Classification of Cuntz-Krieger algebras and Graph C^*-algebras

Gunnar Restorff

University of the Faroe Islands

Graph Algebras: Bridges between graph algebras C^*-algebras and Leavitt path algebras, 21-26 April 2013

Outline of talk

1. Historical remarks
2. Primitive ideal space
3. Filtered K-theory
4. Classification of Cuntz-Krieger algebras
5. Some answers
   - Unital classification
   - Strong classification (external classification)
   - Range results
   - Strong classification, II
   - Phantom Cuntz-Krieger algebras
   - Graph C^*-algebras

Commercials

Two vacant positions at the University of the Faroe Islands:

- Assistant Professor of Mathematics (educating teachers to elementary school). Link to advertisement: http://tinyurl.com/mathjobFO1
- Assistant or Associate Professor of Mathematics (teaching classical undergraduate courses for engineering). Link to advertisement: http://tinyurl.com/mathjobFO2

Commercials

Two vacant positions at the University of the Faroe Islands:

- Assistant Professor of Mathematics (educating teachers to elementary school). Link to advertisement: http://tinyurl.com/mathjobFO1
- Assistant or Associate Professor of Mathematics (teaching classical undergraduate courses for engineering). Link to advertisement: http://tinyurl.com/mathjobFO2

Commercials

Two vacant positions at the University of the Faroe Islands:

- Assistant Professor of Mathematics (educating teachers to elementary school). Link to advertisement: http://tinyurl.com/mathjobFO1
- Assistant or Associate Professor of Mathematics (teaching classical undergraduate courses for engineering). Link to advertisement: http://tinyurl.com/mathjobFO2

Outline

Classification

Consider a class $\mathcal{C}$ of objects from some category.

- If we to each object X in $\mathcal{C}$ associate an object $I(X)$ of some fixed category in such a way that $X \cong Y \Rightarrow I(X) \cong I(Y)$, then we call I an invariant.
- We call such an I a complete invariant if also $I(X) \cong I(Y) \Rightarrow X \cong Y$.
- A functor is always an invariant, and if it is a complete invariant, we call it a classification functor. Not all invariants are functors.
- A functor F is called a strong classification functor if for all objects X and Y every isomorphism from $F(X)$ to $F(Y)$ is induced by an isomorphism from X to Y.

Classification

Consider a class $\mathcal{C}$ of objects from some category.

- If we to each object X in $\mathcal{C}$ associate an object $I(X)$ of some fixed category in such a way that $X \cong Y \Rightarrow I(X) \cong I(Y)$, then we call I an invariant.
- We call such an I a complete invariant if also $I(X) \cong I(Y) \Rightarrow X \cong Y$.
- A functor is always an invariant, and if it is a complete invariant, we call it a classification functor. Not all invariants are functors.
- A functor F is called a strong classification functor if for all objects X and Y every isomorphism from $F(X)$ to $F(Y)$ is induced by an isomorphism from X to Y.

Classification

Consider a class $\mathcal{C}$ of objects from some category.

- If we to each object X in $\mathcal{C}$ associate an object $I(X)$ of some fixed category in such a way that $X \cong Y \Rightarrow I(X) \cong I(Y)$, then we call I an invariant.
- We call such an I a complete invariant if also $I(X) \cong I(Y) \Rightarrow X \cong Y$.
- A functor is always an invariant, and if it is a complete invariant, we call it a classification functor. Not all invariants are functors.
- A functor F is called a strong classification functor if for all objects X and Y every isomorphism from $F(X)$ to $F(Y)$ is induced by an isomorphism from X to Y.

Classification

Consider a class $\mathcal{C}$ of objects from some category.

- If we to each object X in $\mathcal{C}$ associate an object $I(X)$ of some fixed category in such a way that $X \cong Y \Rightarrow I(X) \cong I(Y)$, then we call I an invariant.
- We call such an I a complete invariant if also $I(X) \cong I(Y) \Rightarrow X \cong Y$.
- A functor is always an invariant, and if it is a complete invariant, we call it a classification functor. Not all invariants are functors.
- A functor F is called a strong classification functor if for all objects X and Y every isomorphism from $F(X)$ to $F(Y)$ is induced by an isomorphism from X to Y.

