Monodromy and Arithmetic Groups

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I will talk about hypergeometric functions and the monodromy group associated to them. To set up the notation, I will recall some very elementary results from differential equations.

Differential Equations on the Unit Disc

Let $z \in \Delta$ where Δ be the open unit disc in the plane. Suppose $f_0, \cdots, f_{r-1}$ are holomorphic functions on the disc. Consider the differential equation

$$\frac{d^r X}{dz^r} + f_{r-1}(z)\frac{d^{r-1}X}{dz^{r-1}} + \cdots + f_0(z)X = 0.$$

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Almost the same is true if we assume that $f_i(z)$ have at most a simple pole at 0 but are holomorphic elsewhere on the disc.

Theorem 2

There are $n-1$ linearly independent solutions which are holomorphic on the disc Δ to the foregoing equation.

Differential Equations

Suppose $q \in \Delta^*$ lies in the punctured unit disc (punctured at 0) and for each i, $f_i(q) = \frac{P_i(q)}{q^{n-i}}$, where P_i are holomorphic in q. We write $q = e^{2\pi iz}$ where z is on the upper half plane. By Cauchy's theorem, the equation above has n linearly independent solutions on $\mathfrak{h}$, which are holomorphic on $\mathfrak{h}$.

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The exponential map $\mathfrak{h} \to \Delta^*$ given by $z \mapsto q$ is a covering map and the functions $f_i(q)$ are invariant under the deck transformation group, which is a cyclic group generated by $g_0 : z \mapsto z + 1$.

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Thus the space of solutions X of the differential equation is invariant under the group $g_0^{\mathbb{Z}}$. This action is the "local monodrmy action". If a solution X is actually holomorphic in q even at 0, then the monodromy action is trivial on X.

Local Monodromy

If $f_i(q)$ have at most a simple pole at $q = 0$, then by a result mentioned earlier, the space of holomorphic solutions in z is n dimensional and has an $n - 1$ dimensional subspace which consists of solutions holomorphic in q, on the disc Δ. In particular, the monodromy action on this subspace is trivial. Hence there exists a basis of solutions X, such that the matrix of g_0 is of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & a_r \\ 0 & 1 & 0 & \cdots & 0 & a_{r-1} \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}$$

where $a_1 \neq 0$ is called the exceptional eigenvalue of the local monodromy element g_0. The matrix g_0 is called a **complex refelction.**

Gauss' Hypergeometric Function

Let us begin with Gauss's Hypergeometric function. Let a, b, c be real numbers with c not a non-negative integer. Denote, for an integer $n \geq 0$ by

$$(a)_n = a(a+1)\cdots(a+n-1),$$

the **Pochhammer Symbol**, with $(a)_0 = 1$.

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The **Gauss hypergeometric function** is

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}.$$

Theorem 3

This series converges absolutely and uniformly on compact sets in the region $\mid z \mid < 1$.

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Proof.

This is a simple consequence of the ratio test. □

Analytic Continuation

We may view the open unit disc Δ^* punctured at 0, as a subset of the thrice punctured projective line: $\Delta^* \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The latter is covered by the upper half plane $\mathfrak{h}$ and so we may write $z = \lambda(\tau)$ for $z \in \Delta$, with $\tau \in \lambda^{-1}(\Delta) \subset \mathfrak{h}$. Then it is known that $F(z)$ admits an analytic continuation to the whole of $\mathfrak{h}$.

Differential Equation satisfied by F

Write $\theta = q\frac{d}{dq}$. We will view θ as a differential operator on $C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The Gauss hypergeometric function F satisfies the differential equation

$$q(\theta + a)(\theta + b)F = (\theta + c - 1)\theta F.$$

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On the (two dimensional) space of solutions of this differential equation (viewd as functions on the upper half plane in the variable τ with $q = e^{2\pi i\tau}$), the deck-transformation group Γ operates and hence we get a two dimensional representation of Γ. This is called the monodromy representation of Γ.

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The group Γ may be identified with the fundamental group of the curve C, which is free on two generators g_0 and g_∞, two small loops in C going counterclockwise exactly once around 0 and ∞ respectively.

