Structural Ramsey Theory and Generalized Indiscernibles

Lynn Scow

University of Illinois at Chicago

Neostability Theory BIRS Workshop 2012

Outline

order indiscernibles

- Fix a linear order O and an L-structure M (may assume we are working in a monster model).
- Consider the well-known definition for *indiscernible sequence* (the a_i are same-length tuples from M):

Definition

$(a_i : i \in O)$ is an indiscernible sequence if for all finite n and sequences $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ from O

$$i_1 < \ldots < i_n \text{ and } j_1 < \ldots < j_n \Rightarrow$$

$$\mathrm{tp}^L(a_{i_1}, \ldots, a_{i_n}; M) = \mathrm{tp}^L(a_{j_1}, \ldots, a_{j_n}; M)$$

recast

- Consider O as a structure in its own right, $\mathcal{O} = (O, <)$ in the language $L' = \{<\}$, and rewrite the definition for the purposes of generalization:

Definition

$(a_i : i \in O)$ is an indiscernible sequence if for all finite n and sequences $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ from O,

$$\mathrm{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{O}) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{O}) \Rightarrow$$

$$\mathrm{tp}^{L}(a_{i_1}, \ldots, a_{i_n}; M) = \mathrm{tp}^{L}(a_{j_1}, \ldots, a_{j_n}; M)$$

generalized indiscernibles

- Now we fix an arbitrary L'-structure $\mathcal{I}$ in the place of $\mathcal{O}$.

Definition ([She90])

We say that $(a_i : i \in I)$ is $\mathcal{I}$-*indexed indiscernible* if for all finite n and sequences $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ from I,

$$\mathrm{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{I}) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{I}) \Rightarrow$$

$$\mathrm{tp}^{L}(a_{i_1}, \ldots, a_{i_n}; M) = \mathrm{tp}^{L}(a_{j_1}, \ldots, a_{j_n}; M)$$

- We often fix an index set I and look at a variety of structures $(\mathcal{I})$ that we can put on this set by way of different languages (L').
- In this case, a sequence $(a_i : i \in I)$ may be referred to as L'-*generalized indiscernible*, if it is $\mathcal{I}$-indexed indiscernible for some understood L'-structure on I.

observations

- If $\mathcal{I}$ is an L'-structure and $L^* \subseteq L'$ is some reduct, then any $(a_i : i \in I)$ that is L^*-generalized indiscernible is automatically L'-generalized indiscernible.
- This is because L' "weakens the hypothesis" of a conditional that is already true.
- The other direction is nontrivial.
- Think of $L' = \{<\}$, $L^* = \{\}$ in $M \models T$ for unstable T.
- There are indiscernible sequences that fail to be indiscernible sets.

examples

- These indiscernibles have been used in work of Baldwin-Shelah, Džamonja-Shelah, Laskowski-Shelah, Kim-Kim, and Guingona, among others.
- They have had great utility in studying a variety of tree-properties: e.g. (k-)TP, (k-)TP_1, (k-)TP_2. Hopefully this success may be extended to the case of SOP_1, SOP_2.
- Each of the above properties stipulates the existence of a formula and parameters $(a_i : i \in \beta^{<\lambda})$ exhibiting some consistency-inconsistency pattern, usually indexed by some kind of language you can put on the tree.
- To narrow in on the important aspects of the pattern, one assumes that the witnesses are indiscernible with respect to some appropriate language L' on the tree $\beta^{<\lambda}$.
- That you may assume such indiscernible witnesses exist *retaining the pattern* is often a difficult thing to prove, in and of itself.

restrictions on $\mathcal{I}$

- In the original presentation, it was assumed that

  (†) complete quantifier-free types in $\mathcal{I}$ are equivalent to formulas

  (e.g. L' is finite relational)
- This has been reflected in examples in the literature: e.g.
  - $\mathcal{I}=\beta^{<\omega}$ for $\beta = 2, k, \omega$ and $L' = \{\trianglelefteq, \wedge, <_{\text{lex}}\}$
  - $\mathcal{I}=\mathcal{R}$ a graph for $L' = \{R, <\}$
- In fact, it is a question how general we can make $\mathcal{I}$ and retain the utility of the original order indiscernible sequences.
- Say that $\mathcal{I}$ is *quantifier-free oligomorphic (qfo)* if there are finitely many quantifier-free n-types in $\mathcal{I}$, for each n.
- This is one way to obtain (†).
- With inspiration from the trees case, we focus here on uniformly locally finite structures $\mathcal{I}$ in a finite language ($\Rightarrow$ qfo)

