Optimal Homotopies of Curves on Surfaces

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University of Toronto

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Setting

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- γ_1 and γ_2 are closed curves on M
- H is a homotopy from γ_1 to γ_2 such that $\mathrm{length}(H(t)) \leq L$ for all t

Theorem 1

Theorem (with Y. Liokumovich)

If γ_1 and γ_2 are simple, then for each $\epsilon > 0$, there exists a homotopy $\widetilde{H}$ from γ_1 to γ_2 that is composed of simple curves of length no more than $L + \epsilon$.

Reidemeister Moves

Type A

Reidemeister Moves

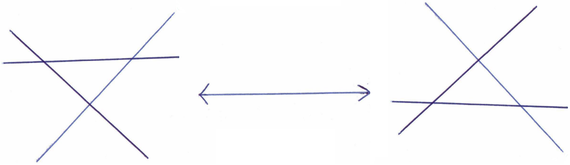
Type A

Type B

Reidemeister Moves

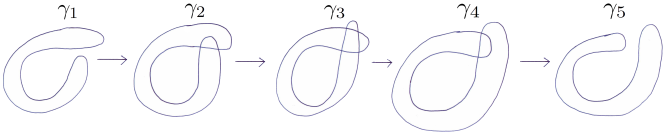
Type A

Type B



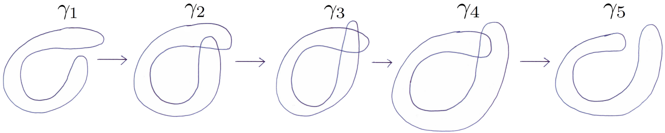
Type C

Example

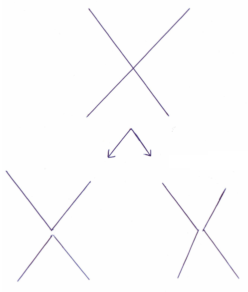


Example Homotopy

Example

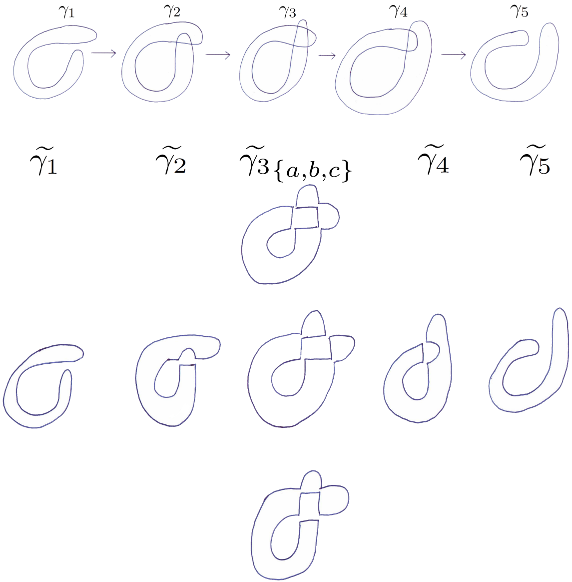


Example Homotopy



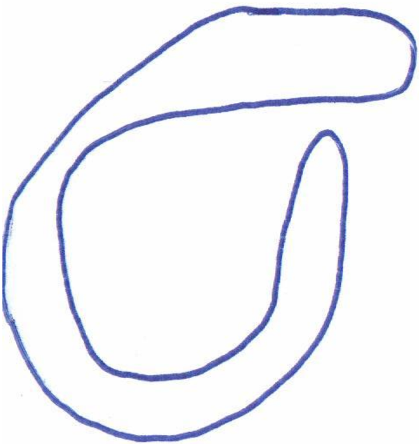
Cutting Vertices

Graph Construction



Graph Construction

$\widetilde{\gamma_1}$



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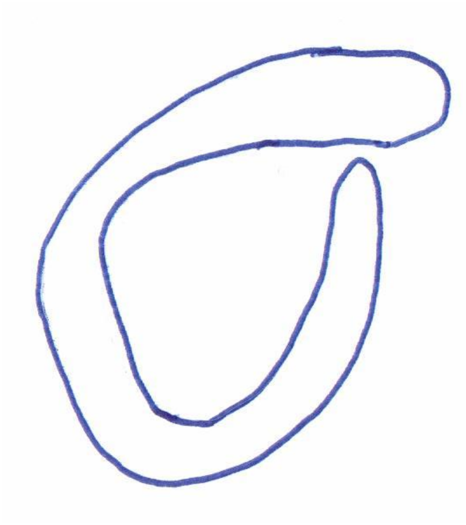

Graph Construction

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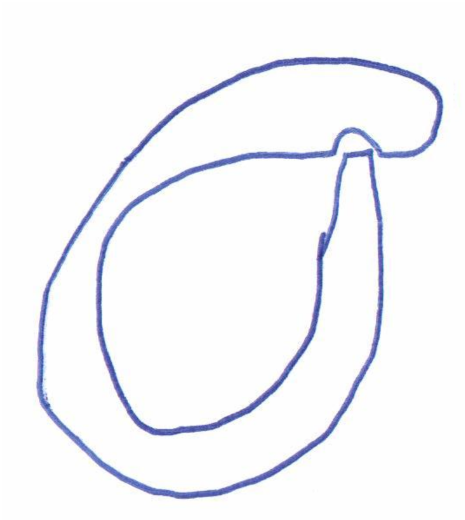

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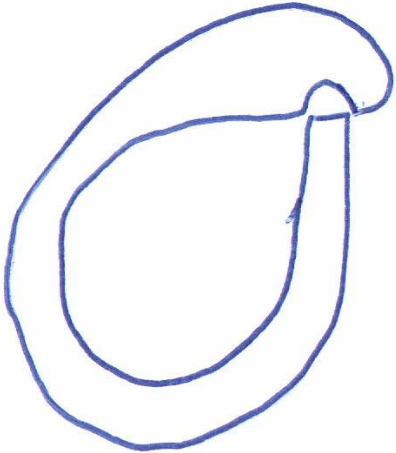

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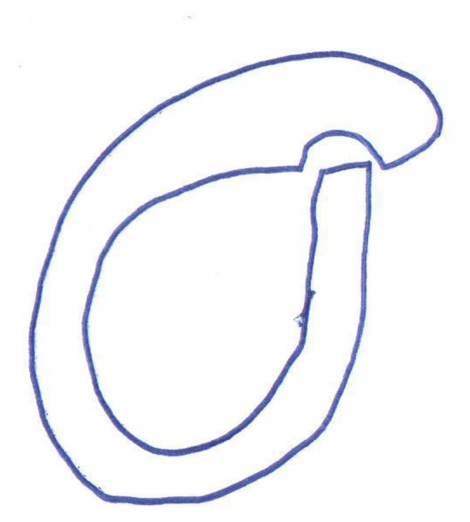
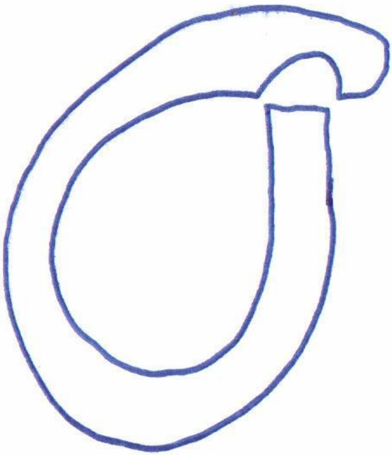


Graph Construction

$\widetilde{\gamma_2}$



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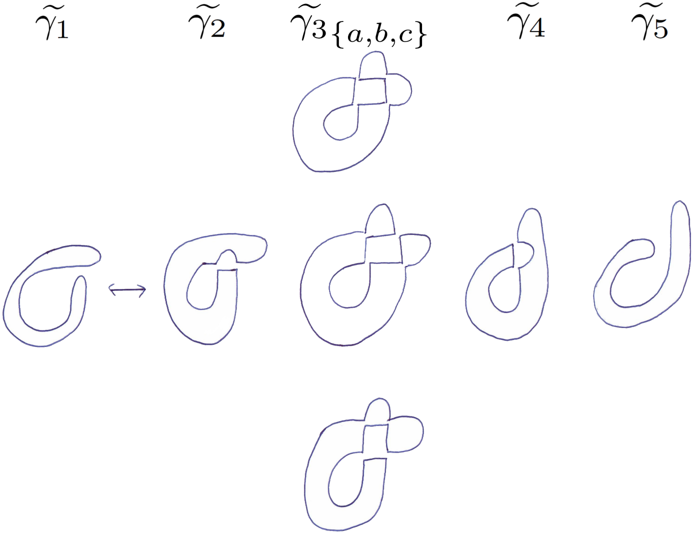
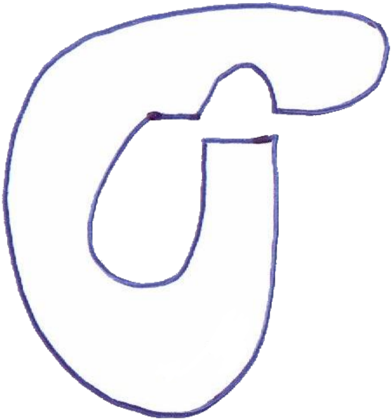


