Equivalence relations on computable structures

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Computable Model Theory
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Main Question

Question

Given a class of structures K and an equivalence relation E, how hard is it to determine when two computable structures from K are E–equivalent?

Motivation

1. Computable model theory;
2. Descriptive set theory;
3. Theory of numberings/computability.

Motivation from Computable Model Theory

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Goncharov and Knight: $I(K)$ is hyperarithmetical
$\iff K^c = Mod^c_\varphi$ for a computable infinitary sentence φ.

Equivalence relations on computable structures

Every binary relation E on structures from K^c can be identified with the set

$$\{(m,n) | m,n \in I(K), \mathcal{M}_m \text{ and } \mathcal{M}_n \text{ are in the relation } E\}.$$

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Goncharov and Knight

Identify relations on computable structures with subsets of ω via indices, and use m–reducibility to compare their complexity.

Motivation from Theory of Numberings

Definition (Ershov, 1970's)

Let E, F be equivalence relations on ω. Then $E \leq F$ if there exists a computable function h, such that for all x, y

$$xEy \iff h(x)Fh(y).$$

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The partial order of inclusion modulo finite sets on $\mathcal{P}(\omega)$ can be embedded into the partial order of Borel equivalence relations modulo Borel reducibility.

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Calvert, Cummings, Knight, S. Miller (Quinn) (2004):
tc–reducibility as an effective analogue of the Borel reducibility.

Combining all together

1. Consider a (nice) class of structures K.
2. Identify K^c with the set $I(K) \subseteq \omega$ of indices of the computable members of K.
3. Identify a relation E on K^c with the binary relation $\{(i,j) | i,j \in I(K) and \mathcal{A}_i E \mathcal{A}_j\} \subseteq \omega^2$.

Definition

Let E, F be equivalence relations on (hyperarithmetical) subsets X, $Y$of ω respectively. Then E is reducible to F, $E \leq F$ if there exists a partial computable function h, such that $X \subseteq dom(h), h(X) \subseteq Y$ and for all $i, j \in X$,

$$iEj \iff h(i)Fh(j).$$

Bi–embeddability and Isomorphism

Theorem (F. and Friedman)

The equivalence relation of bi-embeddability on computable graphs is Σ^1_1 *complete among equivalence relations.*

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Theorem (F., Friedman, Harizanov, Knight, McCoy, Montalbán)

The equivalence relation of isomorphism on computable structures from the following classes is complete for all Σ^1_1 equivalence relations on ω:

1. *graphs and trees,*
2. *torsion–free abelian groups,*
3. *abelian p–groups,*
4. *fields, and others.*

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Question

Complexity of the isomorphism on Boolean algebras?

Computable isomorphism

Theorem (F., Friedman, Nies, 2012)

The computable isomorphism on computable structures from K is a Σ^0_3 complete equivalence relation for the following classes K:

- *trees,*
- *equivalence structures,*
- *Boolean algebras, and others.*

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Theorem (F., Friedman, Nies, 2012)

Many–one equivalence and 1–equivalence on indices of c.e. sets are Σ^0_3 complete for equivalence relations.

Hyperarithmetical isomorphism

Theorem (F., Friedman, Nies, Turetsky, 2013)

For a computable successor ordinal α*, the* Δ^0_α *isomorphism on computable trees is complete for* $\Sigma^0_{\alpha+2}$ *equivalence relations.*

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Conjecture/Theorem (work in progress)

The relation of hyperarithmetical isomorphism is complete for Π^1_1 relations.

Open question

Let *E* be a natural equivalence relation. Assume that for any class *K*, *E* on computable structures from *K* must have complexity Γ (where Γ is $\Sigma^1_1, \Pi^1_1, \Sigma^0_3$, etc.).

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Question

For an arbitrary equivalence relation F of complexity Γ, does there exist a computable infinitary sentence φ such that the relation E on Mod^c_φ is equivalent to F?

Thank you!