Grading of Leavitt Path Algebras and Classifications

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AUSTRALIA

Outline

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- Classification of LPAs via K_0-group

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- Graded rings

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or something like this...

Grothendieck group K_0

Let A be a ring with identity.

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$K_0(A)$ is a pre-ordered abelian group with an order unit $[A]$.

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$$K \oplus K \longrightarrow \mathbb{M}_2(K) \oplus K \longrightarrow \mathbb{M}_3(K) \oplus \mathbb{M}_2(K) \longrightarrow \cdots$$
$$(a, b) \longmapsto (\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a)$$

Classification of Ultramatricial algebras

Theorem (Elliott)

Let R and S be ultramatricial K-algebra. Then $R \cong S$ as K-algebra if and only if

$$\big(K_0(R), K_0(R)_+, [R]\big) \cong \big(K_0(S), K_0(S)_+, [S]\big).$$

Classification of LPAs via K-groups

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$\mathcal{L}(F) \cong \mathbb{M}_3(K)$

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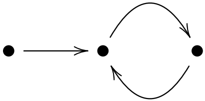

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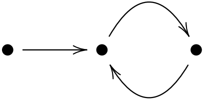

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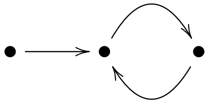

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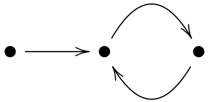
E  $\mathcal{L}(E) \cong \mathbb{M}_3(K[x, x^{-1}])$

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But

$$\mathbb{M}_3(K) \ncong \mathbb{M}_3(K[x, x^{-1}]).$$

So K_0 doesn't seem to classify all types of LPAs.

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• Cuntz, K-theory for certain C^*-algebras, Ann. of Math-1981. 181–197.

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Acyclic:

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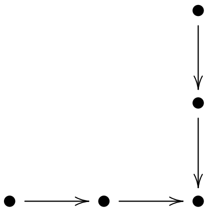
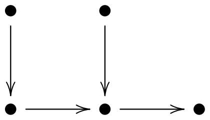


E_2 :



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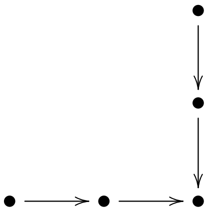

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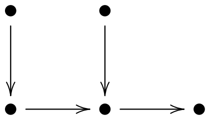

Then $\mathcal{L}(E_1) \cong \mathcal{L}(E_2) \cong \mathbb{M}_5(K)$.

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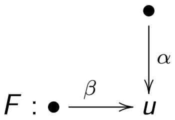

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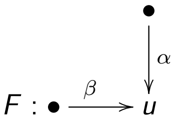

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Consider $(\delta_1, \dots, \delta_n)$, $\delta_i \in \Gamma$. Define a grading on $\mathbb{M}_n(A)$ by

$$\mathbb{M}_n(A)_\lambda = \begin{pmatrix} A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_{\lambda+\delta_2-\delta_2} & \cdots & A_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_{\lambda+\delta_n-\delta_n} \end{pmatrix}.$$

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Denote this matrix ring with this grading by $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$.
We have

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j,$$

Let K be a graded ring concentrated on degree 0. Then

$$\mathbb{M}_3(K)(0,1,1)_0 = \begin{pmatrix} K_0 & K_1 & K_1 \\ K_{-1} & K_0 & K_0 \\ K_{-1} & K_0 & K_0 \end{pmatrix} = \begin{pmatrix} K & 0 & 0 \\ 0 & K & K \\ 0 & K & K \end{pmatrix}$$

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$$K(\alpha, \alpha^*, v \mid v \in E^0, \alpha \in E^1)$$

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2. $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$.
3. $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$.
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$\mathcal{L}_R(E) = \bigoplus_{k\in\mathbb{Z}} \mathcal{L}_R(E)_k$ where,

$$\mathcal{L}_R(E)_k = \Big\{ \sum_i r_i\alpha_i\beta_i^* \mid \alpha_i, \beta_i \text{ paths with finite lengths}, r_i \in R, \text{ and } |\alpha_i| - |\beta_i| = k \text{ for all } i \Big\}.$$

