Descriptive Set Theory and Operator Algebras

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1 Overview of the Field

Descriptive set theory is the theory of definable sets and functions in Polish spaces. It's roots lie in the Polish and Russian schools of mathematics from the early 20^{th} century. A central theme is the study of regularity properties of well-behaved (say Borel, or at least analytic/co-analytic) sets in Polish spaces, such as measurability and selection. In the recent past, equivalence relations and their classification have become a central focus of the field, and this was one of the central topics of this workshop.

One of the original co-organizers of the workshop, Greg Hjorth, died suddenly in January 2011. His work, and in particular the notion of a turbulent group action, played a key role in many of the discussions during the workshop.

Functional analysis is of course a tremendously broad field. In large part, it is the study of Banach spaces and the linear operators which act upon them. A one paragraph summary of this field is simply impossible. Let us say instead that our proposal will be concerned mainly with the interaction between set theory, operator algebras, and Banach spaces, with perhaps an emphasis on the first two items.

An operator algebra can be interpreted quite broadly ([4]), but is frequently taken to be an algebra of bounded linear operators on a Hilbert space closed under the formation of adjoints, and furthermore closed under either the norm topology (C^*-algebras) or the strong operator topology (W^*-algebras). The theory of these algebras began with the work of Murray and von Neumann in the 1930s, and has since expanded to touch much of modern mathematics, including number theory, geometry, ergodic theory, and topology.

The connection between descriptive set theory and functional analysis can be traced back at least as far as Mackey's work on group representations in the 1950s. There he recognized a fundamental obstruction to obtaining a satisfactory decomposition of a unitary representation of a second countable locally compact group as a direct integral of unitary equivalence classes of irreducible representations: the space of infinite-dimensional representations did not naturally carry the structure of a standard Borel space. This led to his "Type I iff smooth dual" conjecture for such groups, ultimately proved with C^*-algebra theory by James Glimm in 1961. These early connections have given way to a range of powerful new results connecting descriptive set theory to ergodic theory and to the theory of C^*- and W^*-algebras, results which have tantalizing prospects for the future.

2 Recent Developments and Open Problems

Recently, there has been great progress at the interface of operator algebras, descriptive set theory, and ergodic theory. Here we single out examples of four types, and explain how our workshop will advance research in each area. (We emphasize the first two types, as they exhibit particularly strong connections between these fields. And of course, the topics covered at the workshop will not be restricted to these four.)

I. Borel reducibility and the complexity of classification problems.

Mackey's opinion that Borel equivalence relations yielding non-standard Borel spaces are simply "unclassifiable" has more recently been countered with a rich theory of cardinality for such relations. Given Polish spaces X and Y carrying equivalence relations E and F, respectively, one says that (X, E) is *Borel reducible* to (Y, F) if there is a Borel map $\Theta : X \to Y$ with the property that

$$xEy \Leftrightarrow \Theta(x)F\Theta(y).$$

In words, assigning invariants to F-classes is at least as difficult as assigning them to E-classes; one writes $E \leq_B F$. There are infinitely many degrees of complexity in this picture, and the relationships between them remain murky in places. There are, however, some standout types: E is *classifiable by countable structures*, if, roughly, it is no more complex than the isomorphism relation on countable graphs; E is *turbulent* if there is a Borel reduction from the orbit equivalence (OE) relation of a turbulent group action into E; E is *below a group action* if there is a Borel reduction from E into the OE relation of a Polish group action (see [15], [2], [14]).

Several results regarding the Borel complexity of C^*- and W^*-algebras and group actions upon them have recently emerged. Sasyk-Törnquist ([29, 30, 28]) have established turbulence for isomorphism of von Neumann factors of all types, while Kerr-Li-Pichot ([20]) have done the same for various types of group actions on standard probablity spaces and the hyperfinite II_1 factor. On the C^*-algebra side, Farah-Toms-Törnquist ([13], [12]) have proved that the isomorphism relation for unital nuclear simple separable C^*-algebras (the primary object's in G. A. Elliott's K-theoretic classification program, [27], [8]) is turbulent yet below a group action, and have established a similar result for metrizable Choquet simplices.

Our workshop will bring together these and other researchers to work on new questions in Borel reducibility, such as assessing the complexity of exact and non-exact C^*-algebras and non-commutative L_p spaces (and perhaps finding therein an instance of the Kechris-Louveau conjecture ([19]) concerning E_1 and group actions), and determining whether various classification functors in functional analysis (K-theory for AF algebras, say) have Borel computable inverses.