Graph Construction

$\widetilde{\gamma_2}$



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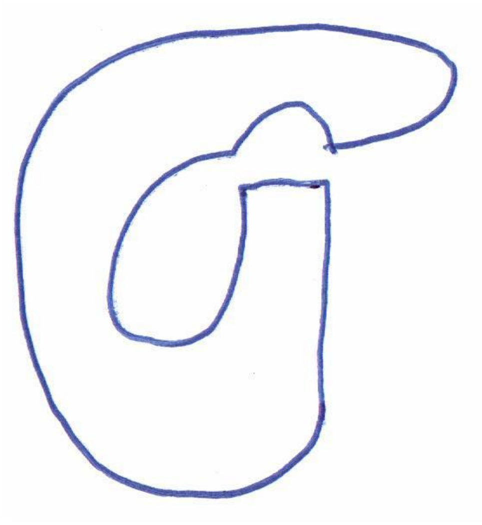

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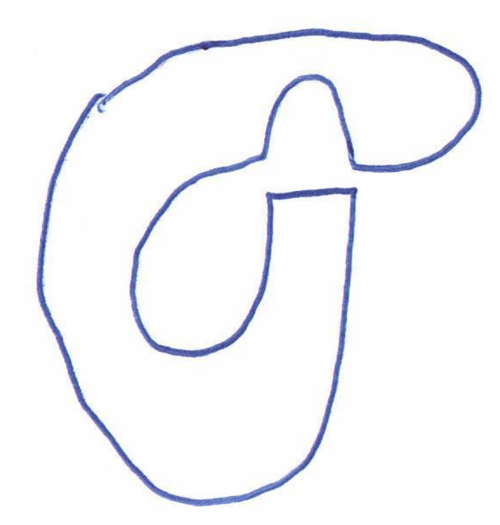

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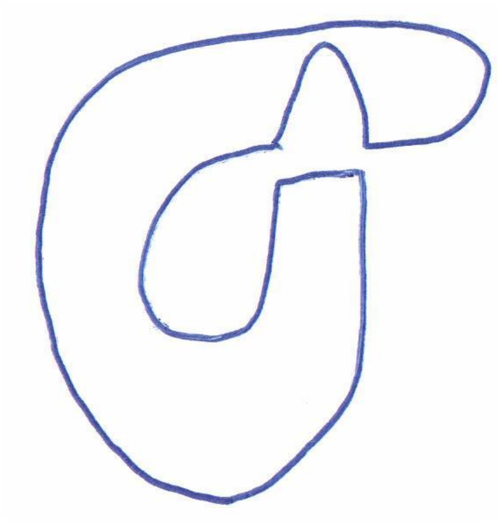

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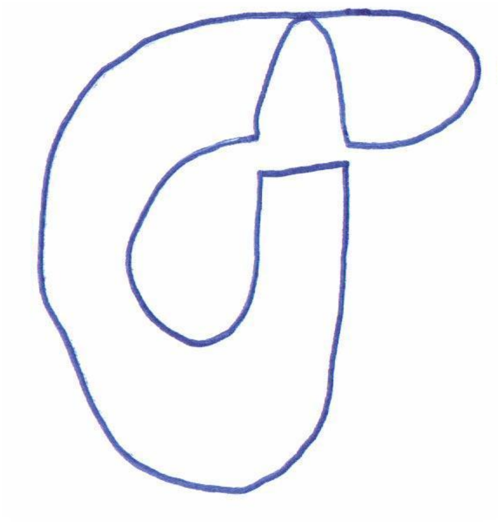

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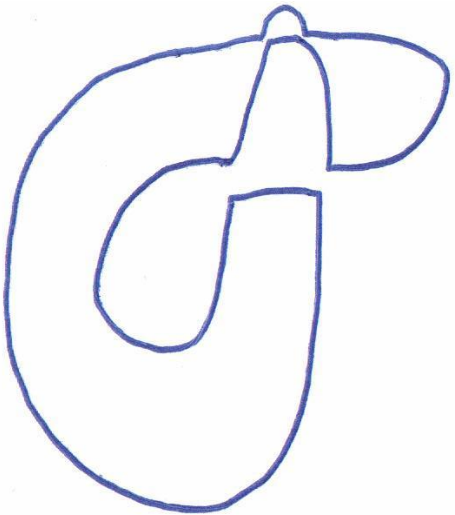

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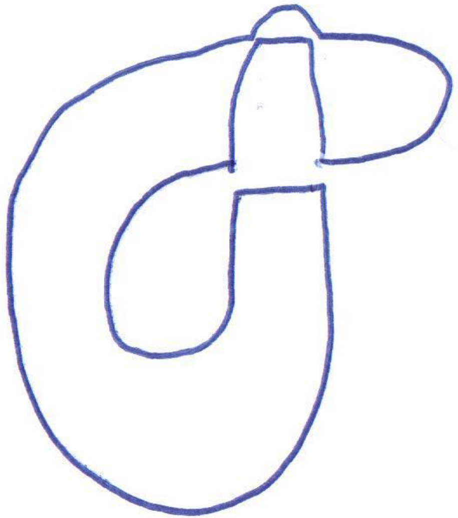

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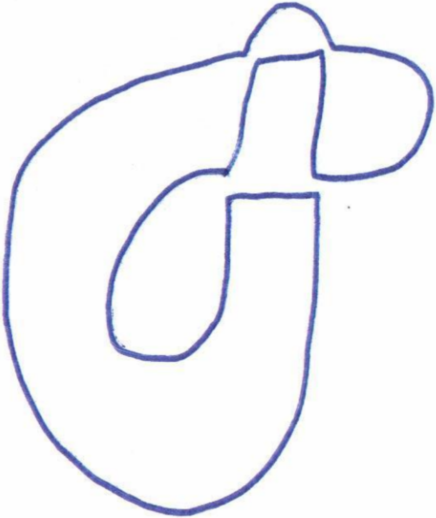

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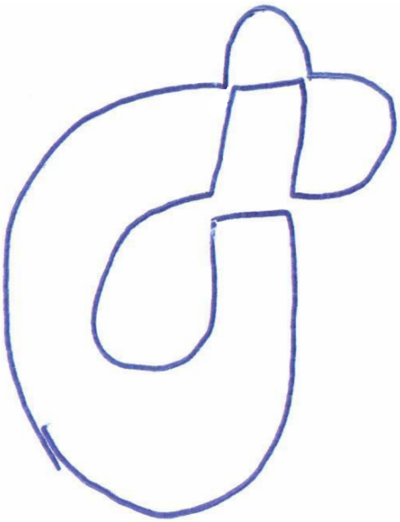
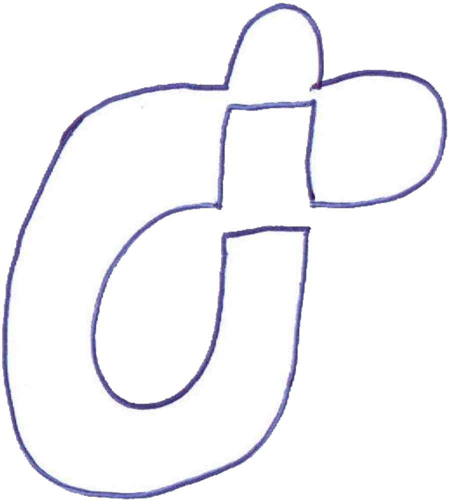


Graph Construction

$\widetilde{\gamma}_{3c}$



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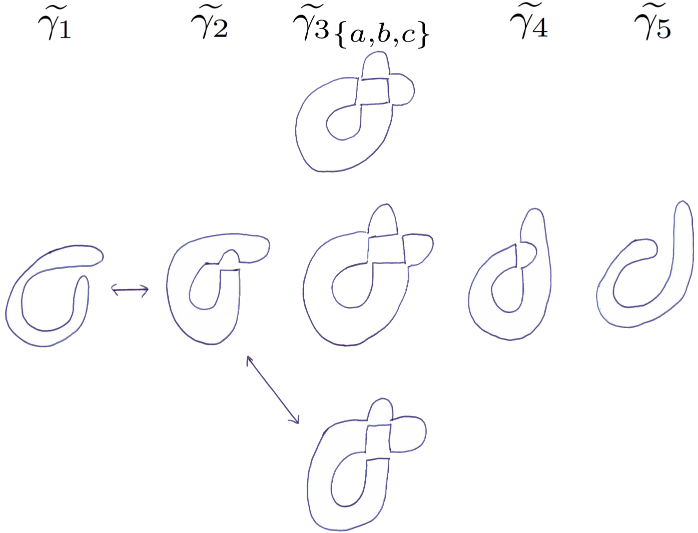