Theorem

E be a finite acyclic graph with sinks $\{v_1, \ldots, v_t\}$. For any sink v_s, let $\{p_i^{v_s} \mid 1 \leq i \leq n(v_s)\}$ be the set of all paths which end in v_s. Then there is a $\mathbb{Z}$-graded isomorphism

$$\mathcal{L}_R(E) \cong_{\mathrm{gr}} \bigoplus_{s=1}^{t} \mathbb{M}_{n(v_s)}(R)\big(|p_1^{v_s}|, \ldots, |p_{n(v_s)}^{v_s}|\big). \tag{1}$$

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$$\mathcal{L}_R(E) \cong_{\mathrm{gr}} \mathcal{L}_R(F)$$

if and only if $k = t$, and after a permutation of indices, $n(v_s) = n(u_s)$ and $\{|p_i^{v_s}| \mid 1 \le i \le n(v_s)\}$ and $\{|p_i^{u_s}| \mid 1 \le i \le n(u_s)\}$ present the same list.

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Let E be a finite graph. The Leavitt path algebra $\mathcal{L}_R(E)$ with coefficients in a ring R is strongly graded if and only if any vertex connects to a cycle.

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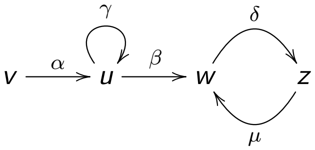

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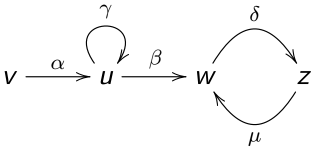

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\begin{aligned}
\alpha = \alpha u = \alpha(\gamma\gamma^* + \beta\beta^*) \\
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Let C_n be a comet with the cycle C of length $n \geq 1$. Let u be a vertex on the cycle C. Eliminate the edge in the cycle whose source is u and consider the set $\{p_i \mid 1 \leq i \leq m\}$ of all paths which end in u. Then

$$\mathcal{L}_K(E) \cong_{\text{gr}} \mathbb{M}_m\left(K[x^n, x^{-n}]\right)\left(|p_1|, \dots, |p_m|\right).$$

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Set of monomials $\{p_i C^k p_j^* \mid 1 \leq i, j \leq n, k \in \mathbb{Z}\}$ is an K-basis of $\mathcal{L}_K(E)$. Define the map

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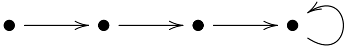
by $\phi(p_i C^k p_j^*) = e_{ij}(x^{kn})$. But $|p_i C^k p_j^*| = kn + |p_i| - |p_j|$ (note that $k \in \mathbb{Z}$). And

$$\deg(\phi(p_i C^k p_j^*)) = \deg(e_{ij}(x^{kn})) = nk + |p_i| - |p_j|.$$

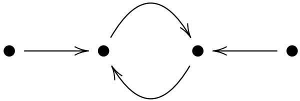
Comets:

E_1 :



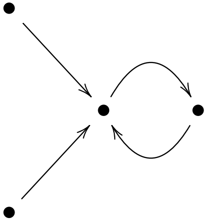
$\mathbb{M}_4(K[x, x^{-1}])(0, 1, 2, 3)_{\text{group ring}}$

E_2 :



$\mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 2)_{\text{skew}}$

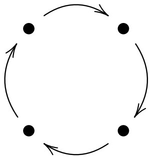
E_3 :



$\mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 1)_{\text{not crossed}}$

and

E_4 :



$\mathbb{M}_4(K[x^4, x^{-4}])(0, 1, 2, 3)_{\text{skew}}$

Arbitrary grading

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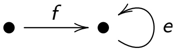
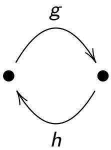
Example

Consider the graphs

E :



F :



Assigning 0 to vertices and 1 to edges in the graphs in the usual manner, we obtain $\mathcal{L}(E) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x, x^{-1}])(0,1)$ whereas $\mathcal{L}(F) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x^2, x^{-2}])(0,1)$ and one can easily observe that $\mathcal{L}_K(E) \ncong_{\mathrm{gr}} \mathcal{L}_K(F)$.

