Herman Gluck Banff, August 2013

# LIPSCHITZ MINIMALITY of GROUP MULTIPLICATION on the THREE-SPHERE

***Haomin Wen***

**The most beautiful maps**

**between beautiful spaces**

**ought to be optimal**

**in some specific mathematical sense,**

**and then characterized by that optimality.**

**Haomin's Theorem.** *The group multiplication map* $m: S^3 \times S^3 \rightarrow S^3$ *is a Lipschitz constant minimizer in its homotopy class, uniquely so up to composition with isometries of domain and range.*

**Remark.** The above theorem is easy (and fun) to prove for $S^1$ .

Haomin's proof for $S^3$ also works for the multiplication map $m: S^7 \times S^7 \rightarrow S^7$ of unit Cayley numbers.

**Lipschitz maps and constants.**

A map $f: X \to Y$ between metric spaces is a ***Lipschitz map*** if there is a constant $C$ such that $d(f(x), f(x')) \leq C\, d(x, x')$ for all $x, x'$ in $X$.

The smallest such constant $C$ is called the ***Lipschitz constant*** of $f$.

There always exists a Lipschitz constant minimizer in the homotopy class of any Lipschitz map between compact metric spaces (by Arzela-Ascoli).

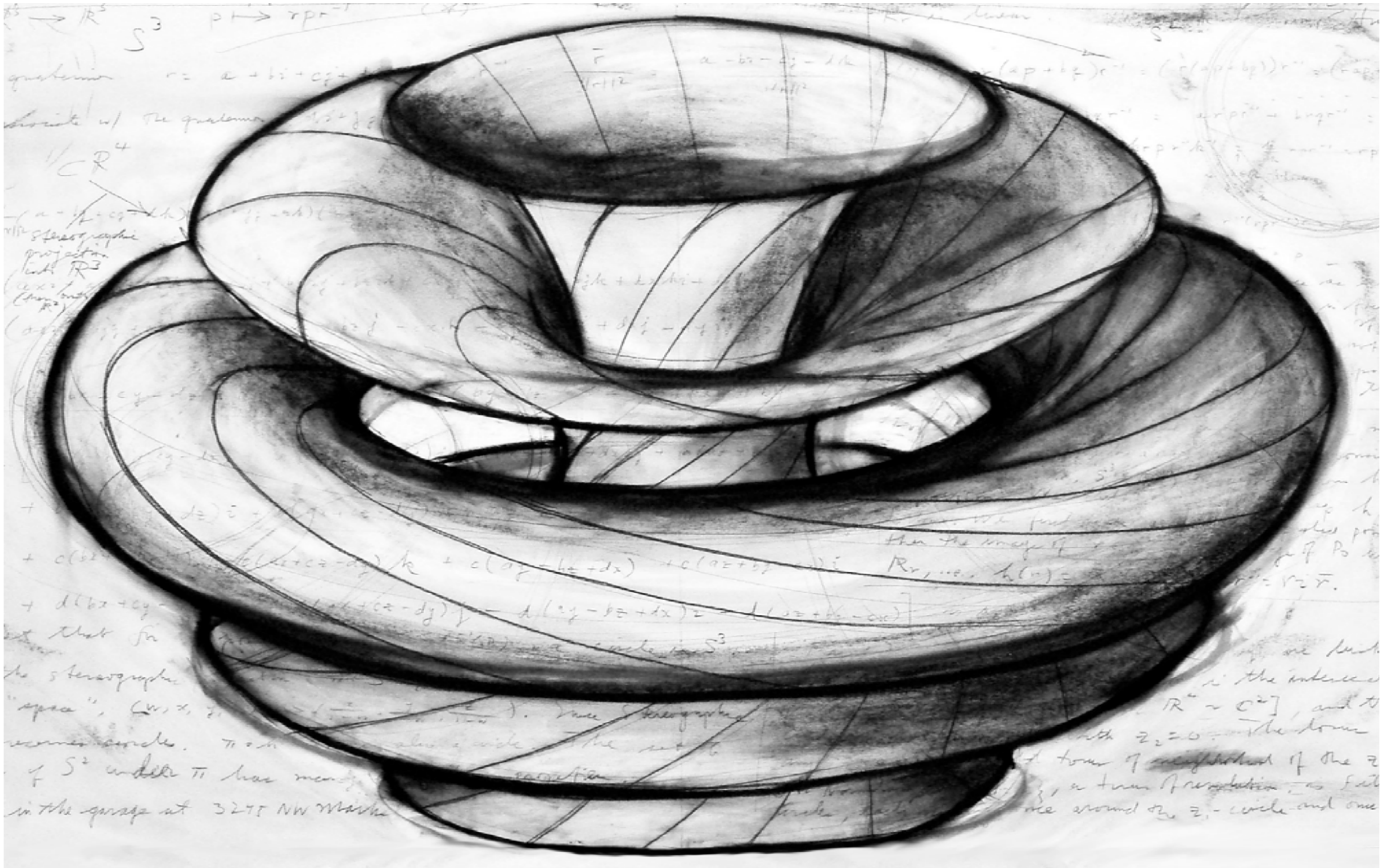
**Background to Haomin's theorem.** Consider the Hopf fibrations of round spheres by parallel great subspheres:

