# Filling multiples of embedded curves

Robert Young
University of Toronto

Aug. 2013

# Filling area

If $T$ is an integral 1-cycle in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ be the minimal area of an integral 2-chain with boundary $T$.

# Filling area

If $T$ is an integral 1-cycle in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ be the minimal area of an integral 2-chain with boundary $T$.
For all $T$, $\mathrm{FA}(2T) \le 2\,\mathrm{FA}(T)$.

# Filling area

If $T$ is an integral 1-cycle in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ be the minimal area of an integral 2-chain with boundary $T$.
For all $T$, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

- If $T$ is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.

# Filling area

If $T$ is an integral 1-cycle in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ be the minimal area of an integral 2-chain with boundary $T$.
For all $T$, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

- If $T$ is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.
- (Federer, 1974) If $T$ is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.

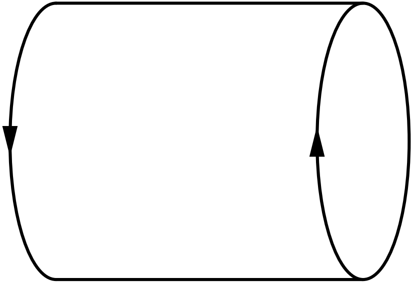
# Filling area

If $T$ is an integral 1-cycle in $\mathbb{R}^n$, let $\mathrm{FA}(T)$ be the minimal area of an integral 2-chain with boundary $T$.
For all $T$, $\mathrm{FA}(2T) \leq 2\,\mathrm{FA}(T)$.

- If $T$ is a curve in $\mathbb{R}^2$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.
- (Federer, 1974) If $T$ is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$.
- (L. C. Young, 1963) For any $\epsilon > 0$, there is a curve $T \in \mathbb{R}^4$ such that
$$\mathrm{FA}(2T) \leq (1 + 1/\pi + \epsilon)\,\mathrm{FA}(T)$$

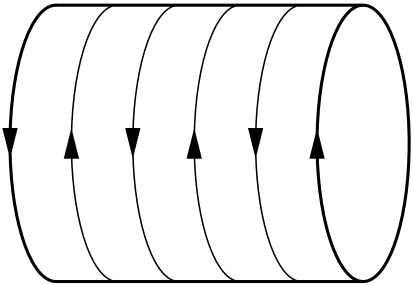
# L. C. Young's example

Let $K$ be a Klein bottle

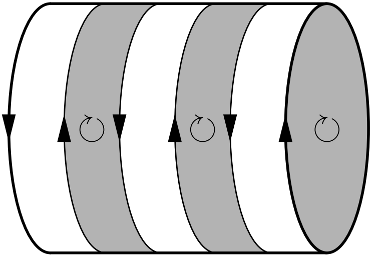


# L. C. Young's example

Let $K$ be a Klein bottle and let $T$ be the sum of $2k+1$ loops in alternating directions.

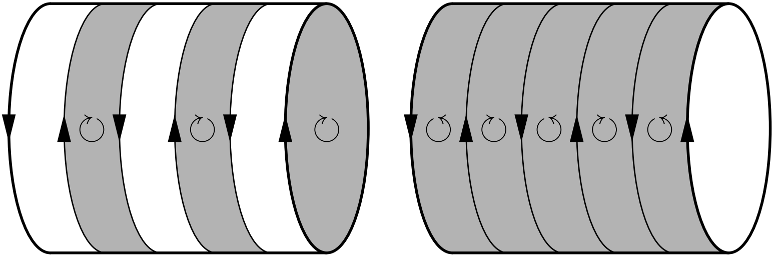


# L. C. Young's example



- $T$ can be filled with $k$ bands and one extra disc $D$
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$

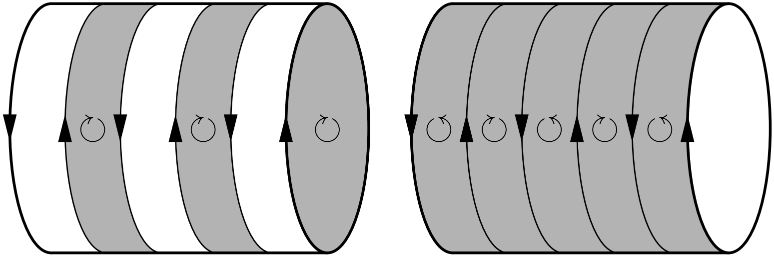
# L. C. Young's example



- $T$ can be filled with $k$ bands and one extra disc $D$
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$

- $2T$ can be filled with $2k+1$ bands
- $\mathrm{FA}(2T) \approx \text{area } K$

# L. C. Young's example



- $T$ can be filled with $k$ bands and one extra disc $D$
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$
- $2T$ can be filled with $2k+1$ bands
- $\mathrm{FA}(2T) \approx \text{area } K$— less than $2\,\mathrm{FA}(T)$ by $2\,\text{area } D$!

