Conformal and CR Geometry

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1 Overview of the Field

Conformal and CR geometry has been a rich topic of study going back to the work of Cartan in the early 20th century. It continues to be the source of interesting new problems as well as to spawn new developments in related areas. It is characterized as a meeting ground of researchers from a wide variety of backgrounds in geometry, analysis, and algebra.

Cartan's algebraic point of view on the topic has inspired the development of the theory of parabolic geometries, curved geometries modeled on homogeneous spaces G/P, where G is a semisimple Lie group and P a parabolic subgroup. This side of the subject draws heavily on representation theory. A general theory of parabolic geometries has been developed, and much recent effort has been devoted to the study of specific examples of the theory beyond the basic examples of the conformal and CR cases. Topics here include BGG sequences and symmetric examples of special geometries. Even for conformal and CR geometry, only recently has important progress been made towards a good understanding of holonomy reduction.

Another side of the subject also with an algebraic flavor concerns the study of asymptotics and its connection with underlying geometric objects. Poincaré-Einstein metrics were introduced formally as a tool to study conformal geometry of the boundary at infinity; an analogous correspondence is manifested in other parabolic geometries. These metrics were subsequently studied as models for the celebrated AdS-CFT correspondence in theoretical physics, and have since evolved to become an arena for a wide class of geometric and analytic questions. Challenges of an analytic flavor are to construct families of such metrics and study their dependence on the conformal structure on the boundary. Many tools that have been developed in this connection (for example the renormalized volume) have found far-reaching applications, some of which were presented in the conference.

Conformal geometry has long been a fertile area for the study of variational problems. The Yamabe problem of finding a constant scalar curvature metric in a conformal class reduces to solving an nonlinear equation involving the conformal Laplacian and remains a model problem inspiring much work in geometric analysis. More recently higher-order and fully nonlinear analogues of this variational problem have been studied, for example in connection with Bransons Q-curvature and the σ_n-curvatures.

The conformal/CR invariant Laplacian and BGG sequences provide new set up for the index theorems. The heat kernel asymptotics of the conformal Laplacian gives a conformal index, and the conformal variation of the functional determinant of conformal Laplacian gives the Q-curvature. These studies can be extend to a lager class of invariant operators. In CR geometry, the de Rham complex can be refined to Rumin complex, which is an example of BGG sequences, and one may define the analytic torsion of the complex. This gives a variant of Ray-Singer metric in CR geometry.

2 The Workshop

The workshop brought together researchers from a variety of backgrounds who find common ground in conformal and CR geometry. There were 42 participants, representing a wide spread of experience levels from senior researchers to postdocs and graduate students. The number of lectures was limited to 22 to leave plenty of time for discussion and collaboration among and between the various research groups present.

A one hour expository lecture by a leading researcher known to be an excellent speaker opened the morning session of each of the three full days. Each of these lectures presented an overview of one aspect of the subject to the diverse audience and provided background for other lectures in that area. Seven lectures of 30 minutes each were delivered by young researchers on their work. The remaining twelve lectures were each of 50 minutes duration and reported on recent progress.

The discussion and interaction during the open periods was lively. It included significant mentoring of junior participants by senior researchers, in which questions were answered in detail and issues raised in the talks were elaborated. Other discussion led to new projects and collaboration. Among such new projects initiated as a result of interations at the workshop were the following.

Bent Ørsted and Yoshihiko Matsumoto started a collaboration on the total Q-curvature for CR manifolds. The talk of Ørsted presented a general framework on the variation of global CR invariants at the flat model, but no examples of CR invariants were discussed. It turns out that an example of such a global invariant is given by the total Q-curvature for partially integrable CR structures, as constructed via Matsumoto's ACHE metric.

Based on the talk of Jeffrey Case on the Q'-curvature on 3-dimensional CR manifolds, Hirachi found a way to define Q'-curvature in terms of the ambient metric that can be formulated in all dimensions. The integral of Q'-curvature gives a global invariant of strictly pseudo convex domains in $\mathbf{C}^n$. This provides another example of an invariant for which the framework of Ørsted can be applied.

3 Talks

The topics covered by the workshop are well-represented by the talks given. We have divided the talks into five sometimes overlapping subject areas.