$S^1 \subset S^3 \to S^2 = CP^1$, $S^1 \subset S^5 \to CP^2$, ..., $S^1 \subset S^{2n+1} \to CP^n$ , ...
$S^3 \subset S^7 \to S^4 = HP^1$, $S^3 \subset S^{11} \to HP^2$, ..., $S^3 \subset S^{4n+3} \to HP^n$ , ...
$S^7 \subset S^{15} \to S^8$ ,

with the nonassociativity of the Cayley numbers responsible for the truncation of the third series.

First one discovered by Hopf in 1931, rest by him in 1935.

All Hopf projections have Lipschitz constant 1 when the base spaces are given the Riemannian submersion metric.



***Hopf fibration of 3-sphere by great circles***
***Lun-Yi Tsai     Charcoal and graphite on paper     2007***

**Thms (with Dennis DeTurck and Pete Storm, 2010).**

**(1)** *Given a Hopf fibration of a round sphere by parallel great subspheres, the projection map to the base space is, up to isometries of domain and range, the unique Lipschitz constant minimizer in its homotopy class.*

Dennis DeTurck

**(2)** *When the fibres of a Hopf fibration are great circles, a unit vector field tangent to these circles is, up to isometries of domain and range, the unique Lipschitz constant minimizer in its homotopy class.*

Pete Storm

## Tracing even further back...

**Theorem (with Wolfgang Ziller, 1986).**
*On $S^3$, the Hopf vector field is volume-minimizing in its homology class in the unit tangent bundle.*

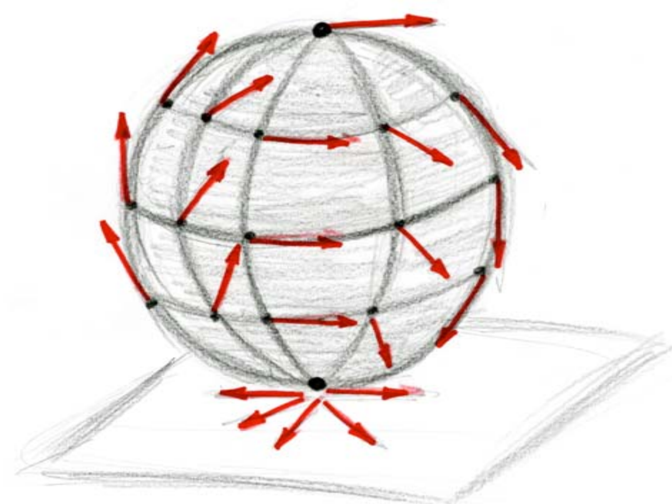
Wolfgang Ziller

**However ...** **(David Johnson, 1988).**

*On $S^5$, the Hopf vector field is **not** volume-minimizing in its homology class, not even a local minimum, though it is a critical "point" of the volume function.*

**Furthermore ...** **(Sharon Pedersen, 1988).**

*As the Hopf vector field on* $S^5$ *, viewed as a cross-section of the unit tangent bundle* $US^5$ *, shrinks there trying to minimize volume, it appears to limit on the image of a vector field with singularities, suggesting that there is no vector field on* $S^5$ *whose image in* $US^5$ *has minimum volume. And likewise for* $S^7$, $S^9$, ...