existence

- One of the first questions we can ask for different pairs $(\mathcal{I}, M)$ is whether an $\mathcal{I}$-indexed indiscernible in M exists.
- Consider qfo $\mathcal{I}$ and $\mathcal{I}$-indexed indiscernible $(a_i : i \in I)$ living in M.
- Let $f : I \to M^k$ send $i \mapsto a_i$.
- For $\emptyset$-definable sets $D \subseteq (M^k)^m$, it must be the case that $f^{-1}(D)$ is a union of quantifier-free m-types in $\mathcal{I}$.
- Thus, the induced structure from M on the indiscernible is a reduct of the language of $\mathcal{I}$.

nonexamples

- $M = (\mathbb{Q}, <)$ does not admit (nontrivial, symmetric) graph indexed indiscernibles, where the indexing language is $L' = \{R\}$.
- The same M does not admit $\mathcal{I}$-indexed indiscernibles, where $\mathcal{I}$ is the structure on $2^{<\omega}$ in $L' = \{\trianglelefteq, \wedge\}$.
- Both problems can basically be fixed by adding a linear order $\{<\}$ to L'.
- In fact, by a previous observation, we always have existence for a linearly-ordered $\mathcal{I}$ by Ramsey's theorem.

$$L^* = \{<\}, \quad L' = \{<, \text{ other relations } \ldots\}$$

- A more interesting question comes out of studying the obstruction on the side of M.

$(\mathcal{I}, M)$

- The relation $(\mathcal{I}, M)$ on structures "M admits an $\mathcal{I}$-indexed indiscernible", is not quite one of interpretability.
- $\mathcal{I}$ is embedded in a power of M by $i \mapsto a_i$, but the set of a_i (the domain) is not usually definable.
- Even if the domain were definable, it is only a reduct of the structure on $\mathcal{I}$ that is necessarily interpreted in M.
- However, it is possible to learn something about M if it admits an $\mathcal{I}$-indexed indiscernible in a non-proper way:
- e.g., if M admits an order(ordered graph)-indexed indiscernible [with maximal age] that is not $\{=\}(\{<\})$-generalized indiscernible, then M is unstable(IP). [She90] ([Sco12])

based on: I

- There is a stronger question beyond existence.
- Suppose we have an $\mathcal{I}$-indexed set of *parameters* in M, $\mathbf{I} = (a_i : i \in I)$. Can we always find an $\mathcal{I}$-indexed indiscernible set $\mathbf{J} = (b_i : i \in I)$ whose structure in M is derived locally from $\mathbf{I}$?
- An $\mathcal{I}$-indexed indiscernible set $\mathbf{J}$ is *based on* $\mathbf{I}$ if:

Definition

for any L-formula $\varphi(x_1, \ldots, x_m)$ and complete quantifier-free L'-type $\eta(v_1, \ldots, v_m)$, if ALL $\bar{j} \models \eta$ from $\mathcal{I}$ satisfy $(a_{j_1}, \ldots, a_{j_m}) \models \varphi$, ...

.

.

.

then all $\bar{i} \models \eta$ have $(b_{i_1}, \ldots, b_{i_m}) \models \varphi$ as well.

- Equivalently, for every finite set Δ of L-formulas, every $\bar{b}_{\bar{i}}$ has its template: there exist $\bar{j}$ with the same qftp as $\bar{i}$ such

based on: II

- This property is easily recognizable in the usual argument that given $\varphi(x;y)$ with "infinite chains", i.e. there exists $(a_i)_{i<\omega}$ with
$$i<j \Rightarrow \varphi(a_i;a_j)$$
we may find order indiscernible witnesses $(b_i : i<\omega)$ such that
$$i<j \Rightarrow \varphi(b_i;b_j)$$
- Basically, we finitely satisfy the type of our indiscernible in the chain of witnesses, and we may write in the condition that (b_i) be a chain in φ, because this property shows up **everywhere** on the qf L'-type $\{v_1 < v_2\}$ in the original set.

modeling property

- The following property is clearly stated for the case of tree-indexed indiscernibles in [DS04].

