 f_0 f_1 \ldots} U_{x_0 g_1}, \; U_{x_0 x_1 g_2}, \; \cdots \in \widehat{\Lambda}_\infty$
- $x_0 \ldots x_{n-1} g_n \to x_0 x_1 \ldots$
- $U_{x_0 x_1 x_2 \ldots} \notin \widehat{\Lambda}_\infty$!!!

Filters

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- In path-space terminology, the anologue of $\widehat{\Lambda}_\infty$ is denoted $\Lambda^{\leq\infty}$ (Definition in RSY2004).
- Define $\partial\widehat{\Lambda}$ to be the filters $U \in \widehat{\Lambda}$ such that for each $\mu \in U$, $E \subset s(\mu)\mathcal{FE}(\Lambda)$, there exists $\nu \in E$ such that $\mu\nu \in x$
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- Then $\partial\widehat{\Lambda} = \{U_x : x \in \partial\Lambda\}$.
- $\overline{\widehat{\Lambda}_\infty} = \partial\widehat{\Lambda}$
- $\widehat{\Lambda}_\infty = \partial\widehat{\Lambda}$ if Λ is *row-finite* and *locally convex*:
- Λ is row-finite if $v\Lambda^m$ is finite for each $v \in \Lambda^0$ and $m \in \mathbb{N}^k$, and
- Λ is locally-convex if for each $i \neq j$, $\mu \in \Lambda^{e_i}$ and $\nu \in r(\mu)\Lambda^{e_j}$, the sets $s(\mu)\Lambda^{e_j}$ and $s(\nu)\Lambda^{e_i}$ are nonempty.

C^*-algebras

- A k-graph Λ is *finitely aligned* if $\mathrm{MCE}(\mu, \nu)$ is finite (possibly empty) for all $\mu, \nu \in \Lambda$.
- Given a finitely aligned k-graph Λ, a *Cuntz-Krieger Λ-family* in a C^*-algebra B is a map $s : \Lambda \to B$ such that each s_λ is a partial isometry, and that
  - CK1. $\{s_v : v \in \Lambda^0\}$ are mutually orthogonal projections,
  - CK2. $s_\mu s_\nu = s_{\mu\nu}$ if $\mu\nu \in \Lambda$,
  - CK3. $s_\mu^* s_\nu = \sum_{\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)} s_\alpha s_\beta^*$, and
  - CK4. $\prod_{\lambda \in E}(s_v - s_\lambda s_\lambda^*) = 0$ for all $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$.
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- $C^*(\Lambda)$ is the universal C^*-algebra for Cuntz-Krieger Λ-families.
- $C^*(\Lambda)$ is nonzero since the representation $S : \Lambda \to \mathcal{B}(\ell^2(\partial\Lambda))$ given by

$$S_\lambda \xi_x = \begin{cases} \xi_{\lambda x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

yields a nonzero Cuntz-Krieger Λ-family.

Diagonal Subalgebra

We call $D_\Lambda := C^*(\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}) \subset C^*(\Lambda)$ the *diagonal* C^*-subalgebra of $C^*(\Lambda)$. One can show that $D_\Lambda = \overline{\mathrm{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$.

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- So $\{\lambda : \phi(s_\lambda s_\lambda^*) = 1\} \in \widehat{\Lambda}$, and so determines a unique path $x \in W_\Lambda$.
- For each $n \leq d(x)$ and $E \in x(n)\mathcal{FE}(\Lambda)$, (CK4) says that $\prod_{\lambda \in E}(s_{x(n)} - s_\lambda s_\lambda^*) = 0$, and it follows that $x \in \partial\Lambda$.

Removing Sources

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Farthing defined a process which, given an row-finite k-graph Λ, constructs a row-finite k-graph Γ with no sources such that $C^*(\Lambda) \sim_{SME} C^*(\Gamma)$. This process extends the non-infinite boundary paths of Λ to infinite paths [F,W].

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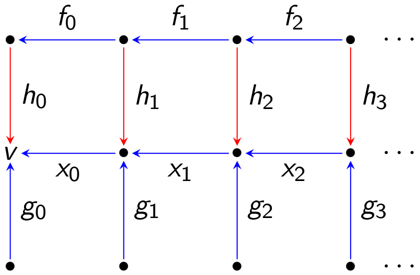

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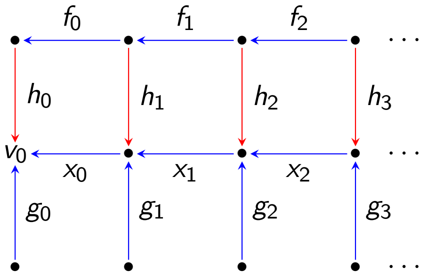

Here, $w_n := x_0 \ldots x_{n-1} g_n$ and any path of degree $(1, \infty)$ are all elements of $\partial\Lambda$. The idea is to extend these paths to be infinite in all directions (degrees, colours,...).

