Long-term behavior of subcritical contact processes

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Outline

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The contact process - definition

Ingredients:

- Finite or countable group Λ
- Infection kernel $a(i,j), i,j \in \Lambda$
  translation invariant, irreducible, $|a| := \sum_{i\in\Lambda} a(0,i) < \infty$
- Recovery rate $\delta \geq 0$

(Λ, a, δ)**-contact process** with states in $\{0,1\}^\Lambda$:

- type 1 at site i induces a type 1 at site j with rate $a(i,j)$
- type 1 at site i becomes a type 0 at rate δ

Remark: Equip $\{0,1\}^\Lambda$ with the product metric.

The contact process - definition

As a process $\eta = (\eta_t)_{t\geq 0}$ taking values in $\mathcal{P} := \{A : A \subset \Lambda\}$ (set of 1's) it has the formal generator

$$\begin{aligned} Gf(A) \quad := \quad & \sum_{i,j\in\Lambda} a(i,j)1_{\{i\in A\}}1_{\{j\notin A\}}\{f(A\cup\{j\}) - f(A)\} \\ & +\delta\sum_{i\in\Lambda} 1_{\{i\in A\}}\{f(A\backslash\{i\}) - f(A)\}. \end{aligned}$$

We write η^A if $\eta_0^A = A$ a.s.

The contact process - elementary properties

Duality:

Consider **reversed infection rates:** $a^\dagger(i,j) := a(j,i)$

- $(\eta_t^A)_{t\geq 0}$: (Λ, a, δ)-contact process
- $(\eta_t^{\dagger\, B})_{t\geq 0}$: $(\Lambda, \mathbf{a}^\dagger, \delta)$-contact process

Then

$$\mathbb{P}[\eta_t^A \cap B \neq \emptyset] = \mathbb{P}[A \cap \eta_t^{\dagger\, B} \neq \emptyset] \qquad A, B \in \mathcal{P}(\Lambda),\ t \geq 0.$$

The contact process - elementary properties

Survival probability:

We say that the (Λ, a, δ)-contact process **survives** if

$$\rho(A) := \mathbb{P}\big[\eta_t^A \neq \emptyset \ \forall t \geq 0\big] > 0$$

for some, and hence for all nonempty A of finite cardinality $|A|$.

We set $\theta := \rho(\{0\})$ and call

$$\delta_\mathrm{c} := \sup\{\delta \geq 0 : \theta > 0\}$$

the **critical recovery rate.**

$\rightarrow \delta > \delta_\mathrm{c}$ **subcritical**

The contact process - elementary properties

Long term behavior:

$$\mathbb{P}\big[\eta_t^{\Lambda} \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

$\overline{\nu}$: **upper invariant law.**

- $\overline{\nu}$ is concentrated on the nonempty subsets of Λ if the $(\Lambda, a^{\dagger}, \delta)$-contact process survives.
- $\overline{\nu} = \delta_{\emptyset}$ if the $(\Lambda, a^{\dagger}, \delta)$-contact process dies out.

The contact process: elementary properties

Exponential growth:

There exists a constant $r = r(\Lambda, a, \delta)$ with $-\delta \le r \le |a| - \delta$ such that

$$r = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[|\eta_t^A|\big] \qquad A \in \mathcal{P}_{fin,+}.$$

Notation:

$$\begin{aligned}
\mathcal{P}_{\mathrm{fin},+} &:= \mathcal{P}_{\mathrm{fin}} \cap \mathcal{P}_+ \\
\mathcal{P}_{\mathrm{fin}} &:= \{A \subset \Lambda : |A| < \infty\} \\
\mathcal{P}_+ &:= \{A \subset \Lambda : |A| > 0\}
\end{aligned}$$

The contact process: elementary properties

Known properties of the exponential growth rate:

- $r(\Lambda, a, \delta) = r(\Lambda, a^\dagger, \delta)$.
- $\delta \mapsto r(\delta)$ is nonincreasing and Lipschitz continuous on $[0, \infty)$ with Lipschitz constant 1.
- If $r > 0$, then the contact process survives.
- $\{\delta \geq 0 : r(\delta) < 0\} = (\delta_{\rm c}, \infty)$.

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Definition of eigenmeasures

A measure μ on $\mathcal{P}_+$ is an **eigenmeasure** of the (Λ, a, δ)-contact process if μ is nonzero, locally finite, and there exists a constant $\lambda \in \mathbb{R}$ such that

$$\int \mu(\mathrm{d}A)\mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+} = e^{\lambda t}\mu \qquad t \geq 0.$$

We call λ the associated **eigenvalue.**

Known properties of eigenmeasures

- **Existence:**
  Each (Λ, a, δ)-contact process has a (spatially) homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue $r = r(\Lambda, a, \delta)$.
- **Scaling and normalization:**
  If $\mathring{\nu}$ is an eigenmeasure , then also $c\mathring{\nu}$ for $c > 0$.
  We normalize: $\int \mathring{\nu}(\mathrm{d}A) 1_{\{0 \in A\}} = 1$
- **Uniqueness:**
  In general not known if $\mathring{\nu}$ is unique.
  Special case: a irreducible, $\overline{\nu}$ nontrivial and $r(\Lambda, a, \delta) = 0$, then $\mathring{\nu}$ is unique and $\mathring{\nu} = c\,\overline{\nu}$.