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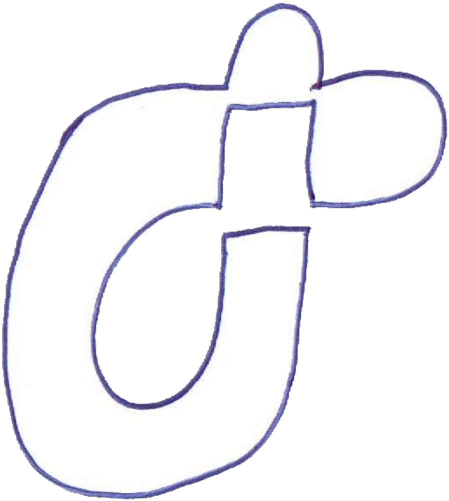

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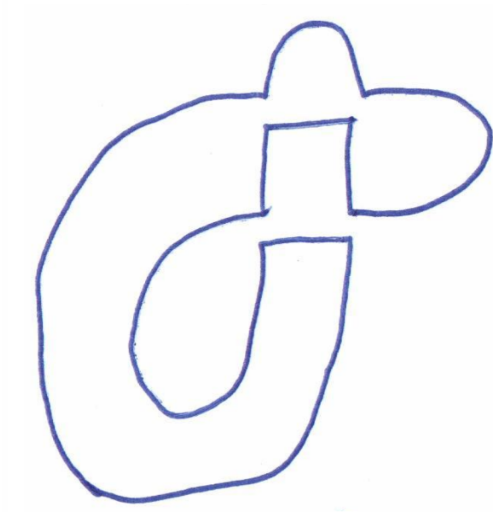

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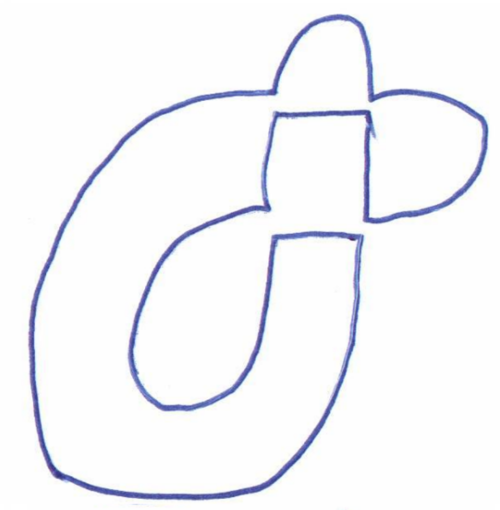

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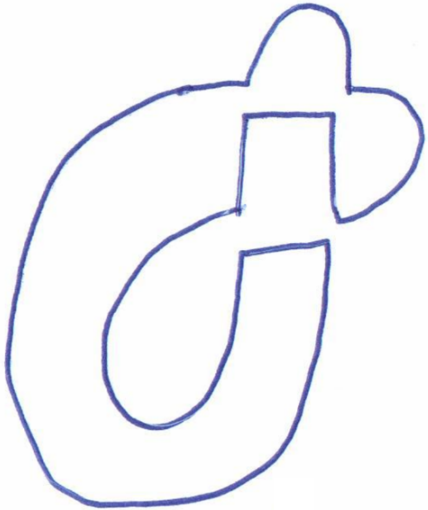

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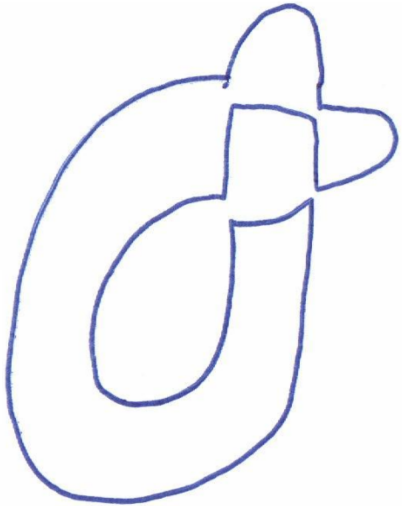

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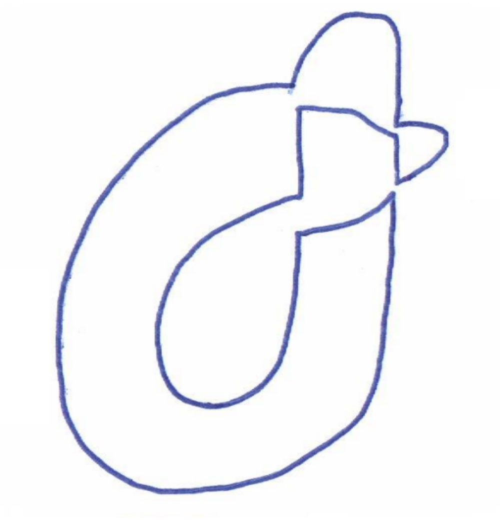

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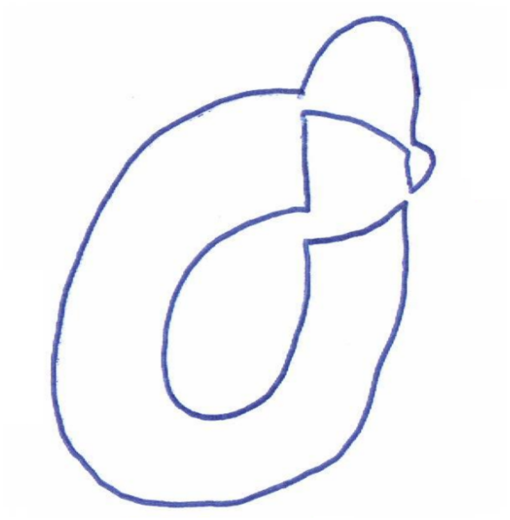

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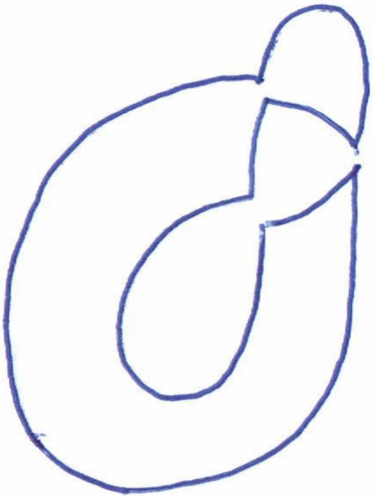

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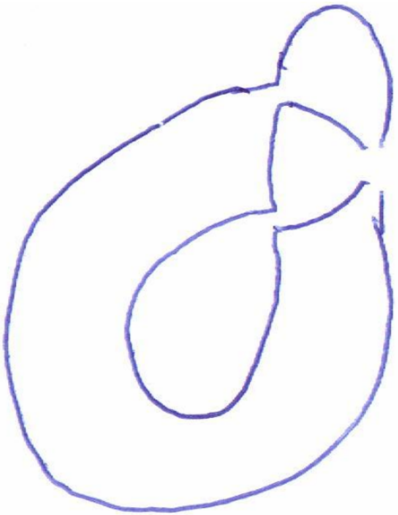

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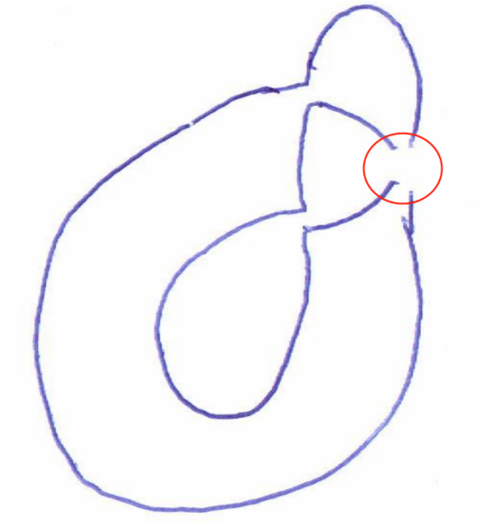
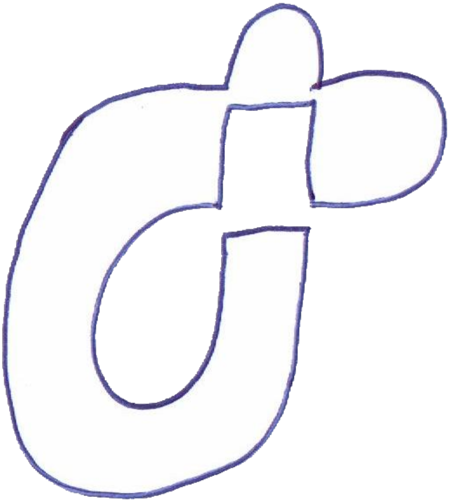


Graph Construction

$\widetilde{\gamma}_{3c}$



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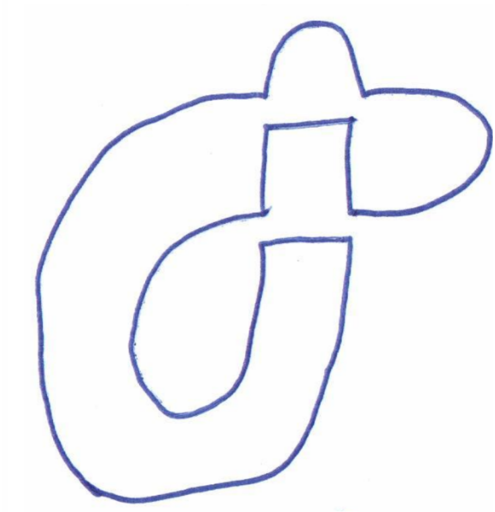