**Try again:**

## Are Hopf vector fields energy minimizers?

**Energy.**

The ***energy*** of a smooth map $f: M \rightarrow N$
beween Riemannian manifolds (with $M$ compact)
is defined by

$$E(f) = 1/2 \int_{x \in M} ||df_x||^2 \, d(vol) ,$$

where $||df_x||^2$ is the sum of the squares of the entries
in a matrix for $df_x$ w.r.t. orthonormal bases.

**Harmonic maps.**

A smooth map $f: M \rightarrow N$ between Riemannian manifolds (with $M$ compact) is ***harmonic*** if it is a critical point of the energy function, that is, if

$$dE(f_t)|_{t=0} = 0$$

for all one-parameter families $\{ f_t \}$ of maps from $M \rightarrow N$ with $f_0 = f$ .

**Hopf vector fields are harmonic maps.**

If a unit vector field is regarded as a map into the unit tangent bundle, then Hopf vector fields on all odd-dimensional spheres are harmonic maps, and on $S^3$ there are no other unit vector fields which are harmonic (Han and Yim, 1996).

But harmonic maps from spheres to compact Riemannian mflds are always unstable (Xin, 1980).

**What if we limit the competition?**

If we now only look at cross-sections of the unit tangent bundle, then the Hopf vector fields $V_H: S^n \to US^n$ are still unstable for $n = 5, 7, 9, \ldots$ (Wood, 1997) ...

...but for $n = 3$ they are stable, and in fact a local minimum (Wood, 1999).

# CONCLUSION

***Energy-minimization doesn't seem to work either.***

## More background: Schmuel Weinberger's question.

Consider maps $f: S^3 \to S^2$ of Hopf invariant $n$ .
Let $L(n)$ be the min Lipschitz constant of all such maps.
Find the asymptotic growth of $L(n)$ as $n \to \infty$ .

Pete Storm brought this question along when he visited Penn in 2006; the first step was to confirm that $L(1) = 1$ .

***We proved more:*** Not only is $L(1) = 1$, but the only maps $f : S^3 \to S^2$ which are homotopic to the Hopf projection and have this minimum Lipschitz constant are the Hopf projection and its compositions with isometries of domain and range.

**General hope:** Many beautiful maps...for example, Riem. submersions of compact homogeneous spaces... are Lipschitz minimizers in their homotopy classes, unique up to composition with isometries of domain and range.

The Hopf projections all have this feature.

***One more known instance.*** The Stiefel projection $V_2R^4 \rightarrow G_2R^4$ is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

***Remark.*** Group multiplication $S^3 \times S^3 \to S^3$ is, up to scale, a Riemannian submersion of compact homogeneous spaces.

In the following pages, we give some excerpts from Haomin's proof of his theorem.

**Back to $S^3$ with a preliminary result.**

**(1)** Group multiplication $S^3 \times S^3 \rightarrow S^3$ has Lipschitz constant $= \sqrt{2}$ .

*This is a matter of observation, which we tackle in a moment.*

**(2)** Any map $S^3 \times S^3 \rightarrow S^3$ homotopic to this has Lipschitz constant $\geq \sqrt{2}$ .

## (1) Group mult $m: S^3 \times S^3 \to S^3$ has $Lip(m) = \sqrt{2}$.

**Proof.** In a Lie group with bi-invariant metric, group mult near all pairs of points are isometric.

Enough to show the differential $m_* : R^3 \times R^3 \to R^3$ has Lipschitz constant (= operator norm) $\sqrt{2}$.

At (identity, identity), $m_*$ = addition in $R^3$.

The matrix A of addition is the $3 \times 6$ matrix $\mathbf{I} \mid \mathbf{I}$.

$Lip(A) = ||A||_{op} = \sqrt{(\text{largest eigenvalue of } A^T A)}$

The eigenvalues of $A^T A$ are computed to be 0, 0, 0, 2, 2, 2, completing the proof.

**Preliminaries to the proof of (2).**

***Definitions.*** A map $f : S^n \to S^n$ is said to be

***even*** if $f(-x) = f(x)$ for all $x \in S^n$ ;

***odd*** if $f(-x) = -f(x)$ for all $x \in S^n$ .

***Easy exercise.*** An even map $S^n \to S^n$ has even degree.

***Theorem (Borsuk).*** An odd map $S^n \to S^n$ has odd degree. (For a proof, see Hatcher, "Algebraic Topology," pp. 174-176.