Classification

Consider a class $\mathcal{C}$ of objects from some category.

- If we to each object X in $\mathcal{C}$ associate an object $I(X)$ of some fixed category in such a way that $X \cong Y \Rightarrow I(X) \cong I(Y)$, then we call I an invariant.
- We call such an I a complete invariant if also $I(X) \cong I(Y) \Rightarrow X \cong Y$.
- A functor is always an invariant, and if it is a complete invariant, we call it a classification functor. Not all invariants are functors.
- A functor F is called a strong classification functor if for all objects X and Y every isomorphism from $F(X)$ to $F(Y)$ is induced by an isomorphism from X to Y.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems.
- Absorption techniques for extensions.
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems.
- Absorption techniques for extensions.
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems.
- Absorption techniques for extensions.
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems.
- Absorption techniques for extensions.
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems (internal classification).
- Absorption techniques for extensions.
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem.

Usually, the first method does only give internal classification results — i.e., it only gives information about C^*-algebras that are known to be Cuntz-Krieger algebras (or graph C^*-algebras). Usually, the second and third methods will give us external classification results. Historically for purely infinite C^*-algebras, results from the first method have preceded the more general results.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems (internal classification).
- Absorption techniques for extensions (usually external classification).
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem (usually external classification).

Usually, the first method does only give internal classification results — i.e., it only gives information about C^*-algebras that are known to be Cuntz-Krieger algebras (or graph C^*-algebras). Usually, the second and third methods will give us external classification results. Historically for purely infinite C^*-algebras, results from the first method have preceded the more general results.

There are mainly three methods that have been used to classify (purely infinite) Cuntz-Krieger algebras — and more recently also graph C^*-algebras. These are

- Techniques from dynamical systems (internal classification).
- Absorption techniques for extensions (usually external classification).
- Kirchberg's isomorphisms theorem for ideal-related KK-theory together with a Universal Coefficient Theorem (usually external classification).

Usually, the first method does only give internal classification results — i.e., it only gives information about C^*-algebras that are known to be Cuntz-Krieger algebras (or graph C^*-algebras). Usually, the second and third methods will give us external classification results. Historically for purely infinite C^*-algebras, results from the first method have preceded the more general results.

Historical remarks

- Cuntz and Krieger introduced the so-called Cuntz-Krieger algebras, and proved a series of results about them. E.g., that the stabilized Cuntz-Krieger algebra is a an invariant of flow equivalence of shifts of finite type. (publ. 1980-81)
- Using the Bowen-Franks groups, Franks classified irreducible shifts of finite type up to flow equivalence. (publ. 1984)
- Cuntz showed that <u>if</u> the Cuntz-Krieger algebras associated with the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are isomorphic, <u>then</u> all simple, purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by their Bowen-Franks groups (or K_0-groups). (publ. 1986)
- Rørdam showed that this is indeed the case. (techniques from dynamical systems, internal classification, publ. 1995)
- Kirchberg and Phillips classified the simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (external, strong classification)

Historical remarks

- Cuntz and Krieger introduced the so-called Cuntz-Krieger algebras, and proved a series of results about them. E.g., that the stabilized Cuntz-Krieger algebra is a an invariant of flow equivalence of shifts of finite type. (publ. 1980-81)
- Using the Bowen-Franks groups, Franks classified irreducible shifts of finite type up to flow equivalence. (publ. 1984)
- Cuntz showed that <u>if</u> the Cuntz-Krieger algebras associated with the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are isomorphic, <u>then</u> all simple, purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by their Bowen-Franks groups (or K_0-groups). (publ. 1986)
- Rørdam showed that this is indeed the case. (techniques from dynamical systems, internal classification, publ. 1995)
- Kirchberg and Phillips classified the simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (external, strong classification)