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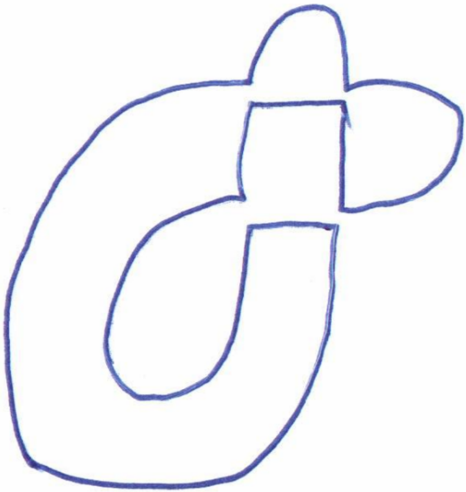

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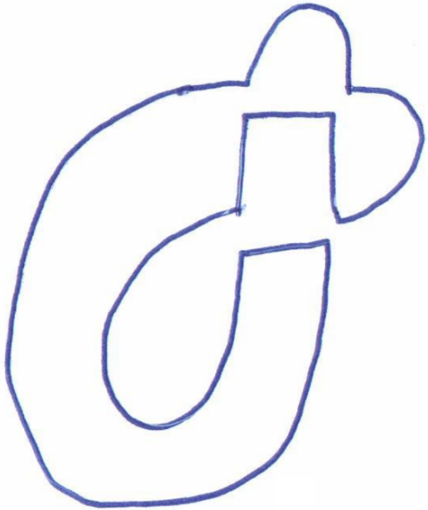

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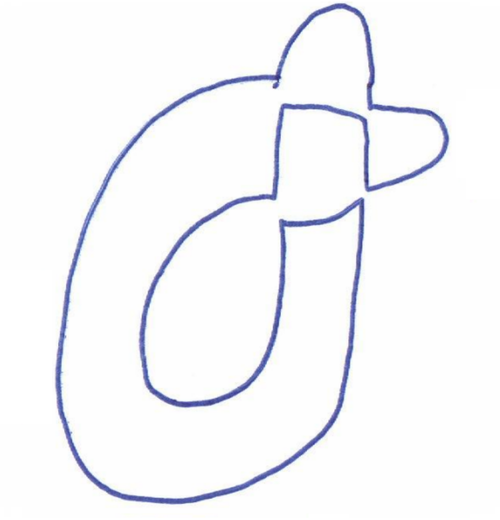

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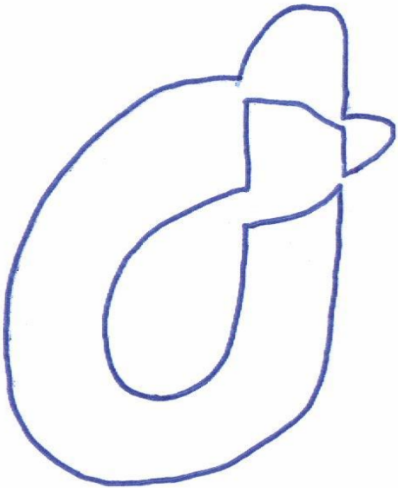

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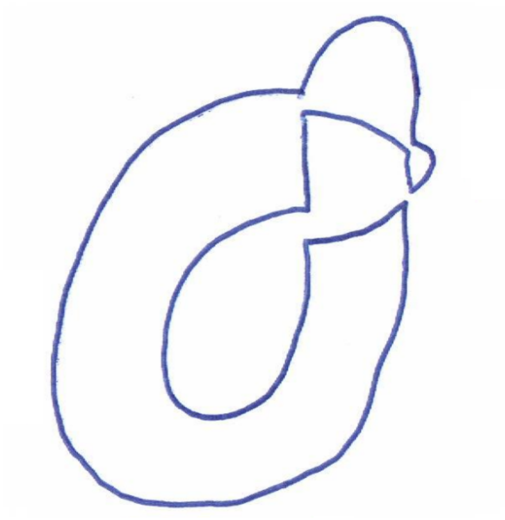

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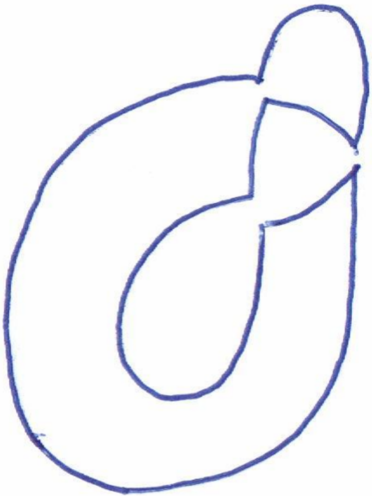

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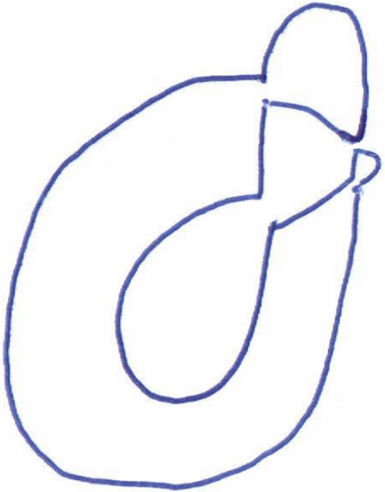

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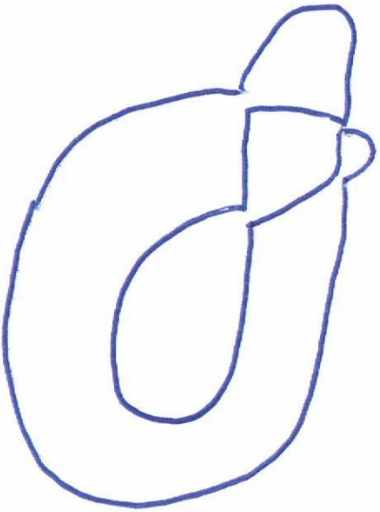

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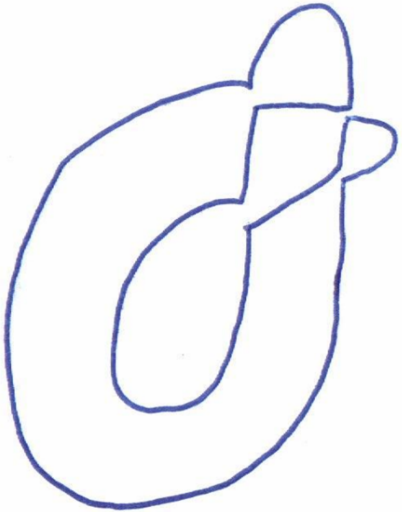

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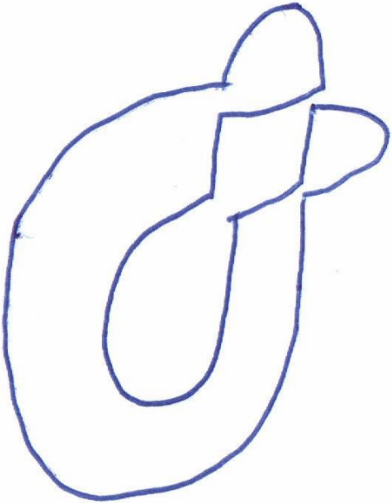
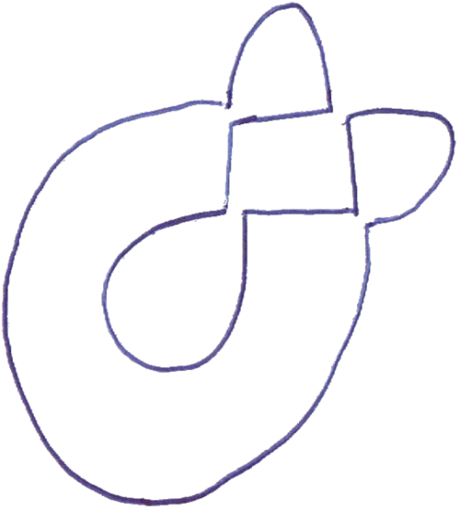


Graph Construction

$\widetilde{\gamma}_{3b}$



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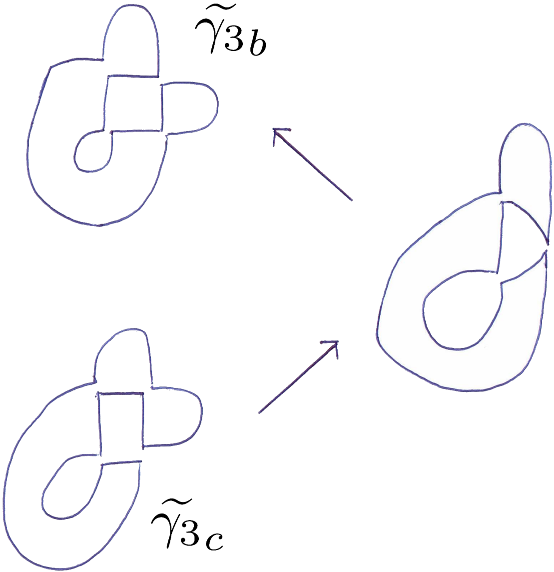