# The main theorem

Q: Is there a $c > 0$ such that $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$?

# The main theorem

Q: Is there a $c > 0$ such that $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$?

## Theorem (Y.)

*Yes! For any $d$, $n$, there is a $c$ such that if $T$ is a $(d-1)$-cycle in $\mathbb{R}^n$, then $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$.*

# From $T$ to $K$

Let $T$ be a $(d-1)$-cycle and suppose that

$$\partial B = 2T.$$

# From $T$ to $K$

Let $T$ be a $(d-1)$-cycle and suppose that

$$\partial B = 2T.$$

Then

$$\partial B \equiv 0 (\bmod 2),$$

so $B$ is a mod-2 cycle.

# From $T$ to $K$

Let $T$ be a $(d-1)$-cycle and suppose that

$$\partial B = 2T.$$

Then

$$\partial B \equiv 0 (\text{mod } 2),$$

so $B$ is a mod-2 cycle.
Let $R$ be an integral cycle such that $B \equiv R (\text{mod } 2)$. Then

$$B - R \equiv 0 (\text{mod } 2)$$
$$\partial \frac{B - R}{2} = \frac{\partial B}{2} = T.$$

# The main proposition

So, to prove the theorem, it suffices to show:

### Proposition

*There is a $c$ such that if $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, then there is an integral cycle $R$ such that $A \equiv R (mod\ 2)$ and $\text{mass}\, R \leq c\, \text{mass}\, A$.*

# The three-dimensional case

If $A$ is a cellular 2-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^3$, let $Z$ be a 3-chain with $\mathbb{Z}/2$ coefficients such that $\partial Z = A$.

# The three-dimensional case

If $A$ is a cellular 2-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^3$, let $Z$ be a 3-chain with $\mathbb{Z}/2$ coefficients such that $\partial Z = A$.
Let $\overline{Z}$ be the "lift" of $Z$ to a chain with integral coefficients.

# The three-dimensional case

If $A$ is a cellular 2-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^3$, let $Z$ be a 3-chain with $\mathbb{Z}/2$ coefficients such that $\partial Z = A$.
Let $\overline{Z}$ be the "lift" of $Z$ to a chain with integral coefficients. Then

$$\partial \overline{Z} \equiv A (\text{mod } 2)$$

# The three-dimensional case

If $A$ is a cellular 2-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^3$, let $Z$ be a 3-chain with $\mathbb{Z}/2$ coefficients such that $\partial Z = A$.
Let $\overline{Z}$ be the "lift" of $Z$ to a chain with integral coefficients. Then

$$\partial \overline{Z} \equiv A (\text{mod } 2)$$

and

$$\text{mass}\, \partial \overline{Z} = \text{mass}\, A,$$

so the proposition holds for $R = \partial \overline{Z}$.

# A crude bound

If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, let $Z$ be a $\mathbb{Z}/2$-chain such that $\partial Z = A$.

# A crude bound

If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, let $Z$ be a $\mathbb{Z}/2$-chain such that $\partial Z = A$. Let $\overline{Z}$ be a lift of $Z$ to a chain with integral coefficients.