Monodromy Representation

The monodromy representation has the property that g_0 fixes the solution F since F is analytic at the puncture 0. One can then describe the monodromy representation by two matrices A and B^{-1} namely the images of g_0 and g_∞. It can be shown that there exists a basis of solutions for which The images of g_0 and g_∞ are of the form $A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -b_0 \\ 1 & -b_1 \end{pmatrix}$.

Suppose that $q \in C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and put $\theta = q\frac{d}{dq}$. Let $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{C}^r, \gamma = (\gamma_1, \cdots, \gamma_{r-1}) \in \mathbb{C}^{r-1}$. We then have the (one variable) generalised hypergeometric function of type ${}_rF_{r-1}$:

$$F(\alpha, \gamma) : q) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\gamma_1)_n \cdots (\gamma_{r-1})_n} \frac{q^n}{n!}.$$

Clausen-Thomae Differential Equation

Theorem 4

The function ${}_rF_{r-1}(q)$ satisfies the differential equation

$$q(\theta+\alpha_1)\cdots(\theta+\alpha_r)F = (\theta+\gamma_1)\cdots(\theta+\gamma_{r-1})\theta F.$$

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$$q(\theta+\alpha_1)\cdots(\theta+\alpha_r)F = (\theta+\gamma_1)\cdots(\theta+\gamma_{r-1})\theta F.$$

Written out, the differential equation may be seen to be of the form

$$\frac{d^rF}{dq^r} + f_{r-1}(q)\frac{d^{r-1}F}{dq^{r-1}} + \cdots + f_0(q)F = 0.$$

Here, $f_i(q)$ are holomorphic on C but have simple poles at $q = 1$. In that case, the local monodromy matrix g_1 is a *complex reflection*.

A Theorem of Levelt

Write $g(X) = \prod_{j=1}^{r}(X - e^{2\pi i\alpha_j})$ and $f(X) = (X-1)\prod_{j=1}^{r-1}(X - e^{2\pi i\beta_j})$. Let A and B be the companion matrices of f, g respectively. We have a representation of $\pi_1(C) =< g_0, g_\infty)$ into $GL_r(\mathbb{C})$ given by $g_0 \mapsto A$ and $g_\infty \mapsto B^{-1}$.

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Theorem 5

(Levelt) There exists a basis of solutions of the differential equation satisfied by the hypergeometric equation ${}_rF_{r-1}$ such that the monodromy representation on the space of solutions of this equation is the above representation.

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Moreover, if $\rho : \Gamma \to GL_r(\mathbb{C})$ is any representation such that the characteristic polynomials of g_0 and g_∞^{-1} are f and g, and such that $g_0 g_\infty$ is a complex reflection, then ρ is equivalent to this representation.

A Theorem of Beukers and Heckman

Suppose now that $f(X)$ and $g(X)$ are reciprocal, have no common factors, and have integral coefficients with $f(0) = g(0) = \pm 1$. We also assume that (f, g) is *primitive pair* i.e. there do not exist polynomials f_1, g_1 and an integer $k \geq 2$ such that $f_1(X^k) = f(X)$ and $g_1(X^k) = g(X)$. Then

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Theorem 6

(Beukers-Heckman) The identity connected component of the Zariski closure of A and B is $Sp_r(\mathbb{C})$ if $f(0) = g(0) = 1$ and SO_r otherwise.

Question

Beukers and Heckman also determine when the monodromy group is finite (this is the same thing as saying that $F(z)$ is an algebraic function). The next question is when the monodromy group an arithmetic group?

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We will say that a subgroup $\Gamma \subset SL_n(\mathbb{Z})$ is an **arithmetic group**, if Γ has finite index in the integral points of its Zariski closure in SL_n. Otherwise, we will say that Γ is **thin.**

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It is hoped that for most of monodromy groups are thin.

Suppose $f, g \in \mathbb{Z}[X]$ have no common root, are primitive of degree r, with $f(0) = g(0) = 1$. Suppose that the difference $f - g$ is monic, or has leading coefficient not exceding two in absolute value. Under these assumptions, we have the

Theorem 7

(S.Singh and V.) The monodromy group $\Gamma(f, g) \subset Sp_r(\mathbb{Z})$ has finite index.

Other Results

There are infinitely many examples (Sarnak-Fuchs-Meiri) for which the real Zariski closure is $SO(r-1,1)$ and the monodromy group is thin (has infinite index in its integral Zariski closure).