Historical remarks

- Cuntz and Krieger introduced the so-called Cuntz-Krieger algebras, and proved a series of results about them. E.g., that the stabilized Cuntz-Krieger algebra is a an invariant of flow equivalence of shifts of finite type. (publ. 1980-81)
- Using the Bowen-Franks groups, Franks classified irreducible shifts of finite type up to flow equivalence. (publ. 1984)
- Cuntz showed that if the Cuntz-Krieger algebras associated with the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are isomorphic, then all simple, purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by their Bowen-Franks groups (or K_0-groups). (publ. 1986)
- Rørdam showed that this is indeed the case. (techniques from dynamical systems, internal classification, publ. 1995)
- Kirchberg and Phillips classified the simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (external, strong classification)

Historical remarks

- Cuntz and Krieger introduced the so-called Cuntz-Krieger algebras, and proved a series of results about them. E.g., that the stabilized Cuntz-Krieger algebra is a an invariant of flow equivalence of shifts of finite type. (publ. 1980-81)
- Using the Bowen-Franks groups, Franks classified irreducible shifts of finite type up to flow equivalence. (publ. 1984)
- Cuntz showed that <u>if</u> the Cuntz-Krieger algebras associated with the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are isomorphic, <u>then</u> all simple, purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by their Bowen-Franks groups (or K_0-groups). (publ. 1986)
- Rørdam showed that this is indeed the case. (techniques from dynamical systems, internal classification, publ. 1995)
- Kirchberg and Phillips classified the simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (external, strong classification)

Historical remarks

- Cuntz and Krieger introduced the so-called Cuntz-Krieger algebras, and proved a series of results about them. E.g., that the stabilized Cuntz-Krieger algebra is a an invariant of flow equivalence of shifts of finite type. (publ. 1980-81)
- Using the Bowen-Franks groups, Franks classified irreducible shifts of finite type up to flow equivalence. (publ. 1984)
- Cuntz showed that <u>if</u> the Cuntz-Krieger algebras associated with the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ are isomorphic, <u>then</u> all simple, purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by their Bowen-Franks groups (or K_0-groups). (publ. 1986)
- Rørdam showed that this is indeed the case. (techniques from dynamical systems, internal classification, publ. 1995)
- Kirchberg and Phillips classified the simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (external, strong classification)

- Huang classified Cuntz-Krieger algebras with finite K-theory. (techniques from dynamical systems, internal classification, publ. 1995)
- Huang classified the Cuntz-Krieger algebras with exactly one non-trivial ideal. (techniques from dynamical systems, internal classification, publ. 1996)
- Rørdam classified extensions of simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (Absorption techniques, external classification, publ. 1997)
- Bonkat showed a UCT for ideal-related KK-theory with exactly one non-trivial ideal, which together with Kirchberg's results gives classification (UCT, strong classification, external classification, publ. 2002)

- Huang classified Cuntz-Krieger algebras with finite K-theory. (techniques from dynamical systems, internal classification, publ. 1995)
- Huang classified the Cuntz-Krieger algebras with exactly one non-trivial ideal. (techniques from dynamical systems, internal classification, publ. 1996)
- Rørdam classified extensions of simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (Absorption techniques, external classification, publ. 1997)
- Bonkat showed a UCT for ideal-related KK-theory with exactly one non-trivial ideal, which together with Kirchberg's results gives classification (UCT, strong classification, external classification, publ. 2002)

- Huang classified Cuntz-Krieger algebras with finite K-theory. (techniques from dynamical systems, internal classification, publ. 1995)
- Huang classified the Cuntz-Krieger algebras with exactly one non-trivial ideal. (techniques from dynamical systems, internal classification, publ. 1996)
- Rørdam classified extensions of simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (Absorption techniques, external classification, publ. 1997)
- Bonkat showed a UCT for ideal-related KK-theory with exactly one non-trivial ideal, which together with Kirchberg's results gives classification (UCT, strong classification, external classification, publ. 2002)

- Huang classified Cuntz-Krieger algebras with finite K-theory. (techniques from dynamical systems, internal classification, publ. 1995)
- Huang classified the Cuntz-Krieger algebras with exactly one non-trivial ideal. (techniques from dynamical systems, internal classification, publ. 1996)
- Rørdam classified extensions of simple, purely infinite, separable, nuclear C^*-algebras in the bootstrap class. (Absorption techniques, external classification, publ. 1997)
- Bonkat showed a UCT for ideal-related KK-theory with exactly one non-trivial ideal, which together with Kirchberg's results gives classification (UCT, strong classification, external classification, publ. 2002)