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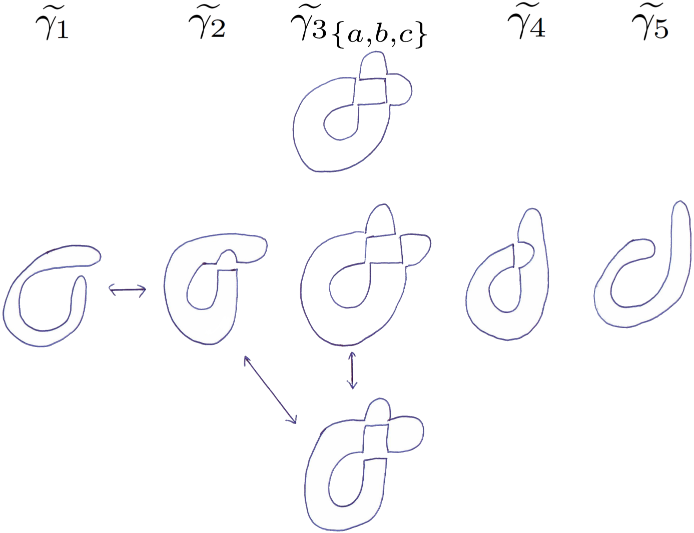
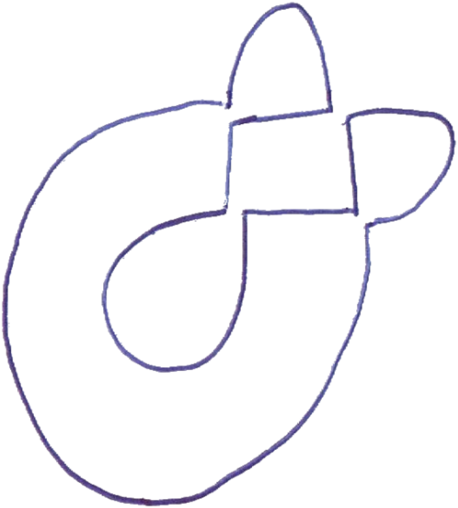


Graph Construction

$\widetilde{\gamma}_{3b}$



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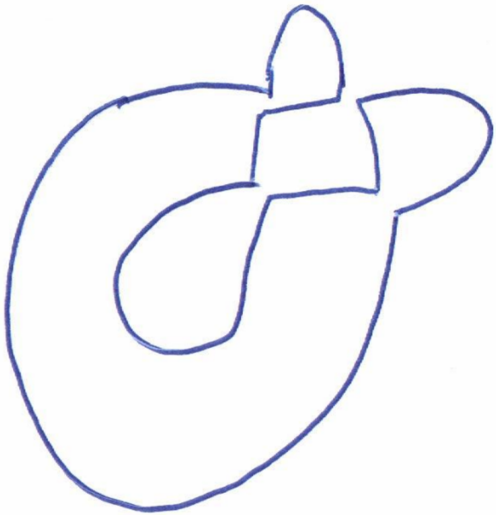

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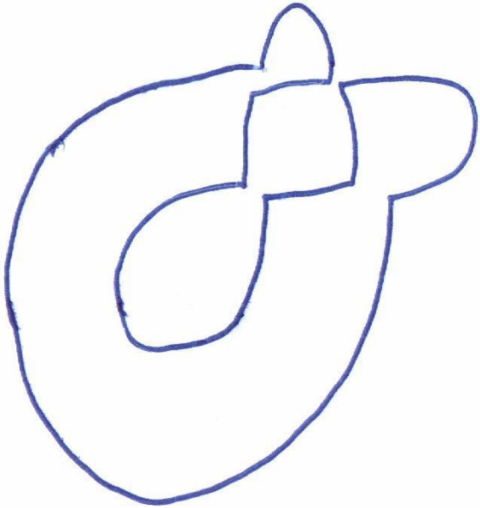

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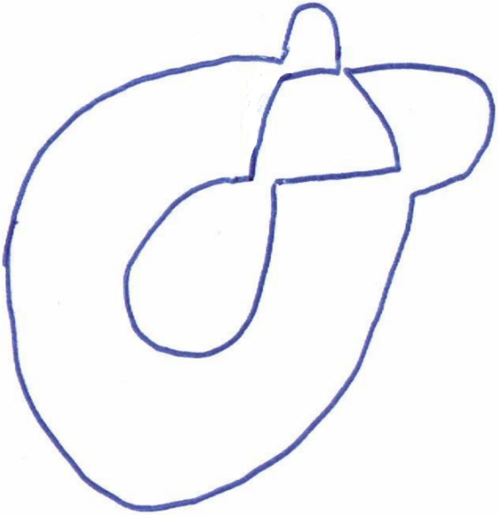

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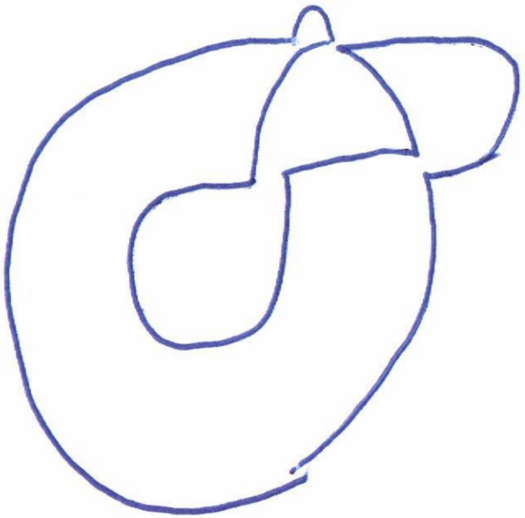

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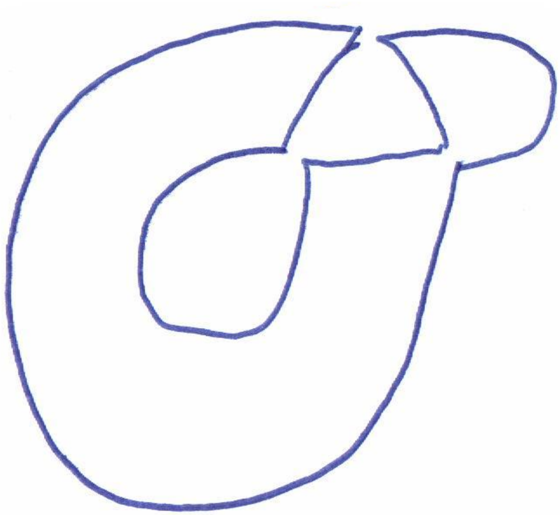

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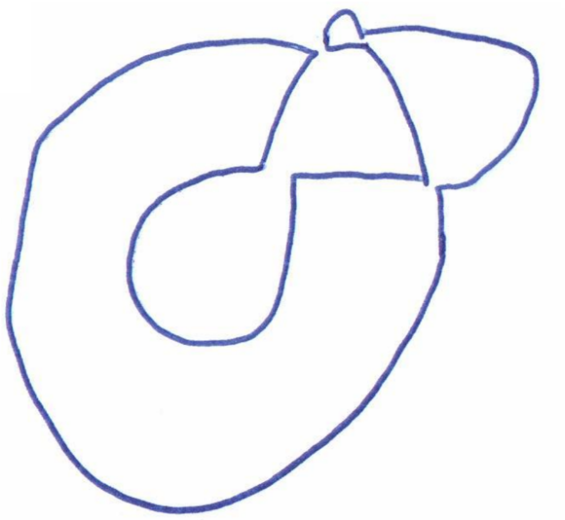

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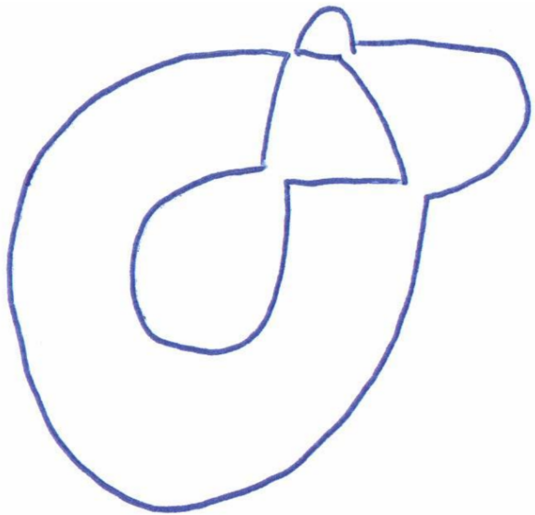