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There are 14 examples of f, g with $g = (X - 1)^4$ (families of Calabi-Yau 3 folds) whose monodromy lies in $Sp_4(\mathbb{Z})$; of these, 7 are thin by Brav-Thomas. The criterion above by Singh and V., shwsh that 3 are arithmetic. The other 4 are unknown.

Sketch of Proof for $n = 4$

Suppose $\Gamma \subset Sp_4(\mathbb{Z}$ is a subgroup. In order that Γ have finite index, it is necessary that Γ is Zariksi dense in Sp_4.

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We need only prove that the reflection subgroup generated by the conjugates of $C = A^{-1}B$ by the elements $1, A, A^2, A^3$ has finite index. But one can show that C, ACA^{-1} and A^2CA^{-2} lie in a maximal parabolic subgroup P and that under the assumption on the leading coefficient of the difference $f - g$ not exceeding two, the group generated by these two elements contain a finite index subgroup of the integral points of the unipotent rdical of P. Now by appealing to the result of Tits, we see that Γ has finite index.

Sketch of Proof

First of all, the elements A and B have the same effect on E_1, e_2, e_3 since they are companion matrices. Hence $C = A^{-1}B$ fixes $e1, e_2, e_3$. Therefore, the conjugate ACA^{-1} also fixes a three dimensional subspace. Hence, in $\mathbb{Q}^4$, the group Δ generated by the three elements C, ACA^{-1} and A^2CA^{-2} has at least a one dimensional space of fixed vectors.

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Now, consider the parabolic subgroup P of SSp_4, which fixes the flag

$$\mathbb{Q}v \subset v^{\perp} \subset \mathbb{Q}^4.$$

It is easy to see that the semi-simple part of the Levi subgroup of P is SL_2. Hence Δ lies in P.

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The condition on coefficients ensures that the projection of the elements C and ACA^{-1} to $SL_2(\mathbb{Q})$ contains the unipotent generators of $SL_2(2\mathbb{Z})$. Hence Δ intersects the unipotent radical of P non-trivially.

Table: List of primitive Symplectic pairs of polynomials of degree 4 (which are products of cyclotomic polynomials), for which arithmeticity follows from Main Theorem

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
1	$\mathbf{X^4-4X^3+6X^2-4X+1}$	$\mathbf{X^4-2X^3+3X^2-2X+1}$	**0,0,0,0**	$\mathbf{\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}}$	$\mathbf{-2X^3+3X^2-2X}$
2	X^4-2X^2+1	$X^4+2X^3+3X^2+2X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$-2X^3-5X^2-2X$
3	X^4-2X^2+1	$X^4+X^3+2X^2+X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{2}{3}$	$-X^3-4X^2-X$
4	X^4-2X^2+1	$X^4+X^3+X^2+X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$-X^3-3X^2-X$
5	X^4-2X^2+1	$X^4-2X^3+3X^2-2X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$2X^3-5X^2+2X$
6	X^4-2X^2+1	$X^4-X^3+2X^2-X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	X^3-4X^2+X
7	X^4-2X^2+1	$X^4-X^3+X^2-X+1$	$0,0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	X^3-3X^2+X
8	$X^4+4X^3+6X^2+4X+1$	$X^4+2X^3+3X^2+2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$2X^3+3X^2+2X$
9	X^4-X^3-X+1	X^4+2X^2+1	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$-X^3-2X^2-X$
10	X^4-X^3-X+1	$X^4+X^3+X^2+X+1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$-2X^3-X^2-2X$
11	X^4-X^3-X+1	$X^4-2X^3+3X^2-2X+1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	X^3-3X^2+X
12	X^4-X^3-X+1	X^4+X^3+X+1	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$-2X^3-2X$
13	X^4-X^3-X+1	$X^4-X^3+2X^2-X+1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$-2X^2$
14	X^4-X^3-X+1	X^4+1	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-X^3-X$
15	X^4-X^3-X+1	$X^4-X^3+X^2-X+1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$-X^2$
16	X^4-X^3-X+1	X^4-X^2+1	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-X^3+X^2-X$
17	$X^4+2X^3+3X^2+2X+1$	X^4+2X^2+1	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$2X^3+X^2+2X$

Table: Continued...