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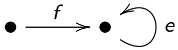

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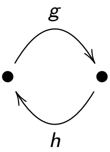

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However assigning 1 for the degree of f and 2 for the degree of e in E and 1 for the degrees of g and h in F,

$\mathcal{L}_K(E) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$ and

$\mathcal{L}_K(F) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$. So with these gradings,

$\mathcal{L}_K(E) \cong_{\text{gr}} \mathcal{L}_K(F)$.

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	degrees	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
M				M_{-1}	M_0	M_1	M_2	
$M(1)$			M_{-1}	M_0	M_1	M_2		
$M(2)$		M_{-1}	M_0	M_1	M_2			

Graded Projective modules

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Let A be a Γ-graded ring and P be a graded A-module. Then the following are equivalent:

1. P is graded and projective;
2. P is graded projective;
3. $Hom_{Gr-A}(P, -)$ is an exact functor in $Gr - A$;
4. P is graded isomorphic to a direct summand of a graded free A-module.

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The group $\mathcal{V}^{\mathrm{gr}}(A)^+$ is called the *graded Grothendieck group* and is denoted by $K_0^{\mathrm{gr}}(A)$, which is a $\mathbb{Z}[\Gamma]$-module.

Example: $\mathbb{Z}$-graded ring $A = K[x^n, x^{-n}]$, where K is a field and $n \in \mathbb{N}$, Example: $\mathbb{Z}$-graded ring $A = K[x^n, x^{-n}]$, where K is a field and $n \in \mathbb{N}$,

$$K_0^{\mathrm{gr}}(A) \cong \bigoplus_n \mathbb{Z},$$

which is a $\mathbb{Z}[x, x^{-1}]$-module, with the action of x on $(a_1, \ldots, a_n) \in \bigoplus_n \mathbb{Z}$ is as follows:

$$x(a_1, \ldots, a_n) = (a_n, a_1, \ldots, a_{n-1}).$$

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Theorem

E finite graph with no sink. Then for $A = \mathcal{L}(E)$ we have

$$K_0^{\mathrm{gr}}(A)/\langle [P] - [P(1)] \rangle \cong K_0(A).$$

Graded Ultramatricial algebra

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Definition

Let A be a Γ-graded field. A Γ-*graded matricial* A-algebra is a graded A-algebra of the form

$$\mathbb{M}_{n_1}(A)(\overline{\delta}_1) \times \cdots \times \mathbb{M}_{n_l}(A)(\overline{\delta}_l),$$

where $\overline{\delta}_i = (\delta_1^{(i)}, \dots, \delta_{n_i}^{(i)})$, $\delta_j^{(i)} \in \Gamma$, $1 \leq j \leq n_i$ and $1 \leq i \leq l$.

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Definition

Let A be a Γ-graded field. Then the ring R is called a Γ-*graded ultramatricial* A-algebra if $R = \bigcup_{i=1}^{\infty} R_i$, where $R_1 \subseteq R_2 \subseteq \dots$ is a sequence of graded matricial A-subalgebras.

Theorem

Let R and S be Γ-graded ultramatricial algebras over a graded field A. Then $R \cong_{\mathrm{gr}} S$ as graded A-algebras if and only if there is an order preserving $\mathbb{Z}[\Gamma]$-module isomorphism

$$\big(K_0^{\mathrm{gr}}(R), K_0^{\mathrm{gr}}(R)_+, [R]\big) \cong \big(K_0^{\mathrm{gr}}(S), K_0^{\mathrm{gr}}(S)_+, [S]\big).$$

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$$\big(K_0^{\text{gr}}(R), K_0^{\text{gr}}(R)_+, [R]\big) \cong \big(K_0^{\text{gr}}(S), K_0^{\text{gr}}(S)_+, [S]\big).$$

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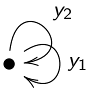
Conjecture. Leavitt path algebras is another class that fits into the above two theorems.

Graded versus non-graded K_0



Set $x_i = y_i^*$.

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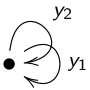

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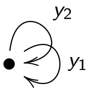

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But considering ϕ as graded homomorphism we get

$$\phi : A \xrightarrow{\cong} A(-1) \oplus A(-1)$$
$$a \mapsto (x_1 a, x_2 a)$$

In same manner $A(i) \cong A(i-1) \oplus A(i-1)$. This gives indication $K_0^{gr}(\mathcal{L}(E)) = \mathbb{Z}[1/2]$.