Outline

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_-(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_-(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_-(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_-(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_{-}(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_{-}(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Primitive ideal space

Definition

Let $\mathfrak{A}$ be a separable C^*-algebra with finitely many ideals, and let X denote the primitive ideal space equipped with the usual hull-kernel topology. Then we have a lattice isomorphism from the open subsets of X to the lattice of ideals of $\mathfrak{A}$. For each element $x \in X$, we let $H(x)$ be the smallest open subset containing x, and we let $H_{-}(x) = H(x) \setminus \{x\}$.

Definition

Using $x \leq y \Leftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}}$ we get a one-to-one correspondence from the T_0-topologies to the partial orders of a fixed finite set. The graph of this relation (i.e., $x \to y$ iff $x \leq y$), is exactly the component graph for a Cuntz-Krieger algebra, so therefore we will use this to illustrate the primitive ideal spaces.

Outline

Full filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with finitely many ideals, and let $X = \mathrm{Prim}(\mathfrak{A})$ denote the primitive ideal space. The full filtered K-theory of $\mathfrak{A}$ is then the collection of the K-groups $K_0(\mathfrak{A}(V \setminus U))$ and $K_1(\mathfrak{A}(V \setminus U))$ for all open subsets $U \subseteq V$ of X together with the homomorphisms

$$\begin{tikzcd}
K_0(\mathfrak{A}(V \setminus U)) \arrow[r] & K_0(\mathfrak{A}(W \setminus U)) \arrow[r] & K_0(\mathfrak{A}(W \setminus V)) \arrow[d] \\
K_1(\mathfrak{A}(W \setminus V)) \arrow[u] & K_1(\mathfrak{A}(W \setminus U)) \arrow[l] & K_1(\mathfrak{A}(V \setminus U)) \arrow[l]
\end{tikzcd}$$

for all open subsets $U \subseteq V \subseteq W$ of X.

Reduced filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with finitely many ideals, and let $X = \mathrm{Prim}(\mathfrak{A})$ denote the primitive ideal space. The reduced filtered K-theory of $\mathfrak{A}$ is then the collection of the K-groups $K_0(\mathfrak{A}(U))$ where U ranges over all $\{x\}$ all $H(x)$ and all non-empty $H_-(x)$ and the K-groups $K_1(\mathfrak{A}(\{x\}))$ whenever $H_-(x) \neq \emptyset$ together with the homomorphisms

$$\begin{tikzcd} K_0(\mathfrak{A}(H_-(x))) \arrow[r] & K_0(\mathfrak{A}(H(x))) \arrow[r] & K_0(\mathfrak{A}(\{x\})) \\ K_1(\mathfrak{A}(\{x\})) \arrow[u] & & \end{tikzcd}$$

for all x satisfying $H_-(x) \neq \emptyset$ and

$$K_0(\mathfrak{A}(H(y))) \longrightarrow K_0(\mathfrak{A}(H_-(x)))$$

whenever $H(y)$ is a proper subset of $H_-(x)$ and $H(y) \subseteq H(z) \subseteq H_-(x)$ implies that $z = y$.

Reduced filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with finitely many ideals, and let $X = \mathrm{Prim}(\mathfrak{A})$ denote the primitive ideal space. The reduced filtered K-theory of $\mathfrak{A}$ is then the collection of the K-groups $K_0(\mathfrak{A}(U))$ where U ranges over all $\{x\}$ all $H(x)$ and all non-empty $H_-(x)$ and the K-groups $K_1(\mathfrak{A}(\{x\}))$ ~~whenever $H_-(x) \neq \emptyset$~~ together with the homomorphisms

$$\begin{tikzcd} K_0(\mathfrak{A}(H_-(x))) \arrow[r] & K_0(\mathfrak{A}(H(x))) \arrow[r] & K_0(\mathfrak{A}(\{x\})) \\ K_1(\mathfrak{A}(\{x\})) \arrow[u] & & \end{tikzcd}$$

for all x ~~satisfying $H_-(x) \neq \emptyset$~~ and

$$K_0(\mathfrak{A}(H(y))) \longrightarrow K_0(\mathfrak{A}(H_-(x)))$$

whenever $H(y)$ is a proper subset of $H_-(x)$ and $H(y) \subseteq H(z) \subseteq H_-(x)$ implies that $z = y$.