1. **Algebraic Aspects and Parabolic Geometries**

Michael Eastwood delivered a one hour lecture presenting an overview of conformal and CR geometry from the perspective of representation theory. He explained Cartan's description of geometry modeled on homogenous spaces G/P, where G is semisimple and P is a parabolic subgroup — such geometries are now called *parabolic geometries*. The examples include conformal and CR geometries. Manifolds with such structures admit invariant operators acting on irreducible bundles induced by the representations of the parabolic P; these operators play essential roles in parabolic geometry. Some of these operators form natural complexes, e.g., the de Rham complex, the deformation complex, the Rumin complex; they can be constructed in a unified manner and are called *Bernstein-Gelfand-Gelfand complexes*. There are other invariant operators that are not operators in the complexes, e.g., Yamabe operators, Paneitz operators and more generally GJMS operators. These are constructed via the Fefferman-Graham ambient metric and/or tractor calculus. These invariant operators are important objects of study and some of the subsequent talks considered various aspects related to them.

The lecture of *Bent Ørsted* served to highlight how the representation theory of conformal and CR geometry can be used to study analytical problems that arise in these fields. He explained (in part based on joint work with N. M. Møller) the application of representation theory in the study of natural functionals in conformal and CR geometry. The functionals may be thought of as certain functions on the space of metrics or complex structures, and he gave examples of stationary points and local extrema. In particular for conformal and CR spheres, he treated determinants of the Yamabe operator and total Q-curvature (the integral of Q-curvature). The problem can be reduced to the study of eigenvalues of intertwinors between the bundles of metrics or infinitesimal deformations thereof and representation theory gives a complete answer for the sphere. Among other things, this provides an alternate proof of a result of Okikiolu on the determinant of Yamabe operator.

Andreas Čap presented results of joint work with Rod Gover concerning projective compactness of affine connections and pseudo-Riemannian metrics. They introduce a parameter measuring an order of projective compactness, which turns out to describe asymptotic behavior of volume growth near the boundary. In the case that the order is 1 or 2, they relate holonomy reductions to the condition that a defining density for the boundary be a solution of a first BGG operator.

Pawel Nurowski reported on joint work with Ian Anderson concerning explicit formulae for Fefferman-Graham ambient metrics of the conformal structures defined by Nurowski from the data of a generic 2-plane field in 5 dimensions. The explicit ambient metrics have remarkable properties in addition to the fact that they are explicit: among other things their expansions can be made to terminate at any desired order.

Katharina Neusser presented a joint work with Robert Bryant, Michael Eastwood and Rod Gover on a general method for constructing complexes of invariant differential operators on manifolds endowed with geometric structures given by bracket generating distributions. For certain cases, using the distribution, the constructed complexes provide fine resolutions of the sheaf of locally constant functions which can serve as alternatives to the de Rham complex. In the case of parabolic geometries, this construction recovers the Bernstein-Gelfand-Gelfand complexes associated to the trivial representation; however, the new constructions are relatively simple and avoid most of the machinery of parabolic geometry.

Matthias Hammerl presented a result of joint work with Čap, Gover and Graham. The result identifies the infinitesimal holonomy of the Fefferman-Graham ambient metric of a conformal structure with the holonomy of the normal connection of its standard tractor bundle. The result generalizes recent work of Graham/Willse concerning ambient extension of parallel tractor fields. The proof uses previous work of Čap/Gover expressing ambient objects in terms of tractors.

Travis Willse gave a talk concerning highly symmetric generic 2-plane fields on 5-manifolds. Cartan solved the equivalence problem for generic 2-plane fields in 5 dimensions, and in the process showed that the maximal possible symmetry group was the split real form of the exceptional group G_2. Willse's work concerned examples with large but sub-maximal symmetry group. He identified explicitly the Fefferman-Graham ambient metric of the associated Nurowski conformal structure and showed that the ambient holonomy as well as the conformal holonomy are both equal to the Heisenberg 5-group.

2. **Geometric Analysis and PDE**

Andrea Malchiodi gave a one hour talk which was an overview of both older and more recent work on the conformal and CR Yamabe problems. While the classical Yamabe problem was the most well-known (and very well-studied) PDE naturally arising in conformal geometry, its resolution left open the question of compactness of the space of solutions in the positive case. The lecture reviewed the recent positive and negative results on this question. It also introduced the analogous question in CR geometry and reviewed the results obtained there.