II. Measure preserving group actions.

Three steadily weaker notions of equivalence for free ergodic actions of countable groups on a standard probability space are conjugacy, orbit equivalence, and von Neumann equivalence (isomorphism of the von Neumann algebra crossed products associated to $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$). The last of these is very weak, as any two actions of ICC amenable groups are equivalent in this sense. Nevertheless Popa has since 2002 developed a remarkable deformation/rigidity theory ([24], [25]) which has allowed him and subsequently many others to establish orbit equivalence from von Neumann equivalence for a wide array of non-amenable groups. This has led to striking results in the descriptive theory of orbit equivalence relations, including the proof (due to Ioana and Epstein, [9], [16]) that every countable non-amenable group admits continuum many orbit inequivalent actions, giving a strong converse to the Connes-Feldman-Weiss Theorem ([6]). More recently, Hjorth has used techniques derived from Popa-Ioana to prove that there are continuum many mutually $\leq_B$-incomparable Borel equivalence relations (see I. above).

We will address several problems in this theory at our workshop, including the extension of von Neumann rigidity to new and larger classes of groups, and the application of these results to gain finer descriptive understanding of orbit equivalence relations.

III. Banach spaces.

The descriptive theory of Banach spaces is another active area under this proposal's umbrella. Rosendal, Ferenczi and Louveau are the prime actors. Recent results include the proof that the isomorphism problem for separable Banach spaces is equivalent to the maximally complicated analytic equivalence relation in the Borel hierarchy, and a partial classification of Banach spaces in terms of minimal subspaces. The second item is part of Gowers' program to classify Banach spaces by finding characteristic spaces present in every space. That program will be pursued further at our workshop.

IV. The structure of C^*-algebras.

C^*-algebra theory has seen many old problems solved lately using set theory as a fundamental tool. (It must be said that the set theory involved is not really descriptive, but we nevertheless have another important interaction between set theory and functional analysis.) These results include the proof by Farah ([10]) and Phillips-Weaver ([23]) that the question of whether all automorphisms of the Calkin algebra are inner is independent of ZFC, and the Akemann-Weaver proof of the consistency of a counterexample to Naimark's problem ("Must a C^*-algebra with only one irreducible representation up to unitary equivalence be isomorphic to the compact operators on some Hilbert space?"), see [1]. Further questions to be addressed at our workshop include the possibility that a solution to Naimark's problem is consistent, and the question of whether the Calkin algebra admits a K_1-reversing automorphism.

3 Timeliness and Relevance.

The progress described above has led to a tremendous amount of new collaboration and dialogue between functional analysts and descriptive set theorists, albeit through a multitude of largely independent projects. That is why a 5-day workshop at BIRS on Descriptive Set Theory and Functional Analysis will be especially effective: we will not only disseminate research and lay the groundwork for progress on major problems in the field–any BIRS workshop should do as much–but also give new coherence to this interdisciplinary field. Success in this last goal will prove particularly helpful to young researchers wanting to enter the field, as they will get a panoramic view of its research and be able to discuss their own research with a cast of faculty never before assembled at a single meeting.

The profile of interdisciplinary research in set theory and functional analysis has been rising steadily. For instance, Texas A&M University hosted a 5-day conference on the topic in August, 2010, and there have been three Appalachian Set Theory Workshops (a NSF funded series) by Kechris, Törnquist and Farah discussing several of the recent results described in this proposal. Our BIRS workshop, however, was an order of magnitude more significant than these events, not least because of the quality of the participants. They included 7 ICM speakers and the present or erstwhile editors of *Journal of the American Mathematical Society*, *Fundamenta Mathematicae*, *Journal of Symbolic Logic*, *Bulletin of Symbolic Logic*, *Canadian Journal of Mathematics*, *Journal of Functional Analysis*, *Pacific Journal of Mathematics*, and *Journal of Operator Theory*.

As for timeliness, we would point out, in addition to the conference activity mentioned above, that most of the significant results motivating this workshop have appeared in the last five years.

4 Schedule of the Workshop.

The following lectures were delivered over the course of the week.

Monday, June18

1. Dima Shlyakhtenko, *Free monotone transport*
2. Bradd Hart, *Model theory of tracial von Neumann algebras*
3. Vern Paulsen, *The Kadison–Singer problem*
4. Asger Törnquist, *A Fraisse-theoretic approach to the Poulsen Simplex*
5. Stuart White, *Perturbations of crossed products*

Tuesday, June19

1. Simon Thomas, *A descriptive view of unitary group representations*
2. Stefaan Vaes, II_1 *factors with a unique Cartan decomposition*
3. Juris Steprans, *Topological centres of group actions*
4. David Kerr, *Independence and entropy in topological dynamics*
5. Ed Effros, *Some personal reflections on QFA (Quantized Functional Analysis)*

Wednesday, June20

1. Tristan Bice, *Calculus of projections in C*-algebras*
2. Justin Moore, *Spatial models of Boolean actions*
3. Cyril Houdayer, *A class of* II_1 *factors with an explicit abelian amenable subalgebra*