Outline

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Classification of Cuntz-Krieger algebras

Theorem (R)

Purely infinite Cuntz-Krieger algebras are classified up to stable isomorphism by the reduced filtered K-theory (and consequently also by the full filtered K-theory).

Questions

- *Can we get unital classification?*
- *Can we get strong classification?*
- *Is it possible to get external classification?*
- *Can we describe the range of the invariants?*
- *Do there exist phantom Cuntz-Krieger algebras.*
- *Is it possible to generalize the results to graph C^*-algebras?*

Outline

Outline

Unital classification

Theorem (Eilers-R-Ruiz)

Essentially, if we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the stabilization is strongly classified by some invariant that includes K_0, then the same invariant together with the class of the unit in K_0 strongly classifies this class (up to unital isomorphism).

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the full filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the reduced filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Unital classification

Theorem (Eilers-R-Ruiz)

Essentially, if we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the stabilization is strongly classified by some invariant that includes K_0, then the same invariant together with the class of the unit in K_0 strongly classifies this class (up to unital isomorphism).

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the full filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the reduced filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Unital classification

Theorem (Eilers-R-Ruiz)

Essentially, if we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the stabilization is strongly classified by some invariant that includes K_0, then the same invariant together with the class of the unit in K_0 strongly classifies this class (up to unital isomorphism).

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the full filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Corollary

If we have a class of unital, separable, nuclear, purely infinite C^*-algebras such that the reduced filtered K-theory strongly classifies the algebras in this class up to stable isomorphism, just throw in the class of the unit in K_0 to get strong unital classification.

Outline

Strong classification (external classification)

Since the (strong) unital classification can be done using strong classification up to stable isomorphism, we will focus on that. To get strong (external) classification, the general idea is to prove a Universal Coefficient Theorem for full filtered K-theory. Then we can lift a full filtered K-theory isomorphism to an ideal related KK-equivalence using this UCT, and then lift that to a $*$-isomorphism using Kirchberg's result. Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear, separable, purely infinite, stable C^*-algebras that are tight over the finite T_0-space X having all its simple quotients being in the bootstrap class.

Theorem (Bonkat)

If $|X| = 2$ and X is non-Hausdorff, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Strong classification (external classification)

Since the (strong) unital classification can be done using strong classification up to stable isomorphism, we will focus on that. To get strong (external) classification, the general idea is to prove a Universal Coefficient Theorem for full filtered K-theory. Then we can lift a full filtered K-theory isomorphism to an ideal related KK-equivalence using this UCT, and then lift that to a $*$-isomorphism using Kirchberg's result. Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear, separable, purely infinite, stable C^*-algebras that are tight over the finite T_0-space X having all its simple quotients being in the bootstrap class.

Theorem (Bonkat)

If $|X| = 2$ and X is non-Hausdorff, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Strong classification (external classification)

Since the (strong) unital classification can be done using strong classification up to stable isomorphism, we will focus on that. To get strong (external) classification, the general idea is to prove a Universal Coefficient Theorem for full filtered K-theory. Then we can lift a full filtered K-theory isomorphism to an ideal related KK-equivalence using this UCT, and then lift that to a $*$-isomorphism using Kirchberg's result.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear, separable, purely infinite, stable C^*-algebras that are tight over the finite T_0-space X having all its simple quotients being in the bootstrap class.

Theorem (Bonkat)

If $|X| = 2$ and X is non-Hausdorff, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Strong classification (external classification)

Since the (strong) unital classification can be done using strong classification up to stable isomorphism, we will focus on that. To get strong (external) classification, the general idea is to prove a Universal Coefficient Theorem for full filtered K-theory. Then we can lift a full filtered K-theory isomorphism to an ideal related KK-equivalence using this UCT, and then lift that to a $*$-isomorphism using Kirchberg's result. Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear, separable, purely infinite, stable C^*-algebras that are tight over the finite T_0-space X having all its simple quotients being in the bootstrap class.

