KMS states on C^*-algebras associated to k-graphs

BIRS workshop "Operator algebras and dynamical systems from number theory"

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Higher-rank Cuntz-Krieger algebras

- Robertson and Steger studied C^*-algbras arising from $\mathbb{Z}^k$ actions on $\tilde{A}_k$-buildings.
- Data consists of k commuting binary matrices such that $A_iA_jA_l$ is binary valued for distinct i, j, l.
- Resulting C^*-algebra generated by copies of the Cuntz-Krieger algebras $\mathcal{O}_{A_i}$ subject to commutation relations encoded by the products A_iA_j.

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- Resulting C^*-algebra generated by copies of the Cuntz-Krieger algebras $\mathcal{O}_{A_i}$ subject to commutation relations encoded by the products $A_i A_j$.
- Kumjian and Pask recognised that such a family of matrices encodes a sort of higher-rank graph:

Definition (KP). A *k-graph* is a countable category Λ with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: whenever $d(\lambda) = m + n$ there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

Notation

- Λ^n denotes $d^{-1}(n)$.
- Factorisation property gives $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$.
- The domain and codomain maps determine maps $s, r : \Lambda \to \Lambda^0$; and then $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all λ.
- Write, for example, $v\Lambda^n$ for $r^{-1}(v) \cap \Lambda^n$.
- $\mathrm{MCE}(\mu, \nu) = \{\lambda : d(\lambda) = d(\mu) \vee d(\nu) \text{ and } \lambda = \mu\mu' = \nu\nu'\}$.
- The coordinate graphs E_i are $E_i = (\Lambda^0, \Lambda^{e_i}, r, s)$; this E_i has adjacency matrix A_i.

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For today:

- Λ is “finite” in the sense that each Λ^n is finite; and
- Λ is strongly connected: each $v\Lambda w \neq \emptyset$.

Connectivity

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- First consequence: suppose that $\Lambda^{e_i} \neq \emptyset$; say $\alpha \in \Lambda^{e_i}$.

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- For $v \in \Lambda^0$, fix $\mu \in v\Lambda r(\alpha)$.
- Factorisation property says $\mu\alpha = \alpha'\mu'$ for some $\alpha' \in v\Lambda^{e_i}$.
- So every $v\Lambda^{e_i} \neq \emptyset$; since Λ^0 is finite, this means each E_i contains a cycle.
- Hence $\rho(A_i) \geq 1$.

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- So every $v\Lambda^{e_i} \neq \emptyset$; since Λ^0 is finite, this means each E_i contains a cycle.
- Hence $\rho(A_i) \geq 1$.
- If $\Lambda^{e_i} = \emptyset$, we can regard Λ as a $(k-1)$-graph; so we can assume wlog that every $\rho(A_i) \geq 1$.

Toeplitz-Cuntz-Krieger families

Definition (KP). Let Λ be a row-finite k-graph with no sources. Then $\mathcal{T}C^*(\Lambda)$ is universal for $\{T_\lambda : \lambda \in \Lambda\}$ such that:

(TCK1) $\{T_v : v \in E^0\}$ is a set of mutually orthogonal projections;

(TCK2) $T_\mu T_\nu = T_{\mu\nu}$ whenever $s(\mu) = r(\nu)$.

(TCK3) $T_\mu^* T_\mu = T_{s(\mu)}$ for all μ, and

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$C^*(\Lambda)$ is the quotient by the ideal generated by

$$\{T_v - \sum_{\mu \in v\Lambda^n} T_\mu T_\mu^* : v \in \Lambda^0, n \in \mathbb{N}^k\}.$$

Spanning elements

- use $\{t_\lambda : \lambda \in \Lambda\}$ for the universal family.
- For $\mu, \nu \in \Lambda$, have $t_\mu^* t_\nu = \sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$, so
$$\mathcal{T}C^*(\Lambda) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : s(\mu) = s(\nu)\}.$$
- Universal property gives $\gamma : \mathbb{T}^k \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ s.t. $\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda$,
- so $r \in [0, \infty)^k$ gives $\alpha^r : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ via $\alpha_t^r = \gamma_{e^{itr}}$.

