Vapnik-Chervonenkis Density

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$$\mathcal{S} \cap A := \{S \cap A : S \in \mathcal{S}\}$$

and call $(A, \mathcal{S} \cap A)$ **the set system on A induced by $\mathcal{S}$.**

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We say A is **shattered by** $\mathcal{S}$ if $\mathcal{S} \cap A = 2^A$.

If $\mathcal{S} \neq \emptyset$, then we define the **VC dimension of $\mathcal{S}$,** denoted by $\mathrm{VC}(\mathcal{S})$, as the supremum (in $\mathbb{N} \cup \{\infty\}$) of the sizes of all finite subsets of X shattered by $\mathcal{S}$. We also decree $\mathrm{VC}(\emptyset) := -\infty$.

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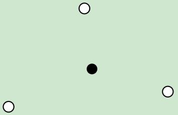
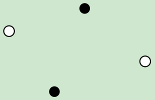


One point in the convex hull of the others



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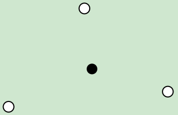

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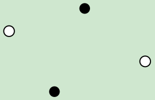

No point in the convex hull of the others

3. Let $\mathcal{S} =$ half spaces in $\mathbb{R}^d$. Then $\mathrm{VC}(\mathcal{S}) = d + 1$.
   (The inequality $\leqslant$ follows from *Radon's Lemma*.)

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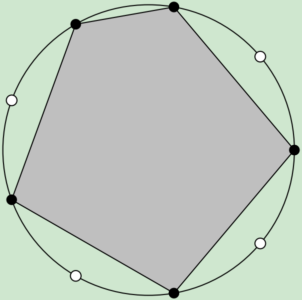

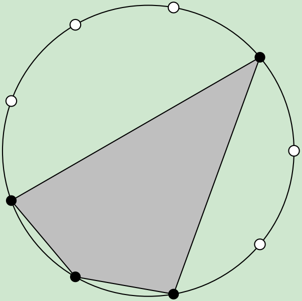

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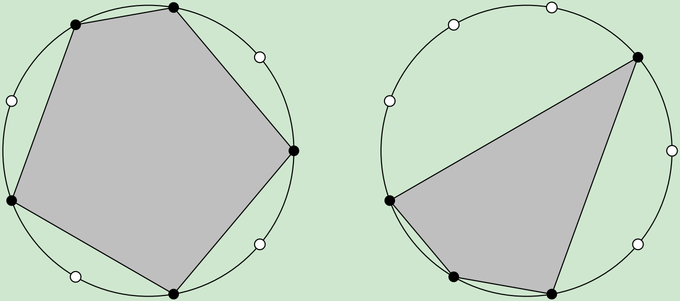

(But $\mathrm{VC}(\{$convex n-gons in $\mathbb{R}^2\}) = 2n + 1$.)

The function

$$n \mapsto \pi_{\mathcal{S}}(n) := \max\left\{ |\mathcal{S} \cap A| : A \in \binom{X}{n} \right\} : \mathbb{N} \to \mathbb{N}$$

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The notion of VC dimension was introduced by Vladimir Vapnik and Alexey Chervonenkis in the early 1970s, in the context of computational learning theory.

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One may now define the **VC density of** $\mathcal{S}$ as

$$\mathrm{vc}(\mathcal{S}) = \begin{cases} \inf\{r \in \mathbb{R}^{>0} : \pi_{\mathcal{S}}(n) = O(n^r)\} & \text{if } \mathrm{VC}(\mathcal{S}) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

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- $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \Rightarrow \mathrm{vc}(\mathcal{S}) = \max\{\mathrm{vc}(\mathcal{S}_1), \mathrm{vc}(\mathcal{S}_2)\}$.

VC duality

Let X be a set (possibly finite). Given $A_1, \dots, A_n \subseteq X$, denote by $S(A_1, \dots, A_n)$ the set of atoms of the Boolean subalgebra of 2^X generated by $A_1, \dots, A_n$: those subsets of X of the form

$$\bigcap_{i \in I} A_i \cap \bigcap_{i \notin I} X \setminus A_i \qquad \text{where } I \subseteq \{1, \dots, n\}$$

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We say that $\mathcal{S}$ is **independent** (in X) if $\pi^*_{\mathcal{S}}(n) = 2^n$ for every n, and **dependent** (in X) otherwise.