Thursday, June 21

1. Jesse Peterson, *Stabilizers of ergodic actions of lattices and commensurators*
2. N. Christopher Phillips, *Outer automorphisms of the Calkin algebra*
3. Problem session
4. Aleksandra Kwiatkowska, *Boolean actions on groups of isometries*
5. Caleb Eckhardt, *Amenable group C*-algebras*
6. Martino Lupini, *Non-classification of automorphisms of C*-algebras up to unitary equivalence*

Friday, June 22

1. Hiroshi Ando, *Finite-type Polish groups and Popa's problem*
2. Todor Tsankov, *Generic representations of abelian groups*
3. Simon Wasserman, *Factorial representations of nonseparable C*-algebras*

5 Questions and Problems.

The following problems were collected during the Thursday problem session.

(1) (Caleb Eckhardt) For a discrete, amenable group Γ; Is $I(\Gamma) := \ker(C^*(\Gamma) \longrightarrow \mathbb{C})$ ever the unique maximal ideal?

(2) (George Elliott) Let Γ be a discrete group. Is Γ type 1 if and only if $C^*(\Gamma)$ has continuous trace?

(3) (Ilijas Farah) What is the complexity of isomorphism of separable C^*-algebras?

Remark 1

An upper bound for this classification problem was found shortly after the workshop (see §6.1).

(4) (N. Christopher Phillips) How does the complexity of isomorphism of separable C^*-algebras compare with the complexity of complete isometric isomorphism of:

- non-self-adjoint subalgebras.
- von Neumann algebras with separable preduals.

Remark 2 (Vern Paulsen) *The complexity is very large; At least as bad as complete isometric isomorphism of all Banach spaces.*

(4') (N. Christopher Phillips) Complexity of complete isometric isomorphism of AF algebras vs. triangular AF algebras.

Remark 3

While AF algebras are classifiable by countable structures (more precisely, by K_0), no reasonable classification of non-selfadjoint algebras is known (see [26]). A proof that the isomorphism relation of triangular algebras is not classifiable by countable structures would show that no reasonable classification is possible.

(5) (Justin Moore) Does $\{C^\infty$ functions $M \longrightarrow U(1)\}$, where M is a compact smooth manifold, have the point realisation property?

Remark 4 *It is known that for all k, $\{C^k$ functions $[0,1] \longrightarrow U(1)\}$ does not have the point realisation property.*

(6) (Vern Paulsen) Let G be a countable discrete group with a pure state on $\ell^\infty(G)$ that uniquely determines a pure state on $L(G)$. Does this extend uniquely to $B(\ell^2(G))$?

Remarks 1

- *This is an open question for all countable, discrete groups G.*
- *It is hard for $\mathbb{Z}$, but might be easier for more complicated groups.*
- *q points (always extend uniquely to $B(\ell^2(G))$) and special filters may be relevant.*
- *Related to $L(G)$ being pavable in the Anderson sense.*
- *Could ask the global question: Do all pure states extend uniquely?*
- *The motivation for this question comes from Kadison–Singer problem ([5]),*

(7) (N. Christopher Phillips) Does $L(\ell^p)/K(\ell^p)$ have outer automorphisms?

Remark 5 *In some cases, the answer is certainly no (e.g. when the quotient is $\mathbb{C}$). $p = \infty$ might be interesting.*

(8) (Ilijas Farah) Is there an analogue of turbulence/some criteria for non-reducibility to unitary conjugacy of normal operators?

(9) (Wilhelm Winter) Does every trace on the countable ultraproduct of unital, separable, tracial C^*-algebras come from an ultraproduct of traces?

Remarks 2 – *(Ilijas Farah) The answer is no if the continuum hypothesis is assumed. The counterexample is commutative and has many traces (uses cardinalities).*

– *The following may be relevant: Unique trace; UHF-algebras; Property SI.*

– *Matui-Sato have positive answers.*

(10) (Bradd Hart) What are the values of σ_n in II_1 factors, where

$$\sigma_n := \sup_{\substack{x_1,\ldots,x_n \\ \|x_j\|\leq 1}} \inf_{\substack{y \\ \|y\|\leq 1}} (\|y^*y-1\|_2 + |\operatorname{tr}(y)| + \sum_{j=1}^{n} \|[x_j, y]\|_2)?$$

Remarks 3 – *This is known for free groups and hyperfinite II_1 factors.*

– *If they are all true (i.e. $\sigma_n = 0$ for all n) then this is property Γ (implies existence of a nontrivial central sequence).*

(11) (Justin Moore) Conjecture: Let $\mathbb{T}$ be a free binary system on one generator, then there is an idempotent finitely additive probability measure on $\mathbb{T}$ with respect to Arens products.