Boundary path starting with h_0

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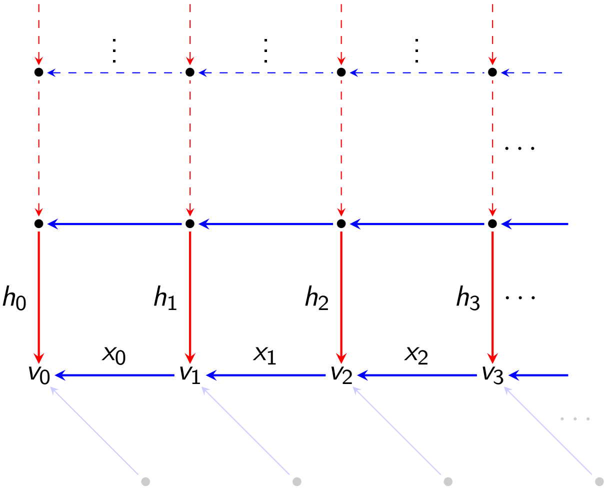
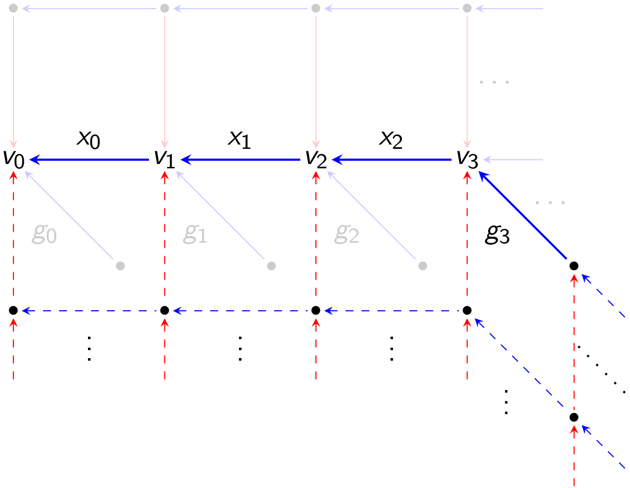
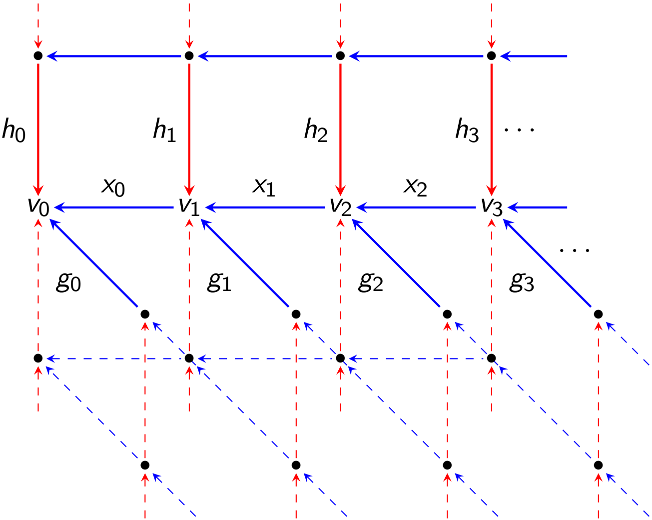


Boundary path w_3



Putting it all together



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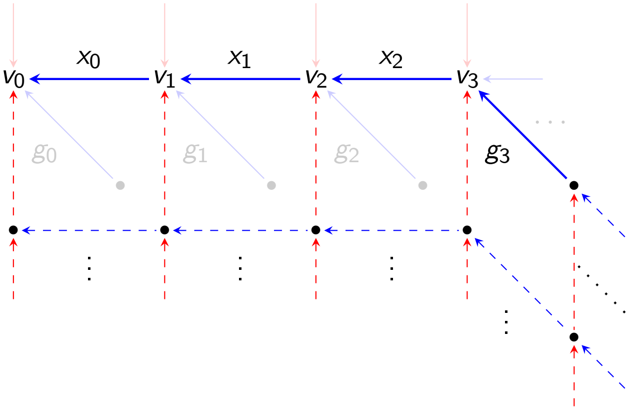

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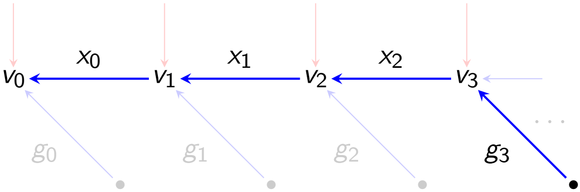

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- Then the following diagram commutes:

$$\begin{array}{ccc} \Lambda^0\Gamma^\infty & \xrightarrow{\quad\pi\quad} & \partial\Lambda \\ \Big\downarrow \eta & & \Big\downarrow h_\Lambda \\ \widehat{pD_\Gamma p} & \xrightarrow{\quad\rho\quad} & \widehat{D_\Lambda} \end{array}$$

  Where η is essentially a restriction of $h_\Gamma : \Gamma^\infty \to \widehat{D_\Gamma}$ to paths with range in Λ^0.