Theorem (Bonkat)

If $|X| = 2$ and X is non-Hausdorff, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Strong classification (external classification)

Since the (strong) unital classification can be done using strong classification up to stable isomorphism, we will focus on that. To get strong (external) classification, the general idea is to prove a Universal Coefficient Theorem for full filtered K-theory. Then we can lift a full filtered K-theory isomorphism to an ideal related KK-equivalence using this UCT, and then lift that to a $*$-isomorphism using Kirchberg's result. Let $\mathfrak{A}$ and $\mathfrak{B}$ be nuclear, separable, purely infinite, stable C^*-algebras that are tight over the finite T_0-space X having all its simple quotients being in the bootstrap class.

Theorem (Bonkat)

If $|X| = 2$ and X is non-Hausdorff, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Meyer-Nest have made a framework, where we get a UCT for full filtered K-theory whenever the projective dimension of the full filtered K-theory is at most 1.

Generalizing results of R and Meyer-Nest, Bentmann and Köhler showed:

Theorem (Bentmann-Köhler)

If X is an accordion space, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Meyer-Nest have made a framework, where we get a UCT for full filtered K-theory whenever the projective dimension of the full filtered K-theory is at most 1.
Generalizing results of R and Meyer-Nest, Bentmann and Köhler showed:

Theorem (Bentmann-Köhler)

If X is an accordion space, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Meyer-Nest have made a framework, where we get a UCT for full filtered K-theory whenever the projective dimension of the full filtered K-theory is at most 1.

Generalizing results of R and Meyer-Nest, Bentmann and Köhler showed:

Theorem (Bentmann-Köhler)

If X is an accordion space, then every isomorphism from the filtered K-theory of $\mathfrak{A}$ to the filtered K-theory of $\mathfrak{B}$ can be lifted to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Consider C^*algebras that are purely infinite, separable, nuclear with all simple subquotients in the Bootstrap class and that tight over one of the following connected T_0-spaces:

$$E\colon \begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd} \qquad F\colon \begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \end{tikzcd}$$

$$39\colon \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad 3F\colon \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$A\colon \begin{tikzcd} \bullet \arrow[dr] & \bullet \arrow[d] & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

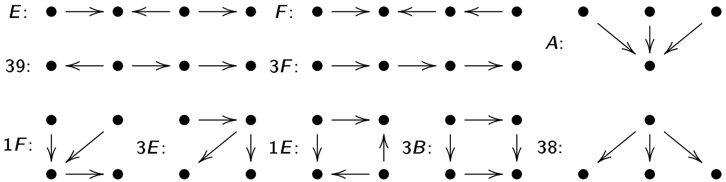
$$1F\colon \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[dl] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad 3E\colon \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl] \arrow[d] \\ \bullet & \bullet \end{tikzcd} \quad 1E\colon \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd} \quad 3B\colon \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad 38\colon \begin{tikzcd} & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \\ \bullet & \bullet & \bullet \end{tikzcd}$$

The spaces $E, F, 39, 3F$ are accordion spaces, so strong classification is alright.

Theorem (Meyer-Nest, Bentmann-Köhler)

Bentmann-Köhler showed that we have counterexamples to classification for $A, 38, 1F, 3E, 1E, 3B$ (using the full filtered K-theory).

Consider C^*algebras that are purely infinite, separable, nuclear with all simple subquotients in the Bootstrap class and that tight over one of the following connected T_0-spaces:



The spaces $E, F, 39, 3F$ are accordion spaces, so strong classification is alright.

Theorem (Meyer-Nest, Bentmann-Köhler)

Bentmann-Köhler showed that we have counterexamples to classification for $A, 38, 1F, 3E, 1E, 3B$ (using the full filtered K-theory).

Theorem (Arklint-R-Ruiz)

Restricted to real rank zero, we have strong classification for the cases $A, 38, 1F, 3E$ and we have counterexamples to classification for $1E$ (using the full filtered K-theory). The case $3B$ is open in the real rank zero case.