Notions of convergence

Let μ_n, μ be locally finite measures on $\mathcal{P}$.

- $\mu_n \to \mu$ **vaguely** ($\mu_n \Rightarrow \mu$) $\Leftrightarrow$

$$\int \mu_n(dA) f(A) \to \int \mu(dA) f(A)$$

  for f continuous, compactly supported.

- For μ_n, μ concentrated on $\mathcal{P}_{\mathrm{fin},+}$, $\mu_n \to \mu$ **locally** $\Leftrightarrow$

$$\mu_n|_{\mathcal{P}_{\mathrm{fin},\,i}} \to \mu|_{\mathcal{P}_{\mathrm{fin},\,i}} \text{ weakly}$$

  where $\mathcal{P}_{\mathrm{fin},\,i} := \mathcal{P}_{\mathrm{fin}} \cap \mathcal{P}_i$ with $\mathcal{P}_i := \{A \in \mathcal{P} : i \in A\}$.

(Local convergence implies vague convergence.)

Uniqueness of and convergence to eigenmeasures

Theorem 1: Sturm, Swart

Let a be irreducible and $r < 0$. Then:

- There exists a **unique** homogeneous eigenmeasure $\mathring{\nu}$ of the (Λ, a, δ)-contact process such that $\int \mathring{\nu}(\mathrm{d}A) 1_{\{0 \in A\}} = 1$.
- $\mathring{\nu}$ has eigenvalue r and is **concentrated on** $\mathcal{P}_{\mathrm{fin}}$.
- If μ is any nonzero, homogeneous, locally finite measure on $\mathcal{P}_+$, then

$$e^{-rt} \int \mu(\mathrm{d}A) \mathbb{P}[\eta_t^A \in \cdot\,]\big|_{\mathcal{P}_+(\Lambda)} \underset{t\to\infty}{\Longrightarrow} c(\mu)\,\mathring{\nu}.$$

  If μ is concentrated on $\mathcal{P}_{\mathrm{fin},+}$ this holds in the sense of local convergence.

Process modulo shifts

Identify sets modulo shifts:

$$\tilde{\mathcal{P}}_{\text{fin}} := \{\tilde{A} : A \in \mathcal{P}_{\text{fin}}\} \qquad \tilde{A} := \{iA : i \in \Lambda\}$$

Let $\tilde{\eta}$ be the on $\tilde{\mathcal{P}}_{\text{fin}}$ induced Markov process:
(Λ, a, δ)**-contact process modulo shifts.**

Transition probabilities:

$$\tilde{P}_t(\tilde{A}, \tilde{B}) = \sum_{\substack{C \in \mathcal{P}_{\text{fin},+} \\ \tilde{C} = \tilde{B}}} P_t(A, C) = m(B)^{-1} \sum_{i \in \Lambda} P_t(A, iB)$$

Connection to quasi-invariance

Let Δ be a $\mathcal{P}_{\mathrm{fin},+}$-valued random variable with

$$\mathring{\nu} = c \sum_{i \in \Lambda} \mathbb{P}\big[i\Delta \in \cdot \big]$$

and $\tilde{\nu}$ the law of Δ modulo shifts.

Theorem 2: Sturm, Swart

Under the assumptions of Theorem 1 the law $\tilde{\nu}$ is a quasi-invariant law for the (Λ, a, δ)-contact process modulo shifts. For any $A \in \mathcal{P}_{\mathrm{fin},+}$

$$\mathbb{P}[\tilde{\eta}_t^A \in \cdot \,|\, \eta_t^A \neq \emptyset] \underset{t\to\infty}{\Longrightarrow} \tilde{\nu},$$

where $\Rightarrow$ denotes weak convergence on $\tilde{\mathcal{P}}_{\mathrm{fin},+}$.

