Operator-algebraic dynamical systems associated to self-similar groups

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Suppose X is a set. For $k \in \mathbb{N} \setminus \{0\}$, X^k is the set of words of length k in letters from X: we set $X^0 = \{\emptyset\}$ and $X^* = \bigcup_{k \geq 0} X^k$.

A *self-similar group* consists of a group G, a finite set X, and a faithful action of G on X^* such that $g \cdot \emptyset = \emptyset$ and, for all $x \in X$ and $g \in G$, there exist unique $y \in X$ and $h \in G$ such that

$$g \cdot (xw) = y(h \cdot w) \quad \text{for all } w \in X^*.$$

Taking $w = \emptyset$ shows that $y = g \cdot x$, and we write $g|_x$ for h, so the defining rule is

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all } w \in X^*.$$

If we view X^* as the vertices in a rooted tree T_X with edges joining each w to each xw, then the axioms imply that G embeds in the group $\operatorname{Aut} T_X$.

Recall that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all } w \in X^*.$$

Then

- $(gh) \cdot (xw) = g \cdot ((h \cdot x)(h|_x \cdot w)) = ((gh) \cdot x)((g|_{h \cdot x} h|_x) \cdot w)$ implies that $(gh)|_x = g|_{h \cdot x} h|_x$;
- $e = e|_x = (g^{-1}g)|_x = g^{-1}|_{g \cdot x} g|_x$ implies that $(g|_x)^{-1} = g^{-1}|_{g \cdot x}$;
- $g \cdot (xyw) = (g \cdot x)(g|_x \cdot (yw)) = (g \cdot x)(g|_x \cdot y)((g|_x)|_y \cdot w)$.

So we define $g|_w := (\cdots((g|_{w_1})|_{w_2})|_{w_3} \cdots)|_{w_k}$, and then

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all } v, w \in X^*.$$

Example 1. Fix an integer $N > 0$, and take $X = \{0, 1, \cdots, N-1\}$. Then $(\mathbb{Z}, X)$ is a self-similar group with the action of the generator γ for $\mathbb{Z}$ defined recursively by

$$\gamma \cdot (iw) = \begin{cases} (i+1)w & \text{if } i < N-1 \\ 0(\gamma \cdot w) & \text{if } i = N-1. \end{cases}$$

The map $v \in X^k \mapsto n_v := \sum_{j=1}^{k} v_j N^{j-1}$ identifies X^k with $\{0, 1, \cdots, N^k - 1\}$, and then the action on X^* is given by

$$m \cdot n_v = m + n_v \pmod{N^k} \quad \text{for } v \in X^k$$

with $m|_v$ characterised by $m + n_v = (m|_v)N^k + m \cdot n_v$. The self-similar action $(\mathbb{Z}, X)$ is an *odometer*.

Example 2. Let $X = \{x, y\}$. Define two automorphisms a, b of the tree T_X recursively by

$$a \cdot (xw) = y(b \cdot w) \qquad a \cdot (yw) = xw$$
$$b \cdot (xw) = x(a \cdot w) \qquad b \cdot (yw) = yw$$

The *Basilica group B* is the subgroup of Aut T_X generated by a and b.

Grigorchuk and Żuk (2002) showed that B has a countable presentation in terms of $\{a, b\}$, and that B does not belong to Day's class of "elementary amenable groups" (containing all finite groups and abelian groups, and closed under extensions, quotients, subgroups and direct limits). Bartholdi and Virág (2005) showed that B is amenable.

Nekrashevych (2005–09): Every SSG (G, X) has a Cuntz-Pimsner algebra $\mathcal{O}(G, X)$ generated by a unitary representation $u : G \to U\mathcal{O}(G, X)$ and a Cuntz family $\{s_x : x \in X\}$ satisfying

$$u_g s_x = s_{g \cdot x} u_{g|_x}.$$

It also has a Toeplitz algebra $\mathcal{T}(G, X)$ in which $\{s_x\}$ is a Toeplitz-Cuntz family (i.e. $s_x^* s_x = 1$ and $1 \geq \sum_x s_x s_x^*$).

Example 1. $\mathcal{O}(\mathbb{Z}, \{0, 1, \cdots N-1\})$ is the Exel crossed product $C(\mathbb{T}) \rtimes_{\alpha, L} \mathbb{N}$ associated to $z \mapsto z^N$.