Definition

Fix an L'-structure I. We say that *$\mathcal{I}$-indexed indiscernibles have the modeling property (MP) in M* if given any parameters $(a_i : i \in I)$ there exist $\mathcal{I}$-indexed indiscernible $(b_i : i \in I)$ (in the monster model) based on the a_i.

- It is possible for M to admit $\mathcal{I}$-indexed indiscernibles, but for $\mathcal{I}$-indexed indiscernibles not to have the modeling property in M.
- For this, we state a necessary condition for $\mathcal{I}$-indexed indiscernibles to have the modeling property.

stretching indiscernibles:I

- First of all, we would like to take the focus away from the structure $\mathcal{I}$ and onto its age.

Definition

By age($\mathcal{I}$) we mean all finitely-generated substructures of $\mathcal{I}$

- We can do this by a lemma that states for L'-structures $\mathcal{I}$, $\mathcal{J}$ with the same age, we may stretch any $\mathcal{I}$-indexed indiscernible onto the index structure $\mathcal{J}$

stretching indiscernibles: II

- More precisely, we have the following:

Lemma ([She90])

Let $\mathcal{I}$ be any L'-structure. If $(a_i : i \in I)$ is an $\mathcal{I}$-indexed indiscernible and age($\mathcal{I}$)=age($\mathcal{J}$), then there exist $\mathcal{J}$-indexed indiscernible $(b_i : i \in J)$ based on the a_i.

- This is stated in CT for the (†) case and w/o the age terminology and for age($\mathcal{J}$) $\subseteq$ age($\mathcal{I}$), but it is the same idea.
- As a proof: the following is f.s. in $(a_i : i \in I)$: $\Gamma(b_j : j \in J) := \{\varphi(b_{j_1}, \dots, b_{j_n}) : n < \omega, \varphi$ from L, and for all $\bar{i}$

  from I with the same qftp as $\bar{j}, \varphi(\bar{a}_{\bar{i}})\}$
- Really, we just need the condition "for all $\bar{i}$ from I with the safe qftp as $\bar{j}$" to not be a vacuous condition, which it will

ramsey classes: I

- Thus, when we are looking at the modeling property, we are really looking at a property about the age $\mathcal{K}$ of a structure $\mathcal{I}$.
- In fact, in the ordered case, the right property is that of being a *Ramsey class*.
- Fix a class $\mathcal{K}$ of finite L'-structures. First we define the *A-substructures of B*:

Definition

For $A, B \in \mathcal{K}$, an *A-substructure of B* is an embedding $f : A \to B$ modulo the equivalence relation of being the same embedding up to an automorphism of A

- In other words, we think of the copy of A as being the range of the embedding map.
- When there is a linear ordering in the language (something to make the structures A rigid) the range can be identified with the embedding.

ramsey classes: II

- Given a finite set X of cardinality k, We refer to a map $c : \binom{C}{A} \to X$ as a *k-coloring of the A-substructures of C.*
- We say that $B' \subseteq C$ is *homogeneous for this coloring* if there is an element $x_0 \in X$ such that $c''\binom{B'}{A} = \{x_0\}$.

Definition

A class $\mathcal{K}$ of finite L'-structures is a *Ramsey class (RC)* if for all $A, B \in \mathcal{K}$ and for all finite k there is a $C \in \mathcal{K}$ such that for any k-coloring of the A-substructures of C, there is a $B' \subseteq C$, isomorphic to B that is homogeneous for this coloring.

- We often write the above as: for all $A, B \in \mathcal{K}$ and k finite there is $C \in \mathcal{K}$ such that

$$C \to (B)_k^A$$

- When we are working with an **age** $\mathcal{K}$ of structures (without finite bound on their cardinality), RC is equivalent to, for all $\mathcal{T}$ with age $\mathcal{K}$, for all $A, B \in \mathcal{K}$, $\mathcal{T} \to (B)^A$

translation

- The following is an adaptation of a similar theorem in [Sco12] concerning finite relational L':

Theorem

Let $\mathcal{I}$ be a locally finite L'-structure for a language $L' \supseteq \{<\}$ such that $\mathcal{I}$ is linearly ordered by $<$. Let $\mathcal{K}$:=age($\mathcal{I}$). $\mathcal{K}$ is a Ramsey class just in case $\mathcal{I}$-indexed indiscernibles have the modeling property.