$\rightarrow$ **Ferrari, Kesten, Martinez '96**

Application

As an application we derive an expression for the derivative of the exponential growth rate

$$r = \lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[|\eta_t^{\{0\}}|\big].$$

Let

$$\begin{aligned} \mu_t &:= \sum_i \mathbb{P}[\eta_t^{\{i\}} \in \cdot\,]\big|_{\mathcal{P}_+} \\ \pi_t &:= \mu_t(\{A : 0 \in A\}) = \sum_i \mathbb{P}[0 \in \eta_t^{\{i\}}] = \mathbb{E}[|\eta_t^{\{0\}}|]. \end{aligned}$$

We use Theorem 1 implying $e^{-rt}\mu_t \underset{t\to\infty}{\Longrightarrow} c\,\mathring{\nu}$ and

$$\frac{\partial r}{\partial \delta} = \frac{\partial}{\partial \delta} \lim_{t\to\infty} \tfrac{1}{t} \log \pi_t.$$

Application

Theorem 3: Sturm, Swart

Under the assumptions of Theorem1 the function

$$\delta \mapsto r(\Lambda, a, \delta)$$

is continuously differentiable on $(\delta_{\rm c}, \infty)$ and satisfies $\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) < 0$ on $(\delta_{\rm c}, \infty)$. Moreover,

$$\frac{\partial}{\partial \delta} r(\Lambda, a, \delta) = -\frac{\int \mathring{\nu}(\mathrm{d}A) \int \mathring{\nu}^{\dagger}(\mathrm{d}B) 1_{\{A \cap B = \{0\}\}}}{\int \mathring{\nu}(\mathrm{d}A) \int \mathring{\nu}^{\dagger}(\mathrm{d}B) |A \cap B|^{-1} 1_{\{0 \in A \cap B\}}},$$

where $\mathring{\nu}$ and $\mathring{\nu}^{\dagger}$ are the eigenmeasures of the (Λ, a, δ)- and $(\Lambda, a^{\dagger}, \delta)$-contact processes, respectively.

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Proof outline Theorem 3

One can show that

$$\begin{aligned}\frac{\partial}{\partial\delta} r(\delta) &= \frac{\partial}{\partial\delta} \lim_{t\to\infty} \frac{1}{t} \log \pi_t(\delta) \\ &= \lim_{t\to\infty} \frac{1}{t} \frac{\partial}{\partial\delta} \log \pi_t(\delta) \\ &= \lim_{t\to\infty} \frac{\frac{1}{t}\frac{\partial}{\partial\delta}\pi_t(\delta)}{\pi_t(\delta)}.\end{aligned}$$

Use local convergence of Theorem 1 for the following expressions:

Proof outline Theorem 3

$$
\begin{aligned}
\pi_t(\delta) &= \sum_j \mathbb{P}[(j,0) \rightsquigarrow (0,t)] = \sum_i \mathbb{P}[(0,0) \rightsquigarrow (i,t)] \\
&= \sum_i \mathbb{P}[\eta_s^{\{0\}} \cap \eta_{t-s}^{\dagger\,\{i\}} \neq \emptyset] \\
&= \sum_{i,j} \mathbb{E}[|\eta_s^{\{0\}} \cap \eta_{t-s}^{\dagger\,\{i\}}|^{-1} 1_{\{j \in \eta_s^{\{0\}} \cap \eta_{t-s}^{\dagger\,\{i\}}\}}] \\
&= \sum_{i,j} \mathbb{E}[|\eta_s^{\{j^{-1}\}} \cap \eta_{t-s}^{\dagger\,\{j^{-1}i\}}|^{-1} 1_{\{0 \in \eta_s^{\{j^{-1}\}} \cap \eta_{t-s}^{\dagger\,\{j^{-1}i\}}\}}] \\
&= \int \mu_{s,\delta}(\mathrm{d}A) \int \mu^{\dagger}_{t-s,\delta}(\mathrm{d}B) |A \cap B|^{-1} 1_{\{0 \in A \cap B\}}
\end{aligned}
$$

Proof outline Theorem 3

With $(0,0) \rightsquigarrow_{(j,s)} (i,t)$ denoting the event of an open path from $(0,0)$ to (i,t) with (j,s) pivotal

$$\begin{aligned}
\frac{1}{t}\frac{\partial}{\partial\delta}\pi_t(\delta) &= -\sum_{i,j}\frac{1}{t}\int_0^t \mathrm{d}s\, \mathbb{P}[(0,0) \rightsquigarrow_{(j,s)} (i,t)] \\
&= -\sum_{i,j}\frac{1}{t}\int_0^t \mathrm{d}s\, \mathbb{P}[(j^{-1},-s) \rightsquigarrow_{(0,0)} (j^{-1}i,t-s)] \\
&= -\sum_{i,j}\frac{1}{t}\int_0^t \mathrm{d}s\, \mathbb{P}[\eta_s^{\{i\}} \cap \eta_{t-s}^{\dagger\,\{j\}} = \{0\}] \\
&= -\frac{1}{t}\int_0^t \mathrm{d}s \int \mu_{s,\delta}(\mathrm{d}A) \int \mu_{t-s,\delta}^{\dagger}(\mathrm{d}B) 1_{\{A\cap B=\{0\}\}}
\end{aligned}$$

Proof outline Theorem 1 - Step 1

Existence of an eigenmeasure concentrated on $\mathcal{P}_{\rm fin}$:

Proposition 1

Let $r < 0$. Then there exists a homogeneous eigenmeasure $\mathring{\nu}$ with eigenvalue r of the (Λ, a, δ)-contact process such that

$$\int \mathring{\nu}(\mathrm{d}A)|A|1_{\{i\in A\}} < \infty \qquad (i \in \Lambda).$$

In particular, $\mathring{\nu}$ is concentrated on $\mathcal{P}_{\rm fin}$.