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19	$X^4+2X^3+3X^2+2X+1$	$X^4+X^3+X^2+X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	X^3+2X^2+X
20	$X^4+2X^3+3X^2+2X+1$	X^4+X^3+X+1	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	X^3+3X^2+X
21	$X^4+2X^3+3X^2+2X+1$	X^4+1	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$2X^3+3X^2+2X$
22	$X^4+2X^3+3X^2+2X+1$	X^4-X^2+1	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$2X^3+4X^2+2X$
23	$X^4+3X^3+4X^2+3X+1$	$X^4+X^3+X^2+X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$2X^3+3X^2+2X$
24	$X^4-2X^3+2X^2-2X+1$	$X^4-2X^3+3X^2-2X+1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$-X^2$
25	$X^4-2X^3+2X^2-2X+1$	X^4+X^2+1	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	$-2X^3+X^2-2X$
26	$X^4-2X^3+2X^2-2X+1$	X^4+1	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-2X^3+2X^2-2X$
27	$X^4-2X^3+2X^2-2X+1$	$X^4-X^3+X^2-X+1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$-X^3+X^2-X$
28	$X^4-2X^3+2X^2-2X+1$	X^4-X^2+1	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-2X^3+3X^2-2X$
29	X^4+2X^2+1	$X^4+X^3+X^2+X+1$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$-X^3+X^2-X$
30	X^4+2X^2+1	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$2X^3-X^2+2X$
31	X^4+2X^2+1	X^4+X^3+X+1	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$-X^3+2X^2-X$
32	X^4+2X^2+1	$X^4-X^3+X^2-X+1$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	X^3+X^2+X
33	$X^4+2X^3+2X^2+2X+1$	$X^4+X^3+X^2+X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	X^3+X^2+X
34	$X^4+2X^3+2X^2+2X+1$	X^4+X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	$2X^3+X^2+2X$

Table: Continued...

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
35	$X^4+2X^3+2X^2+2X+1$	X^4+1	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$2X^3+2X^2+2X$
36	$X^4+2X^3+2X^2+2X+1$	X^4-X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$2X^3+3X^2+2X$
37	$X^4+X^3+2X^2+X+1$	$X^4+X^3+X^2+X+1$	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	X^2
38	$X^4+X^3+2X^2+X+1$	X^4+X^3+X+1	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$2X^2$
39	$X^4+X^3+2X^2+X+1$	X^4+1	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	X^3+2X^2+X
40	$X^4+X^3+2X^2+X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$2X^3+X^2+2X$
41	$X^4+X^3+2X^2+X+1$	X^4-X^2+1	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	X^3+3X^2+X
42	$X^4+X^3+X^2+X+1$	X^4+X^3+X+1	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	X^2
43	$X^4+X^3+X^2+X+1$	X^4+X^2+1	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	X^3+X
44	$X^4+X^3+X^2+X+1$	$X^4-X^3+2X^2-X+1$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$2X^3-X^2+2X$
45	$X^4+X^3+X^2+X+1$	X^4+1	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	X^3+X^2+X
46	$X^4+X^3+X^2+X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$2X^3+2X$
47	$X^4+X^3+X^2+X+1$	X^4-X^2+1	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	X^3+2X^2+X
48	$X^4-3X^3+4X^2-3X+1$	$X^4-X^3+X^2-X+1$	$0,0,\frac{1}{6},\frac{5}{6}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$-2X^3+3X^2-2X$
49	$X^4-2X^3+3X^2-2X+1$	X^4+1	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-2X^3+3X^2-2X$
50	$X^4-2X^3+3X^2-2X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$-X^3+2X^2-X$
51	$X^4-2X^3+3X^2-2X+1$	X^4-X^2+1	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-2X^3+4X^2-2X$

Table: Continued...