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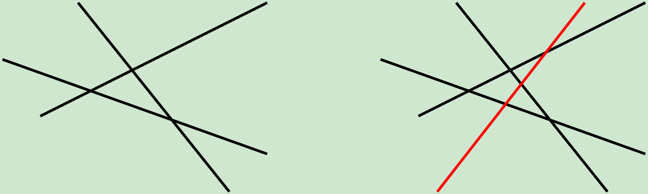

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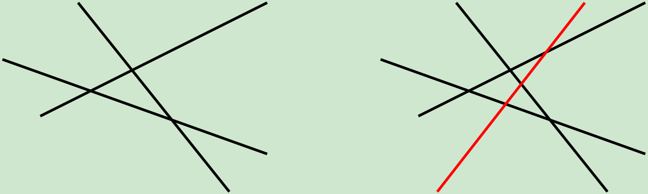

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The function $\pi^*_{\mathcal{S}}$ is called the **dual shatter function of $\mathcal{S}$.**

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Given a finite set $A \subseteq X$ we have a bijection

$$B \mapsto \bigcap_{x \in B} \Phi^*_x \cap \bigcap_{x \in A \setminus B} Y \setminus \Phi^*_x : \quad \mathcal{S}_\Phi \cap A \to S(\Phi^*_x : x \in A).$$

Hence $\pi_\Phi = \pi^*_{\Phi^*}$ and $\pi_{\Phi^*} = \pi^*_\Phi$, and thus

$$\begin{aligned} \mathcal{S}_\Phi \text{ is a VC class} \quad &\Longleftrightarrow \quad \mathcal{S}_{\Phi^*} \text{ is dependent,} \\ \mathcal{S}_{\Phi^*} \text{ is a VC class} \quad &\Longleftrightarrow \quad \mathcal{S}_\Phi \text{ is dependent.} \end{aligned}$$

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Moreover (first noticed by Assouad):

$$\mathcal{S}_\Phi \text{ is a VC class} \quad \iff \quad \mathcal{S}_{\Phi^*} \text{ is a VC class.}$$

We fix:

$\mathcal{L}$: a first-order language,

$x = (x_1, \ldots, x_m)$: object variables,

$y = (y_1, \ldots, y_n)$: parameter variables,

$\varphi(x; y)$: a partitioned $\mathcal{L}$-formula,

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If $\boldsymbol{M} \equiv \boldsymbol{N}$, then $\pi_{\mathcal{S}_\varphi^{\boldsymbol{M}}} = \pi_{\mathcal{S}_\varphi^{\boldsymbol{N}}}$. So, picking $\boldsymbol{M} \models T$ arbitrary, set

$$\pi_\varphi := \pi_{\mathcal{S}_\varphi^{\boldsymbol{M}}}, \quad \mathrm{VC}(\varphi) := \mathrm{VC}(\mathcal{S}_\varphi^{\boldsymbol{M}}), \quad \mathrm{vc}(\varphi) := \mathrm{vc}(\mathcal{S}_\varphi^{\boldsymbol{M}}).$$

The *dual* of $\varphi(x;y)$ is $\varphi^*(y;x) := \varphi(x;y)$. Put

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Recall:

If $\mathrm{VC}(\varphi) < \infty$ then we say that φ is **dependent** in T. The theory T does **not have the independence property** (is **NIP**, or **dependent**) if every partitioned $\mathcal{L}$-formula is dependent in T.

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An important theorem of Shelah (given other proofs by Laskowski and others) says that for T to be NIP it is enough for for every $\mathcal{L}$-formula $\varphi(x;y)$ with $|x| = 1$ to be dependent.

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We do not know an example of a formula φ in a NIP theory with $\mathrm{vc}(\varphi) \notin \mathbb{Q}$.

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$$\mathrm{vc}^{\mathrm{ACF}_0}(\varphi) = \frac{4}{3}; \quad \mathrm{vc}^{\mathrm{ACF}_p}(\varphi) = \frac{3}{2} \text{ for } p > 0.$$

   We do not know an example of a formula φ in a NIP theory with $\mathrm{vc}(\varphi) \notin \mathbb{Q}$.

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2. Growth of π_φ. There is an example of an ω-stable T and an $\mathcal{L}$-formula $\varphi(x;y)$ with $|y| = 2$ and

$$\pi_\varphi(n) = \tfrac{1}{2} n \log n \,(1 + o(1)).$$

3. Uniform bounds on $\mathrm{vc}(\varphi)$.

Uniform bounds on VC density

Some reasons why it should be interesting to obtain bounds on $\mathrm{vc}(\varphi)$ in terms of $|y|$ = number of free parameters:

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Some reasons why it should be interesting to obtain bounds on $\mathrm{vc}(\varphi)$ in terms of $|y|$ = number of free parameters:

1. uniform bounds on VC density often “explain” why certain bounds on the complexity of geometric arrangements, used in computational geometry, are polynomial in the number of objects involved (*example follows later*);
2. connections to strengthenings of the NIP concept: if $\mathrm{vc}(\varphi) < 2$ for each $\varphi(x; y)$ with $|y| = 1$, then T is *dp-minimal*.