Let A be a strongly Γ-graded ring. By Dade's Theorem the functor

$$\begin{aligned} (-)_0 : \text{gr-}A &\to \text{mod-}A_0 \\ M &\mapsto M_0, \end{aligned}$$

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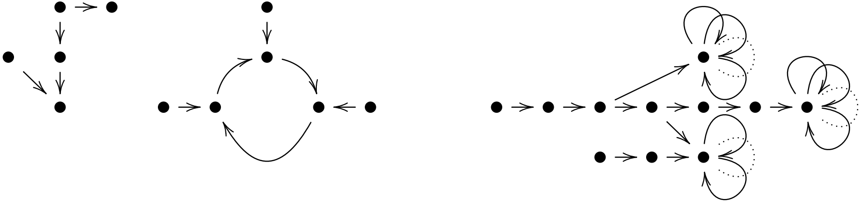
So that it induces an equivalence of categories. This implies that

$$K_i^{\text{gr}}(A) \cong K_i(A_0),$$

for $i \geq 0$.



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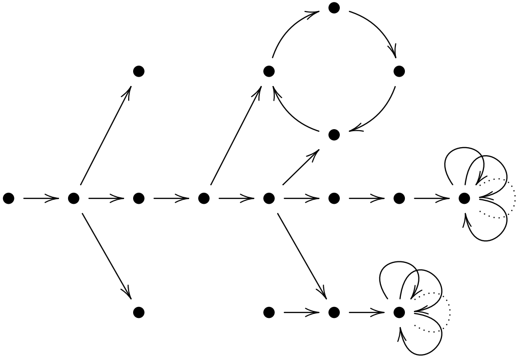

Theorem

Let E and F be polycephaly graphs. Then $\mathcal{L}(E) \cong_{\text{gr}} \mathcal{L}(F)$ if and only if there is a $\mathbb{Z}[x, x^{-1}]$-module isomorphism

$$\big(K_0^{\text{gr}}(\mathcal{L}(E)), [\mathcal{L}(E)]\big) \cong \big(K_0^{\text{gr}}(\mathcal{L}(F)), [\mathcal{L}(F)]\big).$$

Conjecture: Let E and F be finite graphs. Then $\mathcal{L}(E) \cong_{\text{gr}} \mathcal{L}(F)$ if and only if there is an order $\mathbb{Z}[x, x^{-1}]$-module isomorphism

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Theorem (Ara, Pardo)

The conjecture is valid for finite graphs with no sinks and sources.

Relation with symbolic dynamics

E and F finite graphs and A_E and A_F the adjacency matrices.

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$$QGrP(E) \approx QGrP(F)$$

Relation with symbolic dynamics

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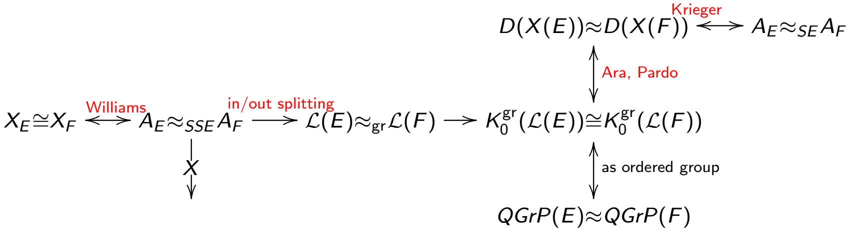

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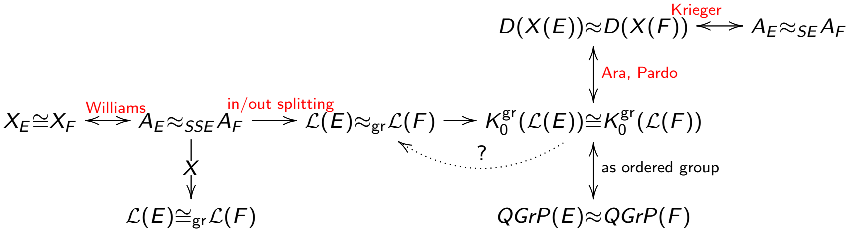

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