Theorem (Bentmann)

There is a Cuntz-Krieger algebra with projective dimension 2 over the space $1E$, so we cannot even hope to just prove that the full filtered K-theory of Cuntz-Krieger algebras (or graph C^*-algebras) have projective dimension 1.

Theorem (Arklint-Bentmann-Katsura)

In the case of real rank zero and free K_1, we have strong classification for the case $1E$.

Theorem (Arklint-R-Ruiz)

Restricted to real rank zero, we have strong classification for the cases $A, 38, 1F, 3E$ and we have counterexamples to classification for $1E$ (using the full filtered K-theory). The case $3B$ is open in the real rank zero case.

Theorem (Bentmann)

There is a Cuntz-Krieger algebra with projective dimension 2 over the space $1E$, so we cannot even hope to just prove that the full filtered K-theory of Cuntz-Krieger algebras (or graph C^*-algebras) have projective dimension 1.

Theorem (Arklint-Bentmann-Katsura)

In the case of real rank zero and free K_1, we have strong classification for the case $1E$.

Theorem (Arklint-R-Ruiz)

Restricted to real rank zero, we have strong classification for the cases $A, 38, 1F, 3E$ and we have counterexamples to classification for $1E$ (using the full filtered K-theory). The case $3B$ is open in the real rank zero case.

Theorem (Bentmann)

There is a Cuntz-Krieger algebra with projective dimension 2 over the space $1E$, so we cannot even hope to just prove that the full filtered K-theory of Cuntz-Krieger algebras (or graph C^*-algebras) have projective dimension 1.

Theorem (Arklint-Bentmann-Katsura)

In the case of real rank zero and free K_1, we have strong classification for the case $1E$.

Outline

Range results

Theorem (Arklint-Bentmann-Katsura)

If we have some finitely generated reduced filtered K-theory such that all the involved K_1-groups are free and the rank of the group that corresponds to $K_1(\mathfrak{A}(\{x\}))$ is less than or equal to the rank of the cokernel of the map from $K_0(\mathfrak{A}(H_-(x)))$ to $K_0(\mathfrak{A}(H(x)))$, for each x, then there exists a unital, purely infinite graph C^*-algebra with this invariant.

If we instead have equality for each x, then the graph algebra can be chosen to be a Cuntz-Krieger algebra.

Theorem (Arklint-Bentmann-Katsura)

If we have some reduced filtered K-theory such that all the involved K_1-groups are free, then there exists a purely infinite graph C^*-algebra with this invariant.

Range results

Theorem (Arklint-Bentmann-Katsura)

If we have some finitely generated reduced filtered K-theory such that all the involved K_1-groups are free and the rank of the group that corresponds to $K_1(\mathfrak{A}(\{x\}))$ is less than or equal to the rank of the cokernel of the map from $K_0(\mathfrak{A}(H_-(x)))$ to $K_0(\mathfrak{A}(H(x)))$, for each x, then there exists a unital, purely infinite graph C^*-algebra with this invariant.

If we instead have equality for each x, then the graph algebra can be chosen to be a Cuntz-Krieger algebra.

Theorem (Arklint-Bentmann-Katsura)

If we have some reduced filtered K-theory such that all the involved K_1-groups are free, then there exists a purely infinite graph C^*-algebra with this invariant.

Range results

Theorem (Arklint-Bentmann-Katsura)

If we have some finitely generated reduced filtered K-theory such that all the involved K_1-groups are free and the rank of the group that corresponds to $K_1(\mathfrak{A}(\{x\}))$ is less than or equal to the rank of the cokernel of the map from $K_0(\mathfrak{A}(H_-(x)))$ to $K_0(\mathfrak{A}(H(x)))$, for each x, then there exists a unital, purely infinite graph C^*-algebra with this invariant.
If we instead have equality for each x, then the graph algebra can be chosen to be a Cuntz-Krieger algebra.

Theorem (Arklint-Bentmann-Katsura)

If we have some reduced filtered K-theory such that all the involved K_1-groups are free, then there exists a purely infinite graph C^*-algebra with this invariant.