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
52	X^4+X^3+X+1	X^4+1	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	X^3+X
53	X^4+X^3+X+1	$X^4-X^3+X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$2X^3-X^2+2X$
54	X^4+X^3+X+1	X^4-X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	X^3+X^2+X
55	X^4+X^2+1	$X^4-X^3+X^2-X+1$	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	X^3+X
56	$X^4-X^3+2X^2-X+1$	X^4+1	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-X^3+2X^2-X$
57	$X^4-X^3+2X^2-X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	X^2
58	$X^4-X^3+2X^2-X+1$	X^4-X^2+1	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-X^3+3X^2-X$
59	X^4+1	$X^4-X^3+X^2-X+1$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	X^3-X^2+X
60	$X^4-X^3+X^2-X+1$	X^4-X^2+1	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-X^3+2X^2-X$

Table: List of primitive Symplectic pairs of polynomials of degree 4 (which are products of cyclotomic polynomials), to which Main Theorem does not apply

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
1*	$X^4-4X^3+6X^2-4X+1$	$X^4+4X^3+6X^2+4X+1$	$0,0,0,0$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$-8X^3-8X$
2	$X^4-4X^3+6X^2-4X+1$	$X^4+2X^3+3X^2+2X+1$	0,0,0,0	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$-6X^3+3X^2-6X$
3*	$X^4-4X^3+6X^2-4X+1$	$X^4+3X^3+4X^2+3X+1$	$0,0,0,0$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$-7X^3+2X^2-7X$
4	$X^4-4X^3+6X^2-4X+1$	X^4+2X^2+1	0,0,0,0	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$-4X^3+4X^2-4X$
5*	$X^4-4X^3+6X^2-4X+1$	$X^4+2X^3+2X^2+2X+1$	$0,0,0,0$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$-6X^3+4X^2-6X$
6	$X^4-4X^3+6X^2-4X+1$	$X^4+X^3+2X^2+X+1$	0,0,0,0	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$-5X^3+4X^2-5X$
7*	$X^4-4X^3+6X^2-4X+1$	$X^4+X^3+X^2+X+1$	$0,0,0,0$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$-5X^3+5X^2-5X$
8*	$X^4-4X^3+6X^2-4X+1$	X^4+X^3+X+1	$0,0,0,0$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$-5X^3+6X^2-5X$
9	$X^4-4X^3+6X^2-4X+1$	X^4+X^2+1	0,0,0,0	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	$-4X^3+5X^2-4X$
10	$X^4-4X^3+6X^2-4X+1$	$X^4-X^3+2X^2-X+1$	0,0,0,0	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$-3X^3+4X^2-3X$
11*	$X^4-4X^3+6X^2-4X+1$	X^4+1	$0,0,0,0$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-4X^3+6X^2-4X$
12	$X^4-4X^3+6X^2-4X+1$	$X^4-X^3+X^2-X+1$	0,0,0,0	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$-3X^3+5X^2-3X$
13*	$X^4-4X^3+6X^2-4X+1$	X^4-X^2+1	$0,0,0,0$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-4X^3+7X^2-4X$
14	$X^4+4X^3+6X^2+4X+1$	X^4-X^3-X+1	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$0,0,\frac{1}{3},\frac{2}{3}$	$5X^3+6X^2+5X$
15	$X^4+4X^3+6X^2+4X+1$	$X^4-2X^3+2X^2-2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$0,0,\frac{1}{4},\frac{3}{4}$	$6X^3+4X^2+6X$
16	$X^4+4X^3+6X^2+4X+1$	X^4+2X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$4X^3+4X^2+4X$
17	$X^4+4X^3+6X^2+4X+1$	$X^4+X^3+2X^2+X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{2}{3}$	$3X^3+4X^2+3X$

Table: Continued...