***Corollary 1.*** If $f : S^n \to S^n$ has even degree, then there is a pair of antipodal points $x$ and $-x$ such that $f(x) = f(-x)$ .

***Proof.*** Suppose not. Then we can homotope $f$ by repulsion such that afterwards $f(-x) = -f(x)$ for *every* $x$ in $S^n$ .

By Borsuk's Theorem, this implies that $f$ has odd degree, contrary to assumption.

***Corollary 2.*** A degree-two map $f : S^n \to S^n$ must have Lipschitz constant $\geq 2$ .

***Proof.*** By Corollary 1, there exists a pair of antipodal points $x$ and $-x$ such that $f(x) = f(-x)$ .

Call this image point $y$ .

Let $x'$ be a point in $S^n$ such that $f(x') = -y$ .

Then $d(x', x) \leq \pi/2$ or $d(x', -x) \leq \pi/2$ ,
yet $d(f(x'), f(x)) = d(f(x'), f(-x)) = \pi$ .

Hence $\mathrm{Lip}(f) \geq 2$ .

**Proof of (2):** Any map $f : S^3 \times S^3 \to S^3$ which is homotopic to the multiplication map $m: S^3 \times S^3 \to S^3$ has $\mathrm{Lip}(f) \geq \sqrt{2}$ .

The restriction of $m$ to the diagonal

$$D(S^3) = \{(x, x): x \in S^3\} \to S^3$$

has degree 2 , so the same must hold for $f$ .

Since $D(S^3)$ is a round 3-sphere of radius $\sqrt{2}$ , it follows from Corollary 2 that $\mathrm{Lip}(f\,|_{D(S^3)}) \geq \sqrt{2}$ .

Hence $\mathrm{Lip}(f) \geq \sqrt{2}$ , as claimed.

***Remark.*** At this point, we know that the multiplication map

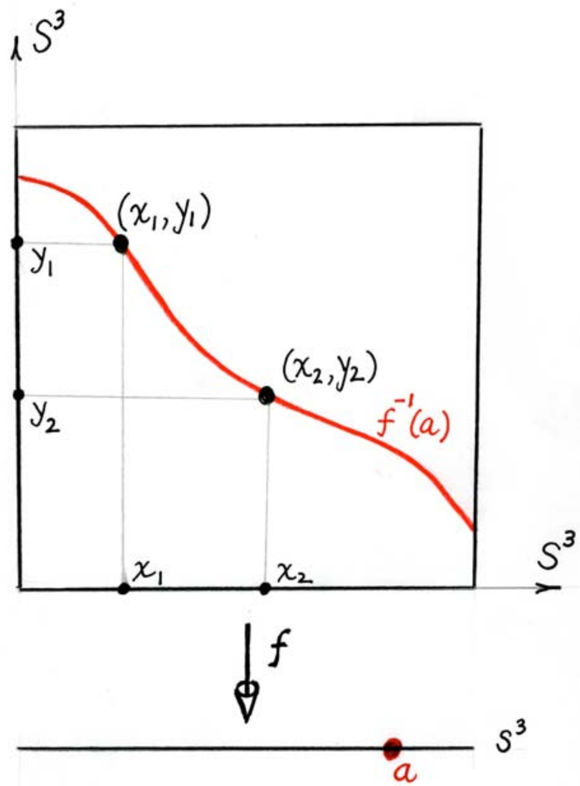
$$m: S^3 \times S^3 \to S^3$$

has the minimum possible Lipschitz constant of $\sqrt{2}$ in its homotopy class.

The issue now is to show that the only other maps in this homotopy class with Lipschitz constant $\sqrt{2}$ are the compositions of $m$ with isometries of domain and range.

## The four steps of Haomin's proof of uniqueness.

**(1)** Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points in $S^3 \times S^3$ which have the same image in $S^3$ under $f$.