Both $\mathcal{O}(G, X)$ and $\mathcal{T}(G, X)$ have natural dynamics α such that $\sigma_t(u_g) = u_g$ and $\sigma_t(s_x) = e^{it} s_x$. So what about the KMS states?

Lemma. (1) Extend s to words by $s_v = s_{v_1} \cdots s_{v_n}$. Then

$$\mathcal{T}(G, X) = \overline{\operatorname{span}}\{s_v u_g s_w^* : v, w \in X^*, g \in G\},$$

and $\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)} s_v u_g s_w^*$.

(2) There are no KMS_β states for $\beta < \ln |X|$.

(3) For $\beta \geq \ln |X|$ a state ϕ is KMS_β if and only if $\phi|_{C^*(u)}$ is a trace and

$$\phi(s_v u_g s_w^*) = \delta_{v,w} e^{-\beta|v|} \phi(u_g).$$

The strategy for finding KMS states (Exel-Laca-Neshveyev) is to look first at β larger than the critical value (here, $\beta > \ln |X|$), and to look for states on the Toeplitz algebra.

Theorem (LRRW) For a normalised trace τ on $C^*(G)$ and $\beta > \ln|X|$, there is a KMS_β state $\psi = \psi_{\beta,\tau}$ on $(\mathcal{T}(G,X),\sigma)$ with

$$\psi(s_v u_g s_w^*) = \delta_{v,w}(1 - e^{-\beta}|X|)e^{-\beta|v|}\sum_{k=0}^{\infty} e^{-\beta k} \sum_{y\in X^k,\, g\cdot y=y} \tau(\delta_{g|_y}).$$

All KMS_β states have this form.

Recall that $\mathcal{T}(G,X)$ is the Toeplitz algebra of a Hilbert bimodule M. The coefficient algebra is $C^*(G)$. As a right module, $M = \bigoplus_{x\in X} C^*(G)$. Say $\{e_x : x \in X\}$ is the usual orthonormal basis. The left action of $C^*(G)$ is the integrated form of a unitary representation $T : G \to U\mathcal{L}(M)$ such that $T_g e_x = e_{g\cdot x}\cdot \delta_{g|_x}$.

To get a concrete representation, we form the Fock bimodule $F(M) = \bigoplus_{j\in\mathbb{N}} M^{\otimes j}$, we take the GNS representation π_τ, and form the induced representation $F(M)\text{-Ind}\,\pi_\tau$ acting in $F(M)\otimes_{C^*(G)} H_\tau$.

$$\psi(s_v u_g s_w^*) = \delta_{v,w}(1 - e^{-\beta}|X|)e^{-\beta|v|} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{y \in X^k,\, g\cdot y = y} \tau(\delta_{g|_y}).$$

Suppose $\tau = \tau_e$ is the usual trace such that $\tau_e(\delta_g) = 0$ for $g \neq e$. Set $F_g^k := \{y \in X^k : g \cdot y = y, g|_y = e\}$. Then

$$\psi_{\beta,\tau_e}(s_v u_g s_w^*) = \delta_{v,w}(1 - e^{-\beta}|X|)e^{-\beta|v|} \sum_{k=0}^{\infty} e^{-\beta k}|F_g^k|.$$

Lemma The sequence $\{|X|^{-k}|F_g^k|\}$ is increasing and converges to c_g, say, with $c_g \in [0, 1)$.

This allows us to compute the limit of $\psi_{\beta,\tau_e}(s_v u_g s_w^*)$ as $\beta \to \ln|X|$ (which is equivalent to $e^{-\beta k} \to |X|^{-k}$).

Theorem (LRRS). There is a $\mathrm{KMS}_{\ln|X|}$ state ψ on $(\mathcal{O}(G, X), \sigma)$ such that

$$\phi(s_v u_g s_w^*) = \delta_{v,w} |X|^{-|v|} c_g.$$

If (G, X) has the property that $\{g|_v : v \in X^*\}$ is finite for every g, then this is the only $\mathrm{KMS}_{\ln|X|}$ state.