Step 1: Eigenmeasure concentrated on $\mathcal{P}_{\rm fin}$

Proof outline of Proposition 1:

Let $\hat{\mu}_\lambda := \int_0^\infty \mu_t\, e^{-\lambda t}\mathrm{d}t$ and $\hat{\pi}_\lambda := \int_0^\infty \pi_t\, e^{-\lambda t}\mathrm{d}t$.

- Swart '09: The measures $\frac{1}{\hat{\pi}_\lambda}\hat{\mu}_\lambda$ $(\lambda > r)$ are relatively compact.

  Each subsequential limit as $\lambda \downarrow r$ is a homogeneous eigenmeasure of the (Λ, a, δ)-contact process, with eigenvalue $r(\Lambda, a, \delta)$.
- We have

$$\limsup_{\lambda\downarrow r} \frac{1}{\hat{\pi}_\lambda} \int \hat{\mu}_\lambda(\mathrm{d}A) 1_{\{0\in A\}} |A| < \infty.$$

Step 2: Uniqueness and vague convergence

For uniqueness and vague convergence it suffices to show for $B \in \mathcal{P}_{\text{fin},+}$

$$e^{-rt} \int \mu P_t(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}} \underset{t \to \infty}{\longrightarrow} c(\mu) \int \mathring{\nu}(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}}.$$

In order to rewrite the right hand side define

$$h_\mu(A) := \int \mu(\mathrm{d}B) 1_{\{A \cap B \neq \emptyset\}} \qquad A \in \mathcal{P}_{\text{fin}}.$$

Step 2: Uniqueness and vague convergence

For $B \in \mathcal{P}_{\text{fin},+}$ the right hand side is

$$\begin{aligned} e^{-rt} h_{\mu P_t}(B) &= e^{-rt} P_t^{\dagger} h_{\mu}(B) = e^{-rt} \tilde{P}_t^{\dagger} \tilde{h}_{\mu}(\tilde{B}) \\ &= \tilde{h}_{\nu}^{\circ}(\tilde{B}) \sum_{\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin},+}} Q_t^{\dagger}(\tilde{B}, \tilde{A}) \frac{\tilde{h}_{\mu}(\tilde{A})}{\tilde{h}_{\nu}^{\circ}(\tilde{A})} \end{aligned}$$

with

$$Q_t^{\dagger}(\tilde{B}, \tilde{A}) := e^{-rt} \frac{\tilde{h}_{\nu}^{\circ}(\tilde{A})}{\tilde{h}_{\nu}^{\circ}(\tilde{B})} P_t^{\dagger}(\tilde{B}, \tilde{A})$$

Step 2: h-transformed Markov process

Proposition 2

$Q_t^{\dagger}(\tilde{A}, \tilde{B})$ are the transition probabilities of an irreducible, positively recurrent Markov process with state space $\tilde{\mathcal{P}}_{\text{fin},+}$.

Because $\frac{\tilde{h}_\mu}{\tilde{h}_{\overset{\circ}{\nu}}}$ can be shown to be bounded we obtain

$$\tilde{h}_{\overset{\circ}{\nu}}(\tilde{B}) \sum_{\tilde{A} \in \tilde{\mathcal{P}}_{\text{fin},+}} Q_t^{\dagger}(\tilde{B}, \tilde{A}) \frac{\tilde{h}_\mu(\tilde{A})}{\tilde{h}_{\overset{\circ}{\nu}}(\tilde{A})}$$

$$\underset{t \to \infty}{\longrightarrow} \quad h_{\overset{\circ}{\nu}}(B)\, c(\mu) = \int \overset{\circ}{\nu}(\mathrm{d}A) 1_{\{A \cap B \neq \emptyset\}}\, c(\mu)$$

Step 3: Local convergence

It suffices to show "**pointwise convergence**":

For $B \in \mathcal{P}_{\text{fin},+}$ we have

$$e^{-rt}\mu P_t(\{B\}) \underset{t\to\infty}{\longrightarrow} c(\mu)\mathring{\nu}(\{B\})$$

using the positively recurrent Markov chain.

Open problems:

Analogous results in the critical and supercritical regime.

Thank you for your attention!