- Thus the sort of age $\mathcal{K}$ in $L' \supseteq \{<\}$ containing only finite structures linearly ordered by $<$, that serves as the age of $\mathcal{I}$ indexing indiscernibles with the MP, is $\mathcal{K}$ that is Ramsey.
- Consider $\mathcal{K}_s$:= all square-free linearly ordered graphs in $L' = \{<, R\}$. By a result in [Neš05], in order to be Ramsey, the reduct of $\mathcal{K}$ to $\{R\}$ would need to have AP. It doesn't.
- Thus, even though all models M admit $\mathcal{I}$-indexed indiscernibles for age($\mathcal{I}$)=$\mathcal{K}_s$, we do not have the maximal

argument I: RC $\Rightarrow$ MP

- For a sequence $\overline{a}$ from I, let $p_{\overline{a}}(\overline{x})$ denote its complete quantifier free type.
- For $A \in \mathcal{K}$ of size n, $p_A(x_1, \ldots, x_n)$ is the *increasing type of* A if $p_A = p_{\overline{a}}$ where $\overline{a}$ is the increasing enumeration of A.
- Note that coloring A-substructures in $\mathcal{I}$ is equivalent to coloring realizations of $p_A(\overline{x})$ (no A gets colored twice, or fails to get colored)
- To show we can find $\mathcal{I}$-indexed indiscernibles based on $\mathbf{I} = (a_i : i \in I)$, we will show that the type of an $\mathcal{I}$-indexed indiscernible is finitely satisfiable in $\mathbf{I}$.
- The type of the indiscernible is of the form: $\Gamma(c_i : i \in I) = \{\varphi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_n}) :$ $\overline{i}, \overline{j}$ are from $I, \mathrm{qftp}^{L'}(i_1, \ldots, i_n) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n)\}$

argument II: $I_0 \subset \mathcal{I}$, Δ finite

- A finite piece of Γ will contain constants c_i whose subscripts only involve a finite list of indices I_0 from I. Only a finite list of L-formulas, φ_l occur – collect these into a finite set, Δ.
- The assignment of $\{$complete Δ-type of $\overline{a}_{\bar{i}}$ $\}$ to $\bar{i}$ is a *type coloring* of $\bar{i}$.
- I_0 contains realizations of only finitely many complete quantifier-free L'-types: $\eta_1, \ldots, \eta_s$ [does not rely on L']
- We need to find a copy B' of $B := \langle I_0 \rangle$ in I and complete Δ-types p_i such that for any $\bar{i}_k \models \eta_k$ from B', $\mathrm{tp}_\Delta(\overline{a}_{\bar{i}_k})$=$p_k$.
- By induction, we only need to do this once, for one η_1.

argument III

- Let A be the element of $\mathcal{K}$ such that any $\bar{i} \models \eta_1$ satisfies $\langle \bar{i} \rangle \cong A$.
- Consider a k-coloring of the A-substructures of I where $k=$ (# Δ-types) as follows:
- for $A \cong A' \subseteq \mathcal{I}$, $c(A') = \mathrm{tp}_\Delta(\overline{a}_{\bar{i}})$ where $\bar{i}$ is A' listed in increasing enumeration.
- Realizations of η_1 occupy a unique place in the linear ordering of A.
- So, in any B' that is homogeneous for the above coloring of A-substructures, the type coloring on $\bar{i} \models \eta_1$ becomes homogeneous.

MP $\Rightarrow$ RC

- Fix a k-coloring on the A-substructures of $\mathcal{I}$ (we want a homogeneous copy of B)
- Let M be a structure housing an I-indexed set of parameters in the following way: $|M| = I$, and $R_l(j_1, \ldots, j_n)$ just in case $p_A(\overline{j})$ and this copy $\overline{j}$ of A is assigned color l in I. The parameters are $(a_i : i \in I)$ such that $a_i = i$.
- In M the R_l are disjoint.
- Take an $\mathcal{I}$-indexed indiscernible $(b_i : i \in I)$ based on the a_i.
- We were looking for B, so take any copy $\overline{i}$ in I, and find the $\overline{a}_{\overline{j}}$ for $\Delta = \{R_1, \ldots, R_k\}$ such that qftp($\overline{i}$)=qftp($\overline{j}$) and

$$\overline{b}_{\overline{i}} \equiv_\Delta \overline{a}_{\overline{j}}$$

- First, $\overline{j} \cong B$. Any copies of A in $\overline{j}$ get colored the same way by the R_l, because $\overline{b}_{\overline{i}}$ says so.

why locally finite?