Remarks 4 – *A positive answer implies that the Thompson group is amenable, and potentially the above is a stronger statement.*

– *Are there algebraic conditions on a binary system that imply the existence of an idempotent measure?*

(12) (Simon Thomas) Conjecture: Unitary equivalence on representations of $\mathbb{F}_2$ is strictly more complicated than unitary equivalence of representations of $\mathbb{Z}$.

(13) (Martino Lupini) Is the relation of conjugacy on the automorphism group of the CAR-algebra Borel?

(14) (Hiroshi Ando) Is it true that $\mathcal{U}(\ell^2)_p = \{u \in \mathcal{U}(\ell^2) : u - 1 \in S^p(\ell^2)\}$ is $\mathcal{U}_{fin}$?

Remark 6 *Shlossberg-Megrelishvili showed that ℓ^p is UR if and only if $1 \leq p \leq 2$.*

6 Scientific Progress Resulting from the Workshop.

This workshop was particularly successful in starting new collaborations and finding solutions to prominent open problems.

6.1 Isomorphism relation of separable C*-algebras

One of the central problems about the complexity of the isomorphism relation of separable C*-algebras is whether it is Borel-reducible to an orbit equivalence relation of a Polish group action. A partial positive answer was given by Farah–Toms–Törniquist ([13]) using a Borel version of Kirchberg's $\mathcal{O}_2$-embedding theorem. A novel approach to this problem was suggested by Vern Paulsen during the workshop. In a rapid email interchange in the week following the workshop a positive answer to this problem was given by Elliott, Farah, Paulsen, Rosendal (who could not attend the workshop but joined the email correspondence), Toms and Törnquist. The construction, while technically simpler than the earlier one by Farah–Toms–Törnuist, is general enough to show that the isometry of non-self-adjoint operator algebras as well as the complete isometry of operator spaces are Borel-reducible to orbit equivalence relations.

6.2 Traces on ultrapowers

Recently Matui-Sato made a conceptual breakthrough in the fine structure of nuclear C*-algebras ([21], [22]). They introduced excision techniques to the stably finite classification programme, which simplifies an important technical argument of Winter ([32], [31]). For separable simple unital nuclear C*-algebras with finitely many extremal traces they found a method for extracting the critical large central sequences required to run these arguments from the central sequences found in tracial von Neumann closures. In this way Matui-Sato where able to very an implication of the important Toms-Winter regularity conjecture for simple unital nuclear C*-algebras with finitely many extremal traces.

Our work has focused on weakening the assumption on the trace space: prior to the BIRS workshop Toms and White had produced an outline for extending Matui-Sato to the case of where the tracial state space has a zero dimensional compact extreme boundary. Whilst at BIRS, Toms-White-Winter were able to find a marriage of these new techniques with earlier techniques of Winter ([32], [31]) and produced a strategy for extending to a finite dimensional compact extreme boundary. Subsequent to the workshop this strategy has been completed and a paper is in preparation.

Tristan Bice has made progress on problem (9) from the list, by constructing a ZFC example of a separable C*-algebra and a trace on its ultrapower that does not arise as an ultraproduct of traces.

6.3 Classification of automorphisms of C*-algebras

At the time of the BIRS meeting, Lupini could prove nonclassification for C*-algebras that contain a central sequence which is not uniformly central. He then asked if all non continuous trace C*-algebras have this property, and George Elliott observed that at least non type I C*-algebras do.

6.4 Model theory of metric structures

A topic that has prominently emerged in the last few years is applications of model theory to operator algebras. Model theory of metric structures was developed by Ben–Ya'acov, Berenstein, Henson and Usvyatsov ([3]). It was adapted to operator algebras in [11]. Bradd Hart gave a well-received talk on applications of this logic to C*-algebras and tracial von Neumann algebras. At the moment it is not clear how these methods can be applied to other operator algebras, most importantly to type III von Neumann algebras. In conversations with Hiroshi Ando and Dima Shlyakhtenko, Hart started developing an approach to this problem.

7 Outcome of the Meeting

From the point of view of the organizers (who have some experience with BIRS workshops!), this was and extremely productive meeting. The number of new collaborations begun and problems solved was very high, and this is largely attributable to having met one of the workshop's original goals: bring together a diverse collection of researchers in functional analysis and set theory in hopes that their collective knowledge will allow them to solve problems hitherto out of reach for either field on its own.

The progress made at the workshop is expected to be pushed further at the Fields Institute Program on Forcing this fall, and in particular at the 5-day workshop on applications of set theory to C^*-algebras from September 10-14. Another event that should solidify gains promised by our BIRS workshop is the Oberwolfach meeting on C^*-algebras, dynamics, and classification in early November. Overall, we consider the workshop to have been very successful.

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