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- so $r \in [0, \infty)^k$ gives $\alpha^r : \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(\Lambda)$ via $\alpha_t^r = \gamma_{e^{itr}}$.
- $\alpha_t^r(t_\mu t_\nu^*) = e^{itr\cdot(d(\mu)-d(\nu))} t_\mu t_\nu^*$, so the $t_\mu t_\nu^*$ are analytic elements.
- both γ and α descend to $C^*(\Lambda)$.

KMS states

- Recall: given $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ and $\beta \in \mathbb{R}$, a state ϕ of A is KMS_β for (A, α) if

$$\phi(ab) = \phi(b\alpha_{i\beta}(a))$$

  whenever $t \mapsto \alpha_t(a), \alpha_t(b)$ have analytic extensions.
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- It always suffices to check this KMS condition on your favourite set of analytic elements with dense linear span.
- Questions:
  - what are the KMS states for $(\mathcal{T}C^*(\Lambda), \alpha^r)$?
  - Which ones factor through $C^*(\Lambda)$?

First observation

- Suppose that ϕ is a KMS_β state of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.
- Universal property of $\mathcal{T}C^*(E_j)$ gives inclusion $\iota : \mathcal{T}C^*(E_j) \to \mathcal{T}C^*(\Lambda)$.
- $\alpha^r(t_f) = e^{itr_j} t_f$ for $f \in \Lambda^{e_j}$.

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- $\alpha^r(t_f) = e^{itr_j} t_f$ for $f \in \Lambda^{e_j}$.
- Put $m^\phi = (\phi(t_v))_{v \in \Lambda^0}$.
- Astrid showed us that then

$$A_i m^\phi \le e^{\beta r_i} m^\phi \quad \text{for all } i \le k.$$

- If ϕ factors through $C^*(\Lambda)$, we have equality.

Perron-Frobenius for commuting matrices

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Proposition (Kumjian-Pask, aHLRS)

(1) If $y \in [0,\infty)^{\Lambda^0} \setminus \{0\}$ *and* $\lambda_1, \ldots, \lambda_k$ *satisfy* $A_i y \leq \lambda_i y$ *for all* i*, then* $y_v > 0$ *for all* v *and* $\lambda_i \geq \rho(A_i)$ *for all* i*; and*

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(2) There is a unique $x^\Lambda \in [0,\infty)^{\Lambda^0}$ with $\|x^\Lambda\|_1 = 1$ which is a common eigenvector of the A_i; and then $A^n := \prod A_n^{n_i}$ satisfies $A^n x^\Lambda = \rho(A^n) x^\Lambda = \prod_{i=1}^k \rho(A_i)^{n_i} x^\Lambda$ for all $n \in \mathbb{N}^k$.

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Corollary

If ϕ *is* KMS_β *for* α^r*, then* $\beta r_i \geq \ln \rho(A_i)$ *for all* i*. If* ϕ *factors through* $C^*(\Lambda)$*, then each* $\beta r_i = \ln \rho(A_i)$*, and* $m^\phi = x^\Lambda$*.*

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Second observation

- If ϕ is KMS$_\beta$, then

$$\phi(t_\mu t_\nu^*) = e^{-\beta r\cdot d(\mu)}\phi(t_\nu^* t_\mu) = e^{-\beta r\cdot(d(\mu)-d(\nu))}\phi(t_\mu t_\nu^*).$$

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- Not so clear what happens if $r \cdot d(\mu) = r \cdot d(\nu)$ but $d(\mu) \neq d(\nu)$.

Proposition (aHLRS)

Suppose that $\beta r_i > \ln \rho(A_i)$ for all i. Then ϕ is KMS_β for $(\mathcal{T}C^*(\Lambda), \alpha^r)$ if and only if

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r\cdot d(\mu)} m^\phi_{s(\mu)} \text{ for all } \mu, \nu. \qquad (*)$$

Proof sketch.

"if" is a calculation. For "only if," need $\phi(t_\mu t_\nu^*) = 0$ if $d(\mu) \neq d(\nu)$ but $r \cdot d(\mu) \neq r \cdot d(\nu)$.