- What about the case when the age of $\mathcal{I}$ does not consist entirely of finite structures.
- Partition properties can become more problematic for infinite structures, e.g.

$$\mathbb{Q} \nrightarrow (\mathbb{Q})_2^{a<b}$$

- Perhaps something like this could be done with restrictions on the colorings.
- Similarly, the requirement that $\mathcal{I}$ be uniformly locally finite in a finite language allows us to take advantage of arguments we made that rely on the qfo property of $\mathcal{I}$.
- What about $||L||$? Useful arguments from structural ramsey theory and topological dynamics focus on the finite/countable case.

closed type

- Can we get the same equivalence of MP and RC in the unordered case?
- Here we make a new definition: let $A \subset \mathcal{I}$ be a finite L'-structure.
- Though there is no linear order in the language, we place an arbitrary order on the structure $\mathcal{I}$. Then any $A \subset \mathcal{I}$ has a "primary ordering" induced by the ordering on $\mathcal{I}$
- Let $\overline{a}$ be the enumeration of A that is increasing according to the primary ordering.

Definition

Let $A \subset \mathcal{I}$ have cardinality n, and $\overline{a}$ its primary enumeration. The *closed type of* A, $c_A(x_1, \ldots, x_n)$ is defined to be $\bigvee_{\sigma \in \mathrm{Aut(A)}} p_{\overline{a}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

- Define the *symmetric type of* A to be $\bigvee_{(\text{ all primary orderings } A' \text{ of } A)} c_{A'}$

case i: we color up to closed types

- We may retain our notions of generalized indiscernibility and modeling property from before.
- However, we know that if the type-coloring is finer than the closed-types of $A \subset \mathcal{I}$, there is no hope of finding the $\mathcal{I}$-indexed indiscernible **in** the original set of parameters.
- This is because there is no good homogeneous set in I: every copy of A contains, in effect, two differently colored copies of itself.
- So we restrict the colorings of tuples $\bar{i}$ from I to colorings of its closed types.
- However, a generalized indiscernible could decide that differently oriented copies of A get colored different types in M – so solving the MP question does not solve the RC question.
- And if $\mathcal{K}$ is a RC, this is no guarantee that we can separate two orientations of A in our generalized indiscernible, even if that is reflected in the initial set of parameters.

case ii: we color up to symmetric types

- We can change our notion of indiscernibility so that two tuples **having the same symmetric type** must map to the same complete type in M, call this a *symmetric indiscernible.*
- Then, solving the MP problem solves the RC problem.
- And if we solve the RC problem, then we can model a coloring that respects closed types at least by a symmetric indiscernible (if not by a generalized indiscernible). [meaning in the end the indiscernible chooses one color for every copy of A, no matter how oriented.]
- Even so, the resulting class is unlikely to be Ramsey.

studying the obstruction in M

- Colorings that break Ramsey theorems often appeal to a ghost ordering on the structure
- We had a few examples of indiscernibles that didn't exist in an ordered structure
- What about a converse: if an indiscernible fails to have the modeling property for a type-coloring (respecting closed types), what does this say about the definable structure of M?

Thanks

Thanks for your attention!

M. Džamonja and S. Shelah.
On $\triangleleft^*$-maximality.
Annals of Pure and Applied Logic, 125(1-3):119–158, 2004.

J. Nešetřil.
Homogeneous structures and ramsey classes.
Combinatorics, Probability and Computing, 14:171–189, 2005.

L. Scow.
Characterization of NIP theories by ordered graph-indiscernibles, 2012.
10.1016/j.apal.2011.12.013.

S. Shelah.
Classification Theory and the number of non-isomorphic models (revised edition).
North-Holland, Amsterdam-New York, 1990.