# A crude bound

If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, let $Z$ be a $\mathbb{Z}/2$-chain such that $\partial Z = A$. Let $\overline{Z}$ be a lift of $Z$ to a chain with integral coefficients. Then

$$\partial \overline{Z} \equiv A (\text{mod } 2)$$

# A crude bound

If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, let $Z$ be a $\mathbb{Z}/2$-chain such that $\partial Z = A$. Let $\overline{Z}$ be a lift of $Z$ to a chain with integral coefficients. Then

$$\partial\overline{Z} \equiv A (\text{mod } 2)$$

and

$$\partial\overline{Z} \lesssim \operatorname{mass} Z.$$

# A crude bound

If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in $\mathbb{R}^n$, let $Z$ be a $\mathbb{Z}/2$-chain such that $\partial Z = A$. Let $\overline{Z}$ be a lift of $Z$ to a chain with integral coefficients. Then

$$\partial \overline{Z} \equiv A (\text{mod } 2)$$

and

$$\partial \overline{Z} \lesssim \text{mass}\, Z.$$

By the isoperimetric inequality for $\mathbb{R}^n$,

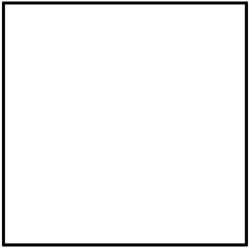
$$\text{mass}\, Z \lesssim (\text{mass}\, A)^{(d+1)/d}.$$

# A $V \log V$ bound

**Proposition (Guth-Y.)**

*If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in the unit grid in $\mathbb{R}^n$, then there is an $R$ such that $A \equiv R \pmod 2$ and*

$$\operatorname{mass} R \lesssim \operatorname{mass} A (\log \operatorname{mass} A).$$

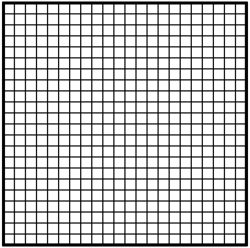


# A $V \log V$ bound

## Proposition (Guth-Y.)

*If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in the unit grid in $\mathbb{R}^n$, then there is an $R$ such that $A \equiv R (\text{mod } 2)$ and*

$$\text{mass}\, R \lesssim \text{mass}\, A (\log \text{mass}\, A).$$

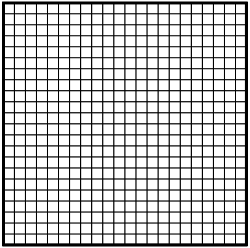


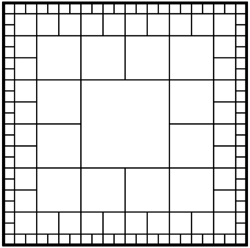
# A $V \log V$ bound

## Proposition (Guth-Y.)

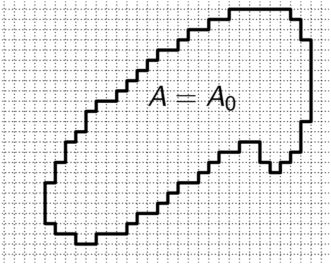
*If $A$ is a cellular $d$-cycle with $\mathbb{Z}/2$ coefficients in the unit grid in $\mathbb{R}^n$, then there is an $R$ such that $A \equiv R (\text{mod } 2)$ and*