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Let $n = (d(\mu) \vee d(\nu)) - d(\mu) > 0$. Combinatorics/induction gives $\phi(t_\mu t_\nu^*) = \sum_{\lambda \in s(\mu)\Lambda^{jn}, \mathrm{MCE}(\mu\lambda, \nu\lambda) \neq \emptyset} \phi(t_{\mu\lambda} t_{\mu\lambda}^*)$ for all j.

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So

$$\begin{aligned} |\phi(t_\mu t_\nu^*)| &\leq \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\mu\lambda}^*) \\ &= e^{-\beta r \cdot (jn + d(\mu))} \sum_w \sum_{\lambda \in s(\mu)\Lambda^{jn} w} \phi(t_w) \end{aligned}$$

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□

KMS states on $\mathcal{T}C^*(\Lambda)$

Theorem (aHLRS)

Suppose that $\beta r_i > \ln \rho(A_i)$ for all i. Then

1. *For $v \in \Lambda^0$, $\sum_{\mu \in \Lambda v} e^{-\beta r \cdot d(\mu)}$ converges to some $y_v > 1$. For $\epsilon \in [0, \infty)^{\Lambda^0}$, $m^\epsilon := \prod_{i=1}^k (1 - e^{-\beta r_i} A_i)^{-1} \epsilon$ satisfies $A_i m^\epsilon \leq e^{\beta r_i} m$ for all i, and $\|m^\epsilon\|_1 = 1$ iff $\epsilon \cdot y = 1$.*

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2. *If $\epsilon \cdot y = 1$, there is a KMS$_\beta$ state ϕ_ϵ such that $\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} m^\epsilon_{s(\mu)}$.*

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2. *If $\epsilon \cdot y = 1$, there is a KMS$_\beta$ state ϕ_ϵ such that $\phi_\epsilon(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} m^\epsilon_{s(\mu)}$.*
3. *$\epsilon \mapsto \phi_\epsilon$ is an affine isomorphism of $\{\epsilon : \epsilon \cdot y = 1\}$ onto the KMS$_\beta$ simplex of $(\mathcal{T}C^*(\Lambda), \alpha^r)$.*

Proof sketch

(1) The terms in $\sum_{\mu \in \Lambda v} e^{-\beta r \cdot d(\mu)}$ are terms in the series expansion of $\prod_{i=1}^{k}(1 - e^{-\beta r_i} A_i)^{-1}$, so the sum converges.

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$$
\begin{aligned}
e^{-\beta r_i} A_i (1 - e^{-\beta r_i} A_i)^{-1} &= \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n+1} \\
&< \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n} = (1 - e^{-\beta r_i} A_i)^{-1}.
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$$e^{-\beta r_i}A_i(1-e^{-\beta r_i}A_i)^{-1} = \sum_{n=0}^{\infty}(e^{-\beta r_i}A_i)^{n+1} < \sum_{n=0}^{\infty}(e^{-\beta r_i}A_i)^{n} = (1-e^{-\beta r_i}A_i)^{-1}.$$

(2) Define $T_\lambda \in \mathcal{B}(\ell^2(\Lambda))$ by $T_\lambda\xi_\mu = \delta_{s(\lambda),r(\mu)}\xi_{\lambda\mu}$. This is a TCK-family, so induces $\pi_T : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda))$.

Proof sketch

(1) The terms in $\sum_{\mu \in \Lambda v} e^{-\beta r \cdot d(\mu)}$ are terms in the series expansion of $\prod_{i=1}^{k}(1 - e^{-\beta r_i} A_i)^{-1}$, so the sum converges. We calculate

$$e^{-\beta r_i} A_i (1 - e^{-\beta r_i} A_i)^{-1} = \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n+1}$$
$$< \sum_{n=0}^{\infty} (e^{-\beta r_i} A_i)^{n} = (1 - e^{-\beta r_i} A_i)^{-1}.$$

(2) Define $T_\lambda \in \mathcal{B}(\ell^2(\Lambda))$ by $T_\lambda \xi_\mu = \delta_{s(\lambda), r(\mu)} \xi_{\lambda\mu}$. This is a TCK-family, so induces $\pi_T : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda))$.