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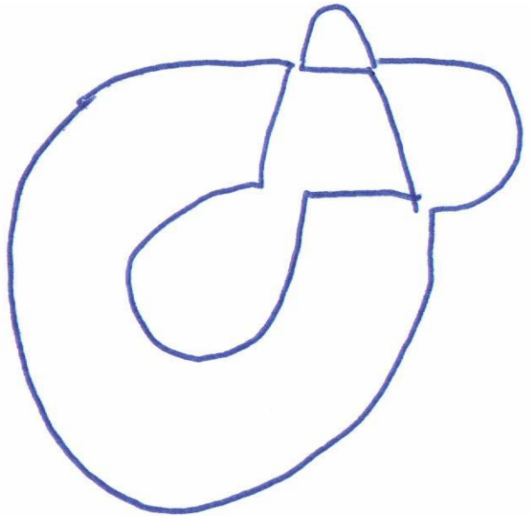

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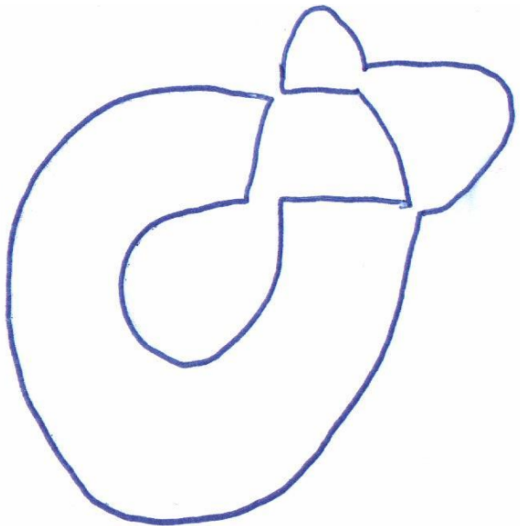
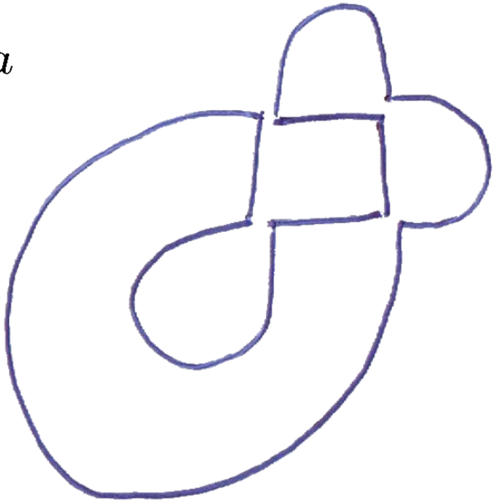


Graph Construction

$\widetilde{\gamma_3}_a$



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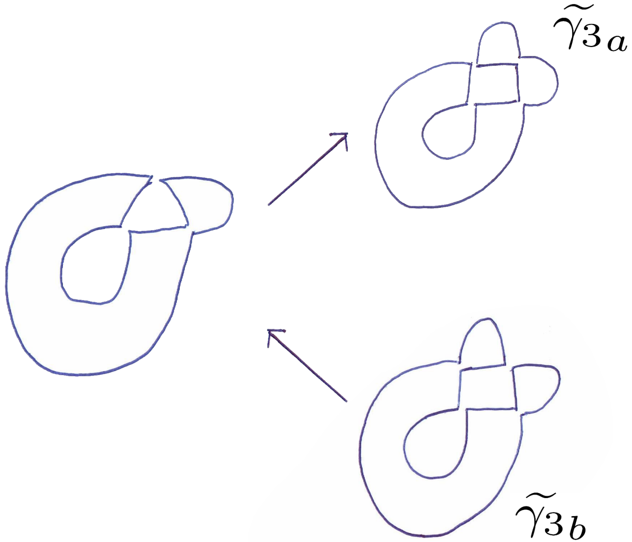

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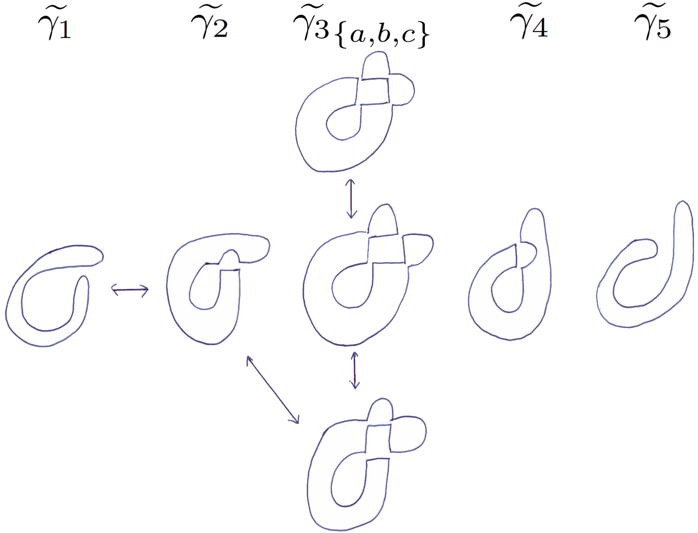
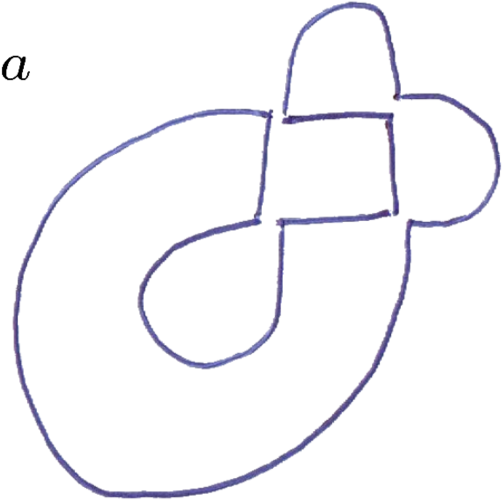


Graph Construction

$\widetilde{\gamma}_{3a}$



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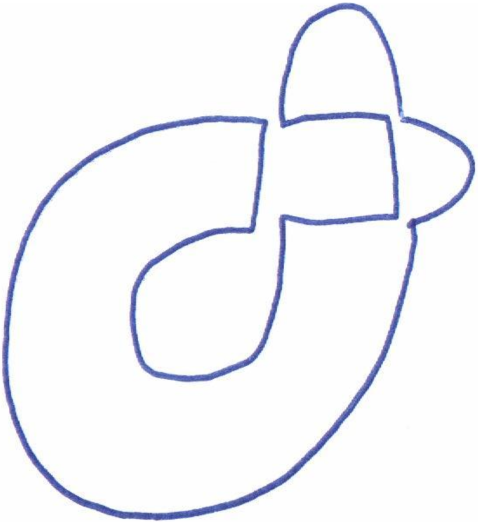

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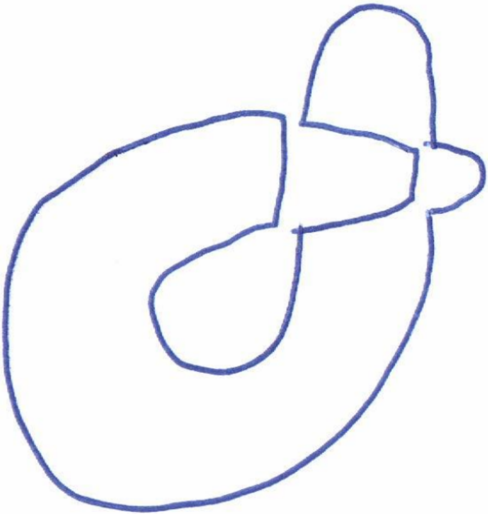

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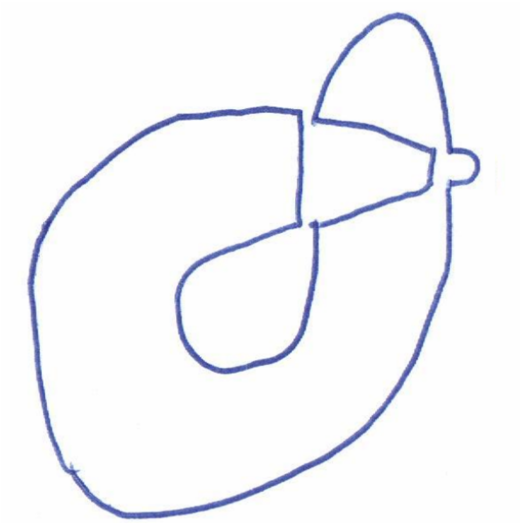