$$\text{mass}\, R \lesssim \text{mass}\, A (\log \text{mass}\, A).$$

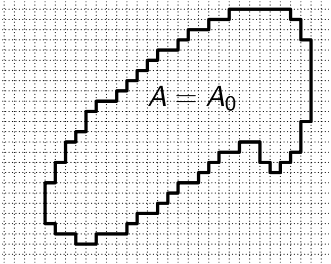


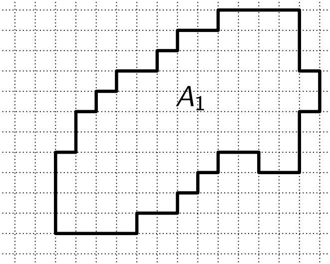


# Filling through approximations

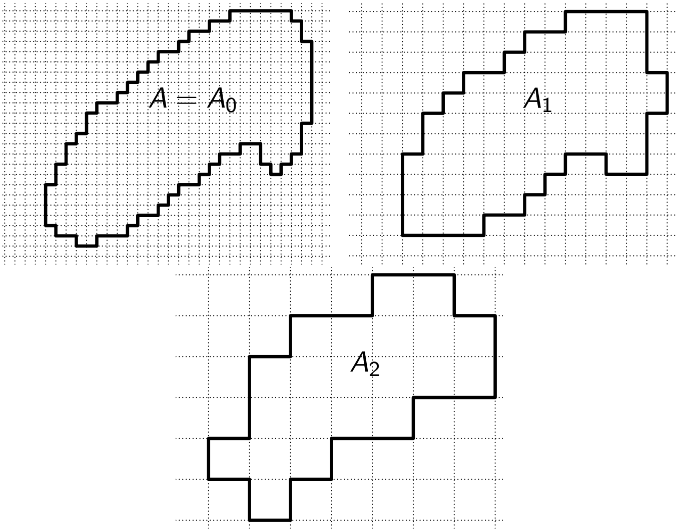


# Filling through approximations

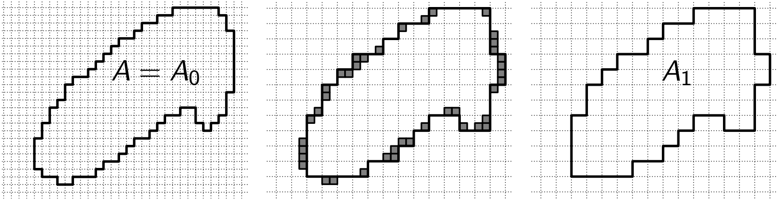


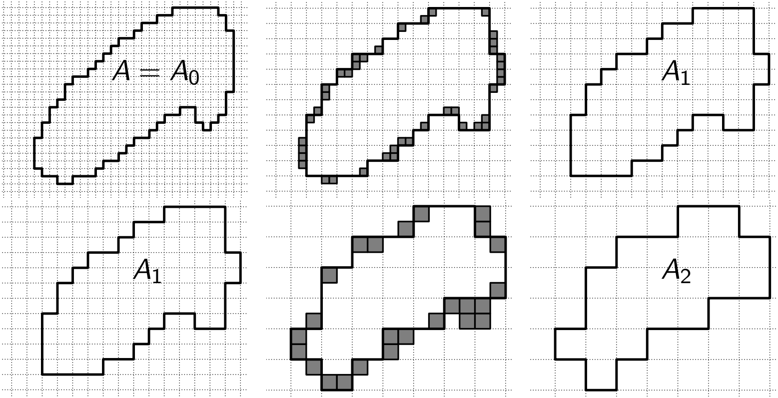
# Filling through approximations



# Filling through approximations



# Filling through approximations



# Regularity and rectifiability

**Definition**

*A set* $E \subset \mathbb{R}^n$ *is Ahlfors d-regular if for any* $x \in E$ *and any* $0 < r < \operatorname{diam} E$,

$$\mathcal{H}^d(E \cap B(x,r)) \sim r^d.$$

**Definition**

*A set* $E \subset \mathbb{R}^n$ *is d-rectifiable if it can be covered by countably many Lipschitz images of* $\mathbb{R}^d$.

# Uniform rectifiability

**Definition (David-Semmes)**

*A set $E \subset \mathbb{R}^n$ is uniformly d-rectifiable if it is d-regular and there is a c such that for all $x \in E$ and $0 < r < \operatorname{diam} E$, there is a c-Lipschitz map $B_d(0, r) \to \mathbb{R}^n$ which covers a $1/c$-fraction of $B(x, r) \cap E$.*

# Sketch of proof

## Proposition

*Every cellular $d$-cycle $A$ in the unit grid with $\mathbb{Z}/2$ coefficients can be written as a sum*

$$A = \sum_i A_i$$

*of $\mathbb{Z}/2$ $d$-cycles with uniformly rectifiable support such that*

$$\sum \operatorname{mass} A_i \leq C \operatorname{mass} A.$$

# Sketch of proof

## Proposition

*Every cellular $d$-cycle $A$ in the unit grid with $\mathbb{Z}/2$ coefficients can be written as a sum*

$$A = \sum_i A_i$$

*of $\mathbb{Z}/2$ $d$-cycles with uniformly rectifiable support such that*

$$\sum \operatorname{mass} A_i \leq C \operatorname{mass} A.$$

## Proposition

*Any $\mathbb{Z}/2$ $d$-cycle $A$ with uniformly rectifiable support is equivalent (mod 2) to an integral $d$-cycle $R$ with*

$$\operatorname{mass} R \leq C \operatorname{mass} A.$$

# Open questions

- What's the relationship between integral filling volume and real filling volume?

# Open questions

- What's the relationship between integral filling volume and real filling volume?
- This suggests that surfaces and embedded surfaces can have very different geometry. What systolic inequalities hold for surfaces embedded in $\mathbb{R}^n$?