Pierre Albin gave a talk on the conformal surgery and compactness of isospectral surfaces. He first reviewed the work of Melrose and Osgood-Phillips-Sarnak on the compactness of isospectral closed surfaces. He then explained the work of Hassel-Zelditch on the notion of relative isospectrality for the Dirichlet Laplacian for exterior domains, making use of the identity of the trace of heat kernels. Next, the work of Borthwick and Perry on isoresonant surfaces (for which the resolvents have the same scattering poles) was recalled. He then introduced the class of bordered surfaces with funnel ends and cusp ends, and formulated the main result (joint work with Aldana and Rochon) stating that for such complete surfaces which coincide outside a compact set, the relatively isospectral ones form a compact set in the C^∞ topology. The main tool is Cheeger-Gromov criteria for compactness of metrics, which requires control of the injectivity radius. To control the cusp ends, he introduced the notion of conformal surgery, in which a delicate analysis is used to verify that the relative determinant behaves correctly under a suitable limit of the conformal surgery. This analysis reduces the compactness result to that of Borthwick-Perry discussed earlier.

Antonio Ache presented recent work on removability of singularities for obstruction-flat metrics with constant scalar curvature. The obstruction tensor, which originates in the work of Fefferman-Graham on the ambient metric, is a higher-order conformally invariant symmetric 2-tensor which exists only in even dimensions. Ache presented analogues of results that are known to hold for minimal submanifolds and for Ricci-flat metrics, using the well-known results of Leon Simon as a point of departure.

Yi Wang presented work showing how the finiteness and suitable smallness of Q-curvature for a complete conformaly flat 4-manifold controls the geometry of the manifold via an isoperimetric inequality. The lecture brought in the theory of quasiconformal mappings to construct appropriate bi-Lipschitz parametrizations.

3. **Poincaré-Einstein Manifolds**

Colin Guillarmou gave a one hour lecture which was an outline of recent work on renormalized volume for Poincaré-Einstein metrics. Starting from a discussion of PE metrics (M, g) as generalizations of hyperbolic space with an induced conformal structure on the sphere at infinity, he discussed the asymptotics of the metric g at the boundary ∂M. He then introduced the renormalized volume functional, which is well-defined when the boundary dimension n is odd. The rest of his lecture was devoted to joint work with S. Moroianu and Schlenker on the case of even-dimensional boundaries. In that case the renormalized volume is a function of the metric in the conformal class at the boundary. Its critical points and extrema relate to the well-studied v_n-functional. He concluded by showing that for even-dimensional boundaries, the space of Cauchy data at the boundary (Dirichlet and Neumann, in the asymptotic expansion at ∂M) of suitably normalized PE metrics is a Lagrangian submanifold of the cotangent space of the space of conformal structures on ∂M.

Maria del Mar González gave a talk on fractional order conformally covariant operators in the setting of Poincaré-Einstein spaces. This is a topic which has generated a lot of activity in the PDE community due to the analytic work of Caffarelli-Sylvestre. She first explained the point of view of Caffarelli-Sylvestre in which they viewed the fractional order equation as a weighted second order equation in the upper half space. She discussed her joint work with Alice Chang in which they extended the energy identity for the fractional order operators whose leading term is of the form Δ^γ, $\gamma \leq 1$, to the setting of Poincaré-Einstein spaces. She then explained her joint work with J. Qing on the solution of the analogue of the Yamabe equation for the fractional order operator. The main result is the existence of energy minimizing solutions when the Sobolev quotient is strictly smaller than that of the hyperbolic space; a particular instance arises when the boundary is non-umbilic somewhere. A key analysis is the extension of the Hopf Lemma to the setting of this equation which provides the basic boundary regularity for the minimizing solution. As a consequence, they obtain a sharp weighted Sobolev trace inequality for such Poincaré-Einstein spaces. A key open question is whether there is a positive mass theorem for this operator. In the last part of the lecture, she briefly outlined on-going joint work with Frank-Monticelli-Tan on an extension of this operator to the CR setting when the CR structure is the boundary at infinity of a complete Kähler-Einstein metric.