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
18	$X^4+4X^3+6X^2+4X+1$	$X^4+X^3+X^2+X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$3X^3+5X^2+3X$
19	$X^4+4X^3+6X^2+4X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$0,0,\frac{1}{6},\frac{5}{6}$	$7X^3+2X^2+7X$
20	$X^4+4X^3+6X^2+4X+1$	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$6X^3+3X^2+6X$
21	$X^4+4X^3+6X^2+4X+1$	X^4+X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}$	$4X^3+5X^2+4X$
22	$X^4+4X^3+6X^2+4X+1$	$X^4-X^3+2X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$5X^3+4X^2+5X$
23	$X^4+4X^3+6X^2+4X+1$	X^4+1	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$4X^3+6X^2+4X$
24	$X^4+4X^3+6X^2+4X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$5X^3+5X^2+5X$
25	$X^4+4X^3+6X^2+4X+1$	X^4-X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$4X^3+7X^2+4X$
26	X^4-X^3-X+1	$X^4+2X^3+2X^2+2X+1$	$0,0,\frac{1}{3},\frac{2}{3}$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$-3X^3-2X^2-3X$
27	$X^4+2X^3+3X^2+2X+1$	$X^4-2X^3+2X^2-2X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$0,0,\frac{1}{4},\frac{3}{4}$	$4X^3+X^2+4X$
28	$X^4+2X^3+3X^2+2X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$0,0,\frac{1}{6},\frac{5}{6}$	$5X^3-X^2+5X$
29	$X^4+2X^3+3X^2+2X+1$	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$4X^3+4X$
30	$X^4+2X^3+3X^2+2X+1$	$X^4-X^3+2X^2-X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$3X^3+X^2+3X$
31	$X^4+2X^3+3X^2+2X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$3X^3+2X^2+3X$
32	$X^4+3X^3+4X^2+3X+1$	$X^4-2X^3+2X^2-2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$0,0,\frac{1}{4},\frac{3}{4}$	$5X^3+2X^2+5X$
33	$X^4+3X^3+4X^2+3X+1$	X^4+2X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$3X^3+2X^2+3X$
34	$X^4+3X^3+4X^2+3X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$0,0,\frac{1}{6},\frac{5}{6}$	$6X^3+6X$

Table: Continued...

No.	$f(X)$	$g(X)$	α	β	$f(X)-g(X)$
35	$X^4+3X^3+4X^2+3X+1$	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$5X^3+X^2+5X$
36	$X^4+3X^3+4X^2+3X+1$	$X^4-X^3+2X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{3}{4},\frac{1}{6},\frac{5}{6}$	$4X^3+2X^2+4X$
37	$X^4+3X^3+4X^2+3X+1$	X^4+1	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$3X^3+4X^2+3X$
38	$X^4+3X^3+4X^2+3X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$4X^3+3X^2+4X$
39	$X^4+3X^3+4X^2+3X+1$	X^4-X^2+1	$\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{2}{3}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$3X^3+5X^2+3X$
40	$X^4-2X^3+2X^2-2X+1$	$X^4+X^3+X^2+X+1$	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$-3X^3+X^2-3X$
41	$X^4-2X^3+2X^2-2X+1$	X^4+X^3+X+1	$0,0,\frac{1}{4},\frac{3}{4}$	$\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{5}{6}$	$-3X^3+2X^2-3X$
42	X^4+2X^2+1	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}$	$0,0,\frac{1}{6},\frac{5}{6}$	$3X^3-2X^2+3X$
43	$X^4+2X^3+2X^2+2X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$0,0,\frac{1}{6},\frac{5}{6}$	$5X^3-2X^2+5X$
44	$X^4+2X^3+2X^2+2X+1$	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$4X^3-X^2+4X$
45	$X^4+2X^3+2X^2+2X+1$	$X^4-X^3+X^2-X+1$	$\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{3}{4}$	$\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}$	$3X^3+X^2+3X$
46	$X^4+X^3+2X^2+X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}$	$0,0,\frac{1}{6},\frac{5}{6}$	$4X^3-2X^2+4X$
47	$\mathbf{X^4+X^3+2X^2+X+1}$	$\mathbf{X^4-2X^3+3X^2-2X+1}$	$\mathbf{\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4}}$	$\mathbf{\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}}$	$\mathbf{3X^3-X^2+3X}$
48	$X^4+X^3+X^2+X+1$	$X^4-3X^3+4X^2-3X+1$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$0,0,\frac{1}{6},\frac{5}{6}$	$4X^3-3X^2+4X$
49	$X^4+X^3+X^2+X+1$	$X^4-2X^3+3X^2-2X+1$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{5}{6}$	$3X^3-2X^2+3X$
50	$X^4-3X^3+4X^2-3X+1$	X^4+1	$0,0,\frac{1}{6},\frac{5}{6}$	$\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8}$	$-3X^3+4X^2-3X$
51	$X^4-3X^3+4X^2-3X+1$	X^4-X^2+1	$0,0,\frac{1}{6},\frac{5}{6}$	$\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}$	$-3X^3+5X^2-3X$