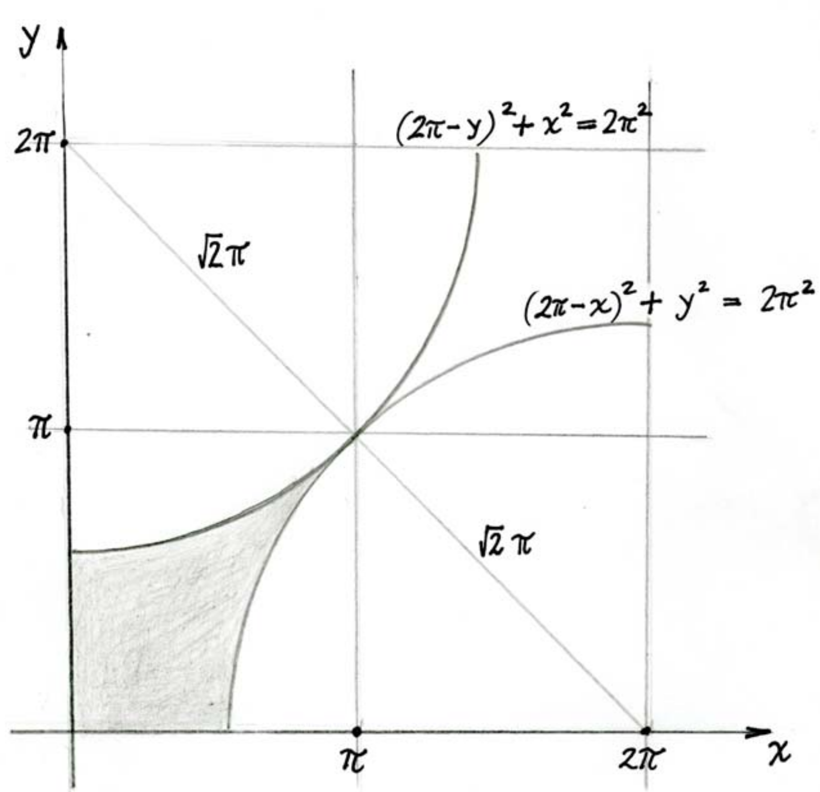
The first step is to prove the following inequalities:

$$(2\pi - d(x_1, x_2))^2 + d(y_1, y_2)^2 \geq 2\pi^2$$

$$(2\pi - d(y_1, y_2))^2 + d(x_1, x_2)^2 \geq 2\pi^2 ,$$

which are at the heart of Haomin's argument.

We graph both inequalities together in the figure below, letting $x = d(x_1, x_2)$ and $y = d(y_1, y_2)$, both in $[0, \pi]$.




The shaded region above consists of the points $(x, y)$ satisfying both inequalities.

**Haomin's approach to proving his inequalities.**

(i) Since the given map $f : S^3 \times S^3 \rightarrow S^3$ is homotopic to the group multiplication $m$, it must take each $S^3 \times b$ *onto* $S^3$. Thus for each $a$ and $b$ in $S^3$, the set $f^{-1}(a) \cap (S^3 \times b)$ must be non-empty.

(ii) Haomin constructs a 3-sphere $S$ through the two given points $(x_1 , y_1)$ and $(x_2 , y_2)$ in $f^{-1}(a)$, homotopic to $S^3 \times b$, so that (i) will hold for $S$ in place of $S^3 \times b$.

(iii) He then shows that if his inequalities are false, the 3-sphere $S$ will be disjoint from $f^{-1}(-a)$, contradicting (i).

**(2)** Use these inequalities to show that each inverse image $f^{-1}(a)$ is the graph of some isometry $h_a: S^3 \rightarrow S^3$, and hence appears inside $S^3 \times S^3$ as a diagonal 3-sphere. There are four steps:

**Step 1.** Let $(x_1, y_1)$ and $(x_2, y_2)$ be points in $f^{-1}(a)$. If $x_2 = -x_1$, then $y_2 = -y_1$, and conversely.

**Step 2.** $f^{-1}(a)$ is the graph of a bijection $h_a: S^3 \rightarrow S^3$.

**Step 3.** The map $h_a: S^3 \rightarrow S^3$ has Lipschitz const $\leq 1$.

**Step 4.** The map $h_a: S^3 \rightarrow S^3$ is an isometry.

**(3)** Show that these diagonal 3-spheres are mutually parallel.

Start with any pair of antipodal points $a$ and $-a$ on the range $S^3$, where their distance apart is $\pi$.

Then the great 3-spheres $f^{-1}(a)$ and $f^{-1}(-a)$ must have distance apart $\geq \pi/\sqrt{2}$ in $S^3 \times S^3$, since the Lipschitz constant of $f$ is $\sqrt{2}$.