Check that $\Delta_\mu := e^{-\beta r \cdot d(\mu)} \epsilon_{s(\mu)}$ satisfies $\sum_{\mu \in \Lambda} \Delta_\mu = 1$. So

$$\phi_\epsilon(a) := \sum_\mu \Delta_\mu (\pi_T(a) \xi_\mu \mid \xi_\mu)$$

is a state; verify (*) to see it's KMS_β.

KMS states on the Cuntz-Krieger algebra

Our proof that $\phi(t_\mu t_\nu^*) = 0$ if $d(\mu) \neq d(\nu)$ but $r \cdot d(\mu) = r \cdot d(\nu)$ breaks down if $\beta r_i = \ln \rho(A_i)$.

No issue if the $\ln \rho(A_i)$ are rationally independent.

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Theorem (aHLRS)

There is a KMS$_\beta$ state for $(C^*(\Lambda), \alpha^r)$ if and only if $\beta r_i = \ln \rho(A_i)$ for all i. The formula $\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu}\rho(A^{d(\mu)})^{-1} x^\Lambda_{s(\mu)}$ always defines such a state. If the $\ln \rho(A_i)$ are rationally independent, then this is the only KMS state for $(C^*(\Lambda), \alpha^r)$.

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Proof.

We saw earlier that $\beta r_i = \ln \rho(A_i)$ is necessary. A weak*-compactness argument proves existence. The uniqueness follows from our calculation

$$\phi(t_\mu t_\nu^*) = e^{-\beta r \cdot d(\mu)} \phi(t_\nu^* t_\mu) = e^{-\beta r \cdot (d(\mu) - d(\nu))} \phi(t_\mu t_\nu^*)$$

earlier. □

Non-uniqueness

- The hypothesis that the $\ln \rho(A_i)$ are rationally independent is needed.
- Let E be the directed graph with one vertex and 2 loops so $C^*(E) = \mathcal{O}_2$.
- Let $\Lambda = \{(\lambda, n) \in E^* \times \mathbb{N}^2 : |\lambda| = n_1 + n_2\}$.
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- Kumjian and Pask prove that $C^*(\Lambda) \cong \mathcal{O}_2 \otimes C(\mathbb{T})$.
- Here $\ln \rho(A_1) = \ln \rho(A_2) = \ln 2$.
- Put $r = (\ln 2, \ln 2)$, and let ϕ be the unique $\mathrm{KMS}_{\ln 2}$ state of $\mathcal{O}_2$.
- Calculations show that $\phi \otimes \psi$ is a KMS_1 state of $C^*(\Lambda)$ for every state ϕ of $C(\mathbb{T})$.

Ground states

- a *ground state* is a state ϕ such that $z \mapsto \phi(a\alpha_z(b))$ is bounded on the upper half-plane for all analytic a, b.
- a KMS$_\infty$-state is a weak*-limit of KMSβ_n states where $\beta_n \to \infty$. On general grounds every KMS$_\infty$ state is a ground state, but not conversely.

Proposition

Suppose each $r_i > 0$. For each probability measure ϵ on Λ^0, there is a ground state of $(\mathcal{T}C*(\Lambda), \alpha^r)$ given by $\phi(t_v) = \epsilon(v)$ for $v \in \Lambda^0$ and $\phi(t_\mu t_\nu^*) = 0$ unless $\mu = \nu = s(\mu)$. These are all of the ground states, and they are all KMS$_\infty$ states.

Ground states

- In the characterisation of ground states, The hypothesis that $r_i > 0$ is needed.
- For example, let $r = (-1, 1)$ and consider $\Lambda = \mathbb{N}^2$ regarded as a 2-graph.
- If ϕ is a state of $\mathcal{T}C^*(\Lambda)$, then

$$\phi(t_{(1,0)}\alpha^r_{x+iy}(t^*_{(1,0)})) = e^{-yr\cdot d((1,0))}\phi(t_0) = e^y\phi(1_{\mathcal{T}C^*(\Lambda)})$$

is not bounded on the upper half plane.

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