The lecture of *Jie Qing* introduced a new aspect of the correspondence between horospherically convex hypersurfaces whose boundary at infinity bounds a domain (at infinity), and complete conformal metrics over these subdomains.

Andreas Juhl gave a lecture concerning his "building blocks" for the GJMS operators. In previous work he showed that the GJMS operators can be expressed by universal formulae in terms of second order building block operators. In this lecture he discussed a relation between the building block operators for a metric g and the corresponding operators for the metric in one higher dimension which arises as the compactification of the associated Poincaré-Einstein metric in normal form.

Rod Gover lectured on collaborative work with Andrew Waldron and Emanuele Latini concerning a calculus for boundary problems on Poincaré-Einstein manifolds. The work studies extension problems for differential forms formulated in terms of solving suitable Proca-type problems in the bulk. Among the many consequences are holographic formulae for the conformally invariant operators and generalized Q-curvatures of Branson-Gover on the boundary at infinity.

4. **CR Geometry**

Sagun Chanillo presented a joint work with Hung-Lin Chiu and Paul Yang on the CR Paneitz operator P_4 on 3-dimensional CR manifolds. P_4 is a unique 4th order CR invariant differential operator with leading part Δ_b^2. This talk gave an intimate link between the spectrum of P_4 and the global embeddability of 3-dimensional compact CR manifolds into complex manifolds, which is a main problem in CR geometry. (For higher dimensions the embeddability is know to be equivalent to the integrability of the CR structure.) Chanillo discussed the proof that if P_4 is non-negative and the CR-Yamabe constant is positive, then the CR manifold is embeddable. He also discussed the converse.

Jan Slovák presented a joint work with Gerd Schmalz on free CR-distributions. These are a geometric structure generalizing generic real n^2-codimensional submanifolds in $\mathbf{C}^{n+n^2}$. This is another instance of parabolic geometry; in the lowest dimensional case $n = 1$, it agrees with usual 3-dimensional CR manifolds. Free CR-distributions admit properties very similar to conformal and CR geometries: for example, the Fefferman construction gives a natural circle bundle with a conformal structure, which is modeled on skew-Hermitian matrices in the same way as the conformal 4-dimensional case. The construction of basic invariants and invariant operators of the geometry is carried out by using the classical exterior calculus and the cohomological data known for the parabolic geometry defined by these distributions.

Jeffrey Case lectured on a collaboration with Paul Yang, in which they considered a variant of the Paneitz operator P_4 on 3-dimensional CR manifolds. Based on the fact that $\ker P_4$ contains all CR pluriharmonic functions, they defined a 4th order differential operator P_4' acting on pluriharmonic functions — it generalizes the operator discovered by Branson, Fontana and Morpurgo on the sphere S^3. As an analog of the derivation of Q-curvature from P_4, they defined a Q'-curvature from P_4' and studied the question of finding a contact form with constant Q'-curvature.

Yoshihiko Matsumoto presented recent results on the asymptotics of asymptotically complex hyperbolic Einstein (ACHE) metrics, the CR counterpart of well-studied asymptotically hyperbolic Einstein metrics. An important aspect regarding ACHE metrics is that the boundary CR structures are just "partially integrable" in general, and it is surprising that a local obstruction to the existence of smooth solutions to the Einstein equation can occur only for non-integrable structures. As an application, he constructed CR invariant powers of the sub-Laplacian and Q-curvature for partially integrable CR manifolds via the Graham-Zworski approach.

5. **Index Theory**

Xianzhe Dai discussed his joint work with X. Huang on intersection R-torsion for manifolds with conical singularities. He first reviewed the half analytic torsion of Cheeger as a conformal invariant for even dimensional manifolds. The Ray-Singer conjecture/theorem says that the analytic torsion equals the R-torsion for closed manifolds. For manifolds with conical singularities, the analytic torsion can be defined using Cheeger's L^2 theory while the R-torsion is defined in terms of the intersection cohomology theory of Goresky-MacPherson. Dai and Huang showed that there is a new geometric contribution from the conical singularity in the Ray-Singer conjecture. They then prove a formula for the intersection R-torsion of a finite cone and use it to introduce a family of spectral invariants which is closely related to Cheeger's half torsion, a concept which has found application in conformal geometry.