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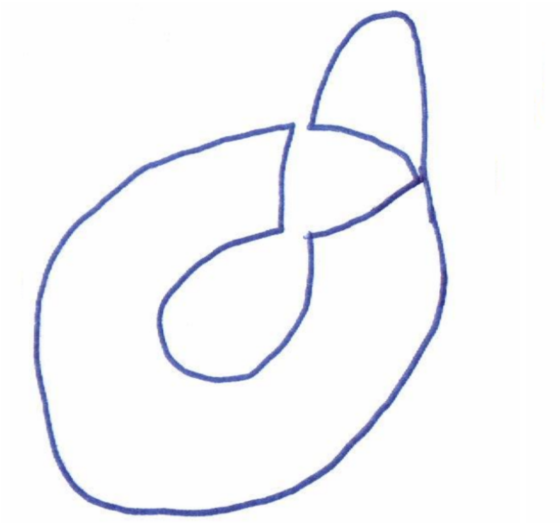
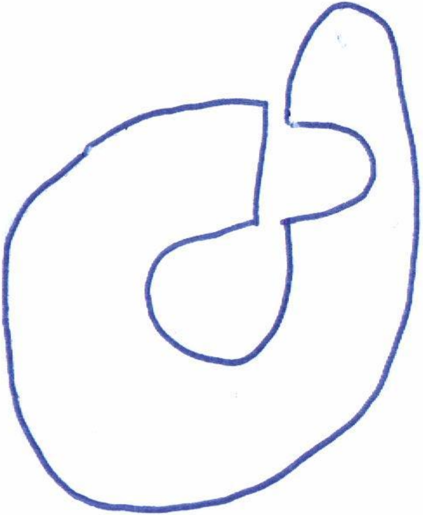


Graph Construction

$\widetilde{\gamma}_4$



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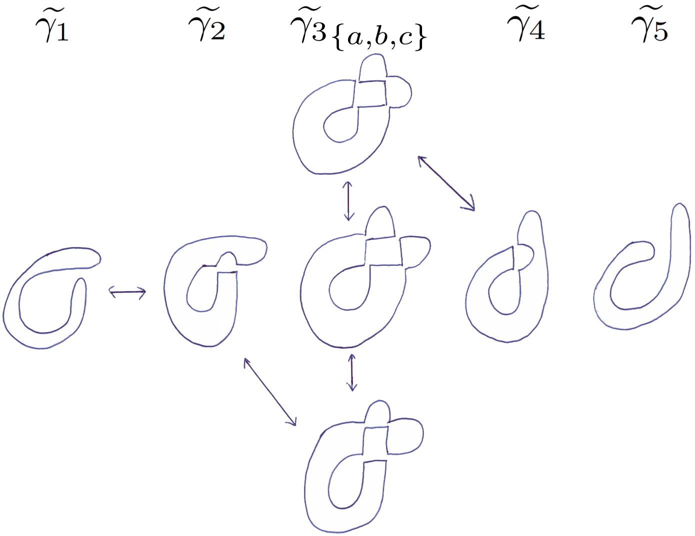

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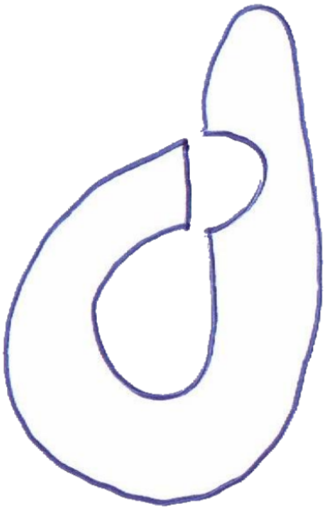

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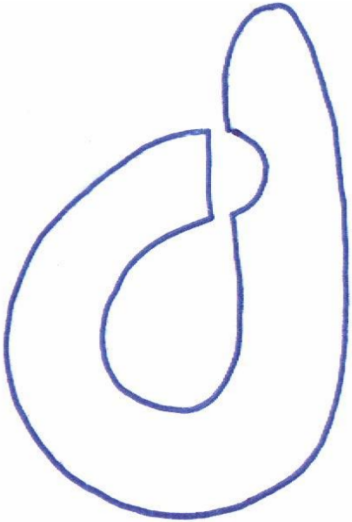

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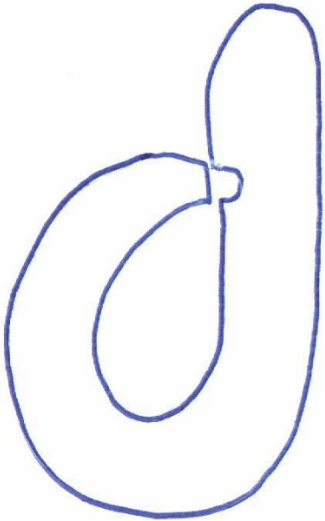
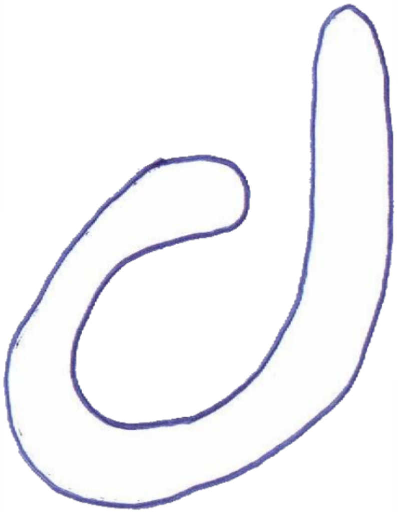
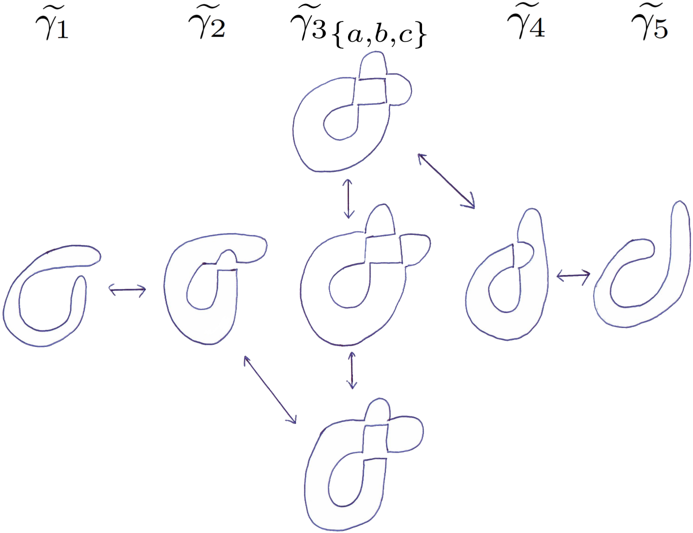


Graph Construction

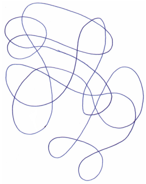
$\widetilde{\gamma}_5$



Graph Construction



Graph Structure



A Complicated Curve

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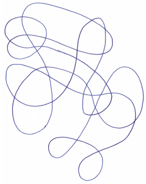

A Complicated Curve

Lemma (Key Lemma)

The first and last vertices have odd degree, and the rest have even degree.

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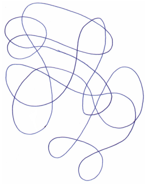

A Complicated Curve

Lemma (Key Lemma)

The first and last vertices have odd degree, and the rest have even degree.

Lemma (Handshaking Lemma)

For any finite indirected graph, the number of vertices with odd degree must be even.

Orientation

If we orient our initial and final curves, then what can be said about the corresponding orientations in the new homotopy?

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Theorem (R. Baer, 1920s)

Given non-contractible simple closed curves γ_1 and γ_2, if they are homotopic, then they are also isotopic.

Theorem 2

Theorem (with Y. Liokumovich)

Let γ be a closed curve on an orientable (M, g). If we can contract 2γ through curves of length at most L, then for any $\epsilon > 0$ we can contract γ through curves of length at most $L + \epsilon$.

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Some remarks:

- This can be seen as a quantitative version of the statement that $\pi_1(M)$ has no elements of order 2 for an orientable 2-dimensional manifold M.
- This theorem is not true if M is of dimension ≥ 4 (embedded projective space).

Theorem 3

Theorem (with R. Rotman)

Let (M, g) be a Riemannian disc with the property that ∂M can be contracted through curves of length no more than L. Then, for any $\epsilon > 0$ and for any $p \in \partial M$, there is a contraction of M through loops based at p of length no more than $L + 2D + \epsilon$. Here, D is the diameter of the disc.

Monotone Homotopy Lemma

Lemma

If we can contract ∂M through loops of length at most L, then we can contract it through a monotone sequence of curves of length no more that $L + \epsilon$.

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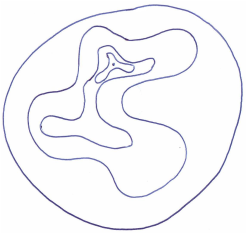

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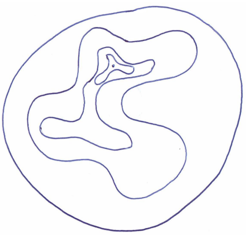

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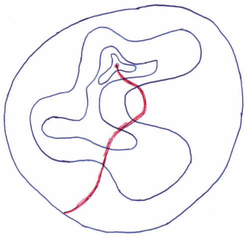
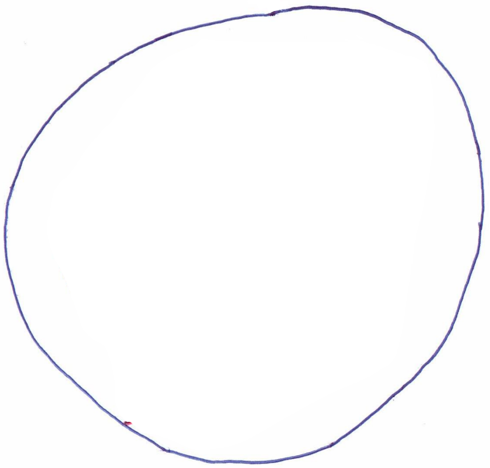


The red curve is a minimal geodesic.