Range results

Theorem (Arklint-Bentmann-Katsura)

If we have some finitely generated reduced filtered K-theory such that all the involved K_1-groups are free and the rank of the group that corresponds to $K_1(\mathfrak{A}(\{x\}))$ is less than or equal to the rank of the cokernel of the map from $K_0(\mathfrak{A}(H_-(x)))$ to $K_0(\mathfrak{A}(H(x)))$, for each x, then there exists a unital, purely infinite graph C^*-algebra with this invariant.
If we instead have equality for each x, then the graph algebra can be chosen to be a Cuntz-Krieger algebra.

Theorem (Arklint-Bentmann-Katsura)

If we have some reduced filtered K-theory such that all the involved K_1-groups are free, then there exists a purely infinite graph C^*-algebra with this invariant.

Outline

Strong classification, II

Theorem (Arklint-R)

There exists a separable, nuclear, purely infinite graph C^*-algebra $C^*(E)$ over a graph E with primitive ideal space of the type $3B$, such that there exists an is automorphism of the full filtered K-theory that cannot be lifted to an automorphism of the algebra $C^*(E) \otimes \mathbb{K}$.

Question

It seems that we can choose the above graph C^*-algebra to be unital and the corresponding graph to be finite. Is it possible to get an example with a Cuntz-Krieger algebra?

Strong classification, II

Theorem (Arklint-R)

There exists a separable, nuclear, purely infinite graph C^*-algebra $C^*(E)$ over a graph E with primitive ideal space of the type $3B$, such that there exists an is automorphism of the full filtered K-theory that cannot be lifted to an automorphism of the algebra $C^*(E) \otimes \mathbb{K}$.

Question

It seems that we can choose the above graph C^*-algebra to be unital and the corresponding graph to be finite. Is it possible to get an example with a Cuntz-Krieger algebra?

Strong classification, II

Theorem (Arklint-R)

There exists a separable, nuclear, purely infinite graph C^*-algebra $C^*(E)$ over a graph E with primitive ideal space of the type $3B$, such that there exists an is automorphism of the full filtered K-theory that cannot be lifted to an automorphism of the algebra $C^*(E) \otimes \mathbb{K}$.

Question

It seems that we can choose the above graph C^*-algebra to be unital and the corresponding graph to be finite. Is it possible to get an example with a Cuntz-Krieger algebra?

Strong classification, II

Theorem (Arklint-R)

There exists a separable, nuclear, purely infinite graph C^*-algebra $C^*(E)$ over a graph E with primitive ideal space of the type $3B$, such that there exists an is automorphism of the full filtered K-theory that cannot be lifted to an automorphism of the algebra $C^*(E) \otimes \mathbb{K}$.

Question

It seems that we can choose the above graph C^*-algebra to be unital and the corresponding graph to be finite. Is it possible to get an example with a Cuntz-Krieger algebra?

Strong classification, II

Theorem (Arklint-R)

There exists a separable, nuclear, purely infinite graph C^*-algebra $C^*(E)$ over a graph E with primitive ideal space of the type $3B$, such that there exists an is automorphism of the full filtered K-theory that cannot be lifted to an automorphism of the algebra $C^*(E) \otimes \mathbb{K}$.

Question

It seems that we can choose the above graph C^*-algebra to be unital and the corresponding graph to be finite. Is it possible to get an example with a Cuntz-Krieger algebra?

Outline

Phantom Cuntz-Krieger algebras

Question

Since the classification of Cuntz-Krieger algebras in general is internal, it is an open question whether there exist separable, nuclear, purely infinite C^*-algebras with all simple subquotiens in the Bootstrap class that has the filtered K-theory isomorphic to the K-theory of a Cuntz-Krieger algebra without being stably isomorphic to a Cuntz-Krieger algebra. Such an algebra is called a phantom Cuntz-Krieger algebra.

Outline

Graph C^*-algebras

There has been (and still is) some progress in extending the classification results to more general graph C^*-algebras, both in the purely infinite case, in the mixed case, and for graphs not satisfying condition (K). [Eilers,R,Ruiz,Sørensen,Tomforde]