Zhiqin Lu spoke about his joint work with Chiung-ju Liu on the Tian-Yau-Zelditch (TYZ) expansion of the Bergman kernel on singular Riemann surfaces. It is well-known that the Bergman kernel defined from the sections of the m-th powers of an ample bundle over a smooth manifold has an expansion in powers of m, called the TYZ expansion. As a first step to generalize the TYZ expansion to a degenerating family of compact Riemann surfaces, they tried to give upper and lower C^0-estimates of the Bergman kernel, which amounts to studying the leading term of the expansion. A proof was given for the case when a single Riemann surface in the family has one ordinary double point.

4 Participants

Antonio Ache, University of Wisconsin
Pierre Albin, University of Illinois Urbana-Champaign
Spyros Alexakis, University of Toronto
Eric Bahuaud, Stanford University
Andreas Čăp, University of Vienna
Jeffrey Case, Princeton University
Alice Chang, Princeton University
Sagun Chanillo, Rutgers University
Jih-Hsin Cheng, Academica Sinica Taipei
Xianzhe Dai, University of Calfornia Santa Barbara
Luca di Cerbo, Duke University
Michael Eastwood, Australian National University
Maria del Mar González, Universitat Politècnica de Catalunya
A. Rod Gover, University of Auckland
Robin Graham, University of Washington
Colin Guillarmou, École Normale Supérieure Paris
Matthew Gursky, University of Notre-Dame
Matthias Hammerl, University of Vienna
Kengo Hirachi, University of Tokyo
Dmitry Jakobson, McGill University
Thalia Jeffres, Wichita State University
Andreas Juhl, Uppsala University
Zhiqin Lu, University of Calfornia Irvine
Farid Madani, Universität Regensburg
Andrea Malchiodi, SISSA
Yoshihiko Matsumoto, University of Tokyo
Stephen McKeown, University of Washington
Brendan McLellan, Center for Quantum Geometry of Moduli Spaces
Katharina Neusser, Australian National University
Pawel Nuroski, University of Warsaw
Bent Ørsted, Aarhus University
Jie Qing, University of Calfornia Santa Cruz
Nicholas Reichert, Princeton University
Katja Sagerschnig, Australian National University
Jan Slovák, Masaryk University
Petr Somberg, Charles University
Vladimír Souček, Charles University
George Sparling, University of Pittsburgh
Dennis The, Australian National University
Yi Wang, Princeton University
Travis Willse, Australian National University
Paul Yang, Princeton University

5 Talks

The following is the list of talks, in the order they were delivered.

Colin Guillarmou: On the renormalized volume
Jie Qing: Hypersurfaces in hyperbolic space and conformal metrics on domains in sphere
Pierre Albin: Compactness of relatively isospectral sets of surfaces
Xianzhe Dai: Cheeger's half torsion and cone
Zhiqin Lu: On the Tian-Yau-Zelditch expansion on Riemann surfaces
Michael Eastwood: Conformal and CR geometry from the parabolic viewpoint
Andreas Juhl: On the building blocks of GJMS-operators
Bent Ørsted: Extremal properties of natural functionals in conformal and CR geometry
Pawel Nurowski: More explicit Fefferman-Graham metrics with G_2 holonomy
Jan Slovák: Free CR-distributions
Andreas Čap: Projective compactness
Rod Gover: Conformal geometry, holography, and boundary calculus
Matthias Hammerl: Ambient and conformal holonomy
Kathatina Neusser: Some complexes of differential operators
Andrea Malchiodi: Recent progress on the Yamabe problem
Sagun Chanillo: Embedding CR 3-manifolds
Maria del Mar González: Fractional order operators in conformal geometry
Yi Wang: Quasiconformal mappings, isoperimetric inequality and finite total Q-curvature
Antonio Ache: Asymptotics in the study of obstruction-flat metrics
Yoshihiko Matsumoto: Asymptotics of ACH-Einstein metrics
Jeffrey Case: A Paneitz-type operator for CR pluriharmonic functions
Travis Willse: Highly symmetric generic 2-plane fields on 5-manifolds and Heisenberg 5-group holonomy