But the open nbhd of radius $\pi/\sqrt{2}$ about a great 3-sphere in $S^3 \times S^3$ is almost the entire space, and omits only the orthogonal great 3-sphere, which must be $f^{-1}(-a)$ .

In like spirit, if we pick another point $b \in S^3$ , we learn that $f^{-1}(b)$ must be parallel to $f^{-1}(a)$ at distance $d(a, b) / \sqrt{2}$ from it, completing the argument that the great 3-sphere fibres of $f$ are parallel to one another.

**(4)** Use classical results to finish the proof.

**Proposition (Y-C Wong 1961, Joseph Wolf 1963).** *Any fibration of an open set on* $S^3 \times S^3$ *by parallel great 3-spheres extends to a fibration of all of* $S^7(\sqrt{2})$ *by parallel great 3-spheres, and any two of these are isometric to one another.*

It follows that any two fibrations of $S^3 \times S^3$ by parallel great 3-spheres can be taken, one to the other, by an isometry of $S^3 \times S^3$ .

Perform this isometry, so that now the fibres of $f: S^3 \times S^3 \rightarrow S^3$ coincide with the fibres of the multiplication map $m: S^3 \times S^3 \rightarrow S^3$ .

Since $f$ and $m$ are now both Riemannian submersions (up to scale) of $S^3 \times S^3 \rightarrow S^3$ having the same fibres, the map of $S^3$ to itself which takes $f(x, y)$ to $m(x, y)$ is an isometry.

This completes the proof of Haomin's theorem.

**What's next?**

**Test question #1:** Try to show that the bundle map

$$SO(n) \rightarrow S^{n-1}$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

This can be shown for $n \leq 4$ on the basis of known results, so the first challenge is for $n = 5$ .

**Test question #2:** Show that the projection map

$$SU(3) \rightarrow S^5$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

**Test question #3:** Show that the projection map of the Stiefel bundle

$$V_2R^n \rightarrow G_2R^n$$

is a Lipschitz constant minimizer in its homotopy class, unique up to composition with isometries of domain and range.

This is also known for $n \leq 4$ , so the first challenge is for $n = 5$ .

**Test question #4: Geometry of real Grassmann mflds.**

Let $G_kR^n$ = set of oriented k-planes thru origin in $R^n$ .

$G_kR^n = SO(n) / \big(SO(k) \times SO(n-k)\big) = k(n-k)$ dim'l mfld.

$$
\begin{array}{ccccccccccc}
G_5R^6 & \subset & G_5R^7 & \subset & G_5R^8 & \subset & G_5R^9 & \subset & G_5R^{10} & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_4R^5 & \subset & G_4R^6 & \subset & G_4R^7 & \subset & G_4R^8 & \subset & G_4R^9 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_3R^4 & \subset & G_3R^5 & \subset & G_3R^6 & \subset & G_3R^7 & \subset & G_3R^8 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_2R^3 & \subset & G_2R^4 & \subset & G_2R^5 & \subset & G_2R^6 & \subset & G_2R^7 & \subset & \ldots \\
\cup & & \cup & & \cup & & \cup & & \cup & & \\
G_1R^2 & \subset & G_1R^3 & \subset & G_1R^4 & \subset & G_1R^5 & \subset & G_1R^6 & \subset & \ldots
\end{array}
$$

$$
\begin{array}{ccccc}
G_3R^4 & \subset & G_3R^5 & \subset & G_3R^6 \\
\cup & & \cup & & \cup \\
G_2R^3 & \subset & G_2R^4 & \subset & G_2R^5 \\
\cup & & \cup & & \cup \\
G_1R^2 & \subset & G_1R^3 & \subset & G_1R^4
\end{array}
$$

The 9-dim'l Grassmann manifold $G_3R^6$ has the rational homotopy type of $S^4 \times S^5$, and the subGrassmannian $G_2R^4$ generates its 4-dim'l homology.

But (with Dana Mackenzie and Frank Morgan, 1995) ... $G_2R^4$ is only a local volume-minimizer in its homology class in $G_3R^6$, not a global volume-minimizer.

***Test question #4.*** Is the inclusion of $G_2R^4$ in $G_3R^6$ a Lipschitz minimizer in its homotopy class?