Monotone Homotopy Lemma Continued



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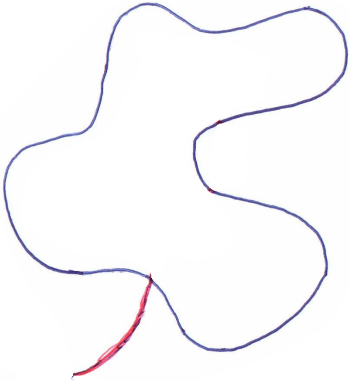

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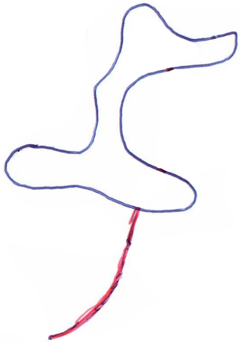

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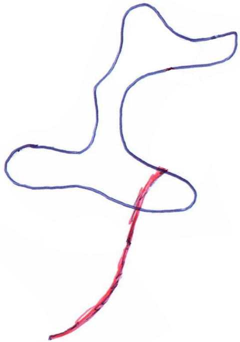

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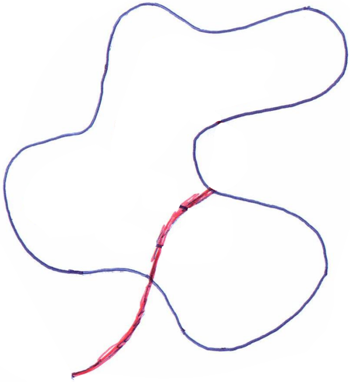

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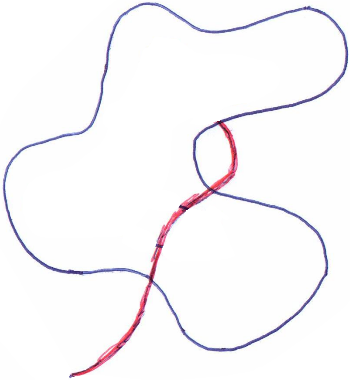

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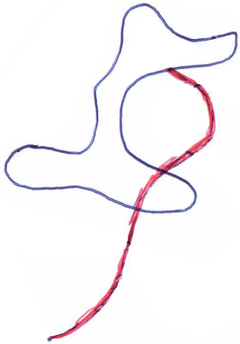

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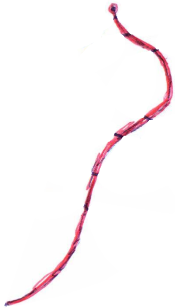
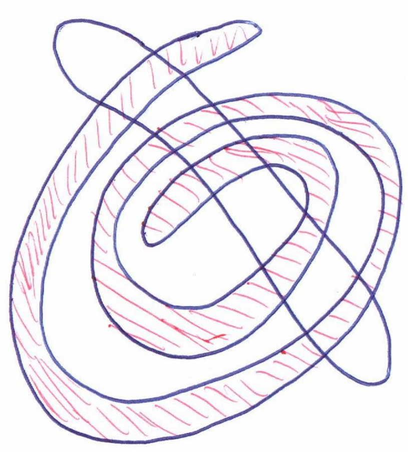


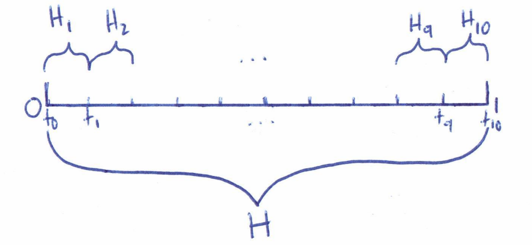
Monotone Homotopy Lemma Continued

Difficulties



How do we make this homotopy monotone?

Proof of Monotone Lemma



By local considerations, we can find a sequence

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

and monotone homotopies $H_1, \ldots, H_n$ such that the following properties are true:

- H_i is defined on $[t_{i-1}, t_i]$.

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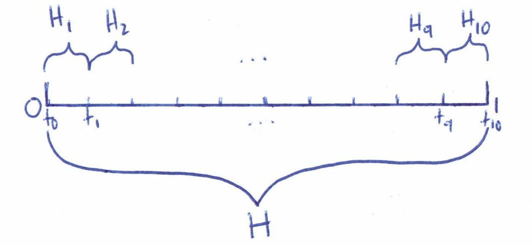

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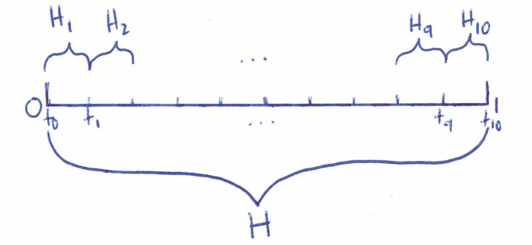

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- $H_0(0)$ is ∂M.
- $H_n(1)$ is a point.

Nested

We also want each H_i and H_{i+1} to be *nested*. This means that $H_i(t_i)$ lies in the closure of the interior of $H_{i+1}(t_i)$.

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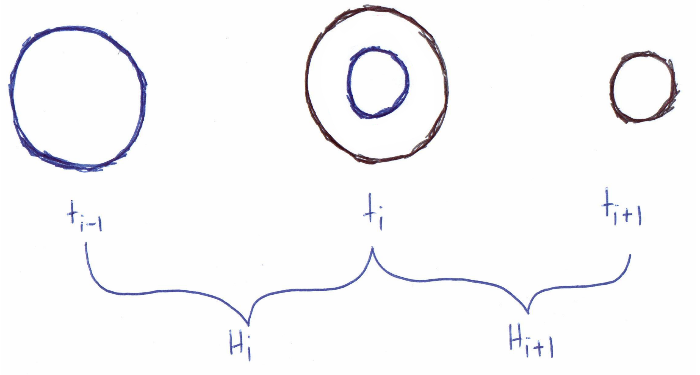

Proof of Monotone Lemma - Continued

We employ a technique that will allow us to glue H_i and H_{i+1} into a single monotone homotopy defined on $[t_{i-1}, t_{i+1}]$ which is still nested with respect to the rest of the H_j.

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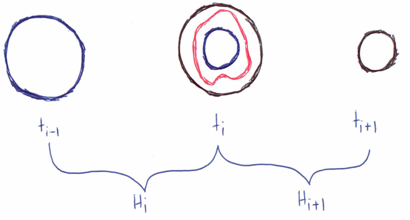

The red curve is the smallest non-contractible curve.

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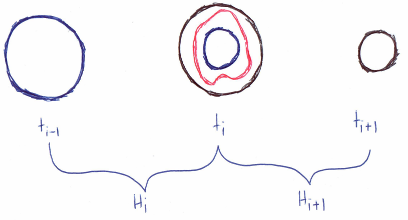

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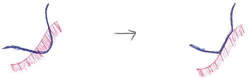

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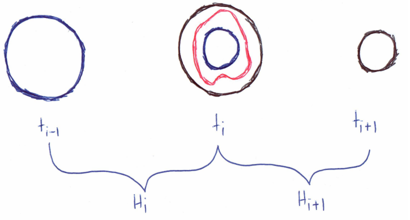

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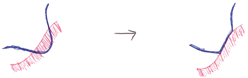

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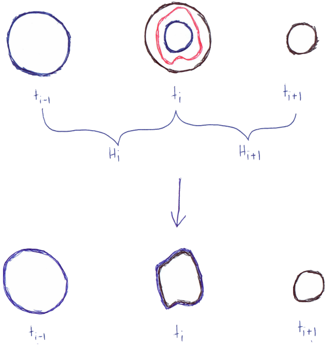

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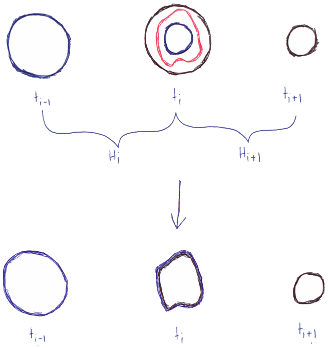
Proof of Monotone Lemma - Continued



The two homotopies can now be glued together.

Successively applying this method allows us to glue all of the monotone homotopies into a single monotone homotopy, concluding the proof.

Remarks

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- By modifying these methods we can prove Theorem 3 for contractible simple closed curves on any 2-dimensional Riemannian manifold. In this case the based loops have length at most $3L + 2D + \epsilon$.

Conclusions

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Thanks for your attention!