Since the group algebra $C^*(G)$ sits inside $\mathcal{O}(G, X)$, we have:

Corollary. There is a trace τ on $C^*(G)$ such that $\tau(\delta_g) = c_g$ for $g \neq e$.

When (G, X) has the finite-state property above, this trace has previously appeared in work of Planchat. But our formula for the values of τ on generators is different, and turns out to be easier to compute.

We want to compute the values $\phi(u_g)$ of our trace. We know that ϕ is the restriction of a $\mathrm{KMS}_{\ln|X|}$ state on $\mathcal{O}(G, X)$. So for each k

$$\phi(u_g) = \phi\Big(u_g \sum_{w \in X^k} s_w s_w^*\Big) = \sum_{w \in X^k} \phi(s_{g \cdot w} u_{g|_w} s_w^*)$$
$$= \sum_{w \in X^k,\ g \cdot w = w} |X|^{-k} \phi(u_{g|_w}).$$

So it suffices to compute $\phi(u_{g|_w})$ for big words w. Many interesting SSGs are *contracting*: there is a finite set (the *nucleus*) $\mathcal{N}$ such that, for all $g \in G$, restrictions of g on sufficiently long words are all in $\mathcal{N}$. For example, the odometers have nucleus $\{e, \gamma, \gamma^{-1}\}$. The basilica group (generated as automorphisms of $\mathcal{T}_X$ for $X = \{x, y\}$ by recursively defined automorphisms a, b), has nucleus

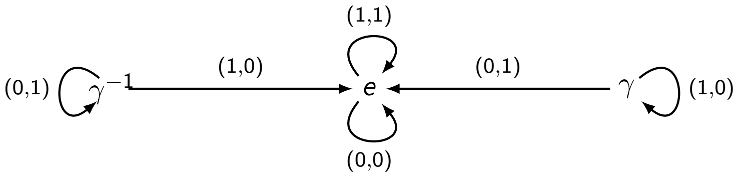
$$\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}.$$

So we want to compute $\phi(u_g)$ for g in the nucleus.

Suppose $S \subset G$ such that $g \in S \implies g|_v \in S$ for all v. The *Moore diagram* of S is the labelled directed graph with vertex set S, and for each $g \in S$ and $x \in X$ an edge from g to $g|_x$ labelled:

$$g \xrightarrow{(x, g\cdot x)} g|_x$$

Example. For the odometer (G, X) with $N = 2$ and $S = \{e, \gamma, \gamma^{-1}\}$, we have:

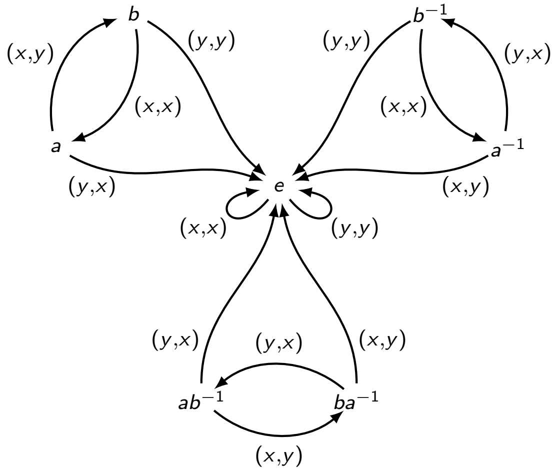


Recall that we are interested in
$F_g^k := \{y \in X^k : g \cdot y = y, g|_y = e\}$ and $c_g := \lim_{k \to \infty} |X|^{-k} |F_g^k|$.

A pair g, y such that $g \cdot y = y$ gives a path in the Moore diagram of G from g to $g|_y$ with labels (y_1, y_1), $(y_2, y_2), \cdots (y_n, y_n)$. We call a path where all the labels have the form (x, x) a *stationary path*.

A pair g, y such that $g \cdot y = y$ and $g|_y = e$ gives a stationary path from g to e. So given g, we want to count the stationary paths in the Moore diagram from g to e.

The Moore diagram for the nucleus of the basilica group:



For $k \geq 1$, half the paths of length k from b to e are stationary.
So $\phi(u_b) = \frac{1}{2}$, $\phi(u_e) = 1$, and $\phi(u_g) = 0$ for $g = a$ or ab^{-1}.