Theorem

Suppose T expands the theory of linearly ordered sets, and assume that T is **weakly o-minimal,** *i.e., in every $\boldsymbol{M} \models T$, every definable subset of M is a finite union of convex subsets of M.*

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It is more convenient to work with π^*, and thus we need to show

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$\Delta(x; y)$: a finite non-empty set of partitioned $\mathcal{L}$-formulas;

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If T is NIP then we set

$$\pi^{*}_{\Delta}(t) := \max\left\{ |S^{\Delta}(B)| : B \in \binom{M^{|y|}}{t} \right\},$$
$$\mathrm{vc}^{*}(\Delta) := \inf\left\{ r \in \mathbb{R}^{>0} : \pi^{*}_{\Delta}(t) = O(t^{r}) \right\}.$$

Uniform bounds on VC density

Definition (adapted from Guingona)

Δ has **uniform definability of types over finite sets** (**UDTFS**) in M with m parameters if there are families of $\mathcal{L}$-formulas

$$\mathcal{F}_i = \big(\varphi_i(y; y_1, \ldots, y_m)\big)_{\varphi \in \Delta} \qquad (i \in I = \text{a finite set})$$

such that for every finite $B \subseteq M^{|y|}$ and $q \in S^{\Delta}(B)$ there are $b_1, \ldots, b_m \in B$ and $i \in I$ such that $\mathcal{F}_i(y; b_1, \ldots, b_m)$ defines q.

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- If we don't care about the number of extra parameters m, then we can always achieve $|I| = 1$ and $|\Delta| = 1$.
- On the other hand: if Δ has a uniform definition $\mathcal{F} = (\mathcal{F}_i)_{i \in I}$ for Δ-types with m parameters, then

$$|S^\Delta(B)| \leqslant |I| \cdot |B|^m \qquad \text{for every finite } B.$$

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Suppose that M has the $\mathrm{VC}\,m$ **property***, i.e., any $\Delta(x;y)$ with $|x|=1$ has UDTFS in M with m parameters.*

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Then every $\Delta(x;y)$ has UDTFS in M with $m|x|$ parameters.

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The same proof applies to quasi-o-minimal theories (e.g., Presburger Arithmetic).

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Uniform bounds on VC density

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Suppose $M = \mathbb{Q}_p$ is the field of p-adic numbers, construed as a structure in the language of rings.

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We also have results stating that in certain stable theories T we have linear bounds on VC density, not obtained via the $\mathrm{VC}\,m$ property.

Theorem

Let A be an infinite abelian group. T.f.a.e.:

1. $\mathrm{vc}(\varphi)$ *for* $\varphi(x;y)$ *with* $|y| = 1$ *is bounded;*
2. *there is some* d *such that* $\mathrm{vc}(\varphi) \leqslant d|y|$ *for each* $\varphi(x;y)$*;*
3. *there are only finitely many* p *such that* $A[p]$ *or* A/pA *is infinite, and for all* p *there are only finitely many* n *such that*

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If A has finite exponent then it has the $\mathrm{VC}\, d$ property (explicit d). The proof involves some combinatorics with distributive lattices.

Uniform bounds on VC density

A general theorem is:

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Suppose T does not have the finite cover property and finite U-rank $\mathrm{U}(T)$. Then $\mathrm{vc}(\varphi) \leqslant |y|\,\mathrm{U}(T)$ for every $\varphi(x;y)$.

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Example ($\mathcal{L}$ = language of rings, $K \models \mathrm{ACF}$)

Choose $\varphi(x;y)$ so that $\mathcal{S}^K_\varphi$ = all zero sets (in K^m) of polynomials in m indeterminates over K of degree $\leqslant d$.

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$$\pi^*_\varphi(t) = \left\{\begin{array}{l}\text{maximum number of non-empty} \\ \text{Boolean combinations of } t \text{ hypersur-} \\ \text{faces in } K^m \text{ of degree} \leqslant d.\end{array}\right\} = \pi_{\varphi^*}(t) = O(t^m)$$

Question

Let $f\colon A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, be L-**Lipschitz** (where $L \in \mathbb{R}^{\geqslant 0}$), i.e.,

$$||f(x) - f(y)|| \leqslant L \cdot ||x - y|| \qquad \text{for all } x, y \in A.$$

Can one extend f to an L-Lipschitz map $\mathbb{R}^m \to \mathbb{R}^n$?

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The usual proofs of this theorem all use some sort of transfinite induction. (A classical explicit construction by MacShane & Whitney only yields an $L\sqrt{n}$-Lipschitz extension.)

Theorem A (A.-Fischer, Proc. LMS 2011)

Let $\boldsymbol{R} = (R, 0, 1, +, \times, <, \dots)$ be a **definably complete** expansion of an ordered field: every non-empty definable subset of R which is bounded from above has a supremum.

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The proof of this theorem used convex analysis and is based on a relationship between Lipschitz maps and monotone set-valued maps (Minty; more recently, Bauschke & Wang).

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Another crucial ingredient (in the case where $R \neq \mathbb{R}$) is a definable version of a classical theorem of Helly:

Theorem B (A.-Fischer, Proc. LMS 2011)

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Let $\boldsymbol{R}$ be a definably complete expansion of an ordered field. Let $\mathcal{C}$ be a definable family of closed bounded *convex* subsets of R^n. Suppose $\mathcal{C}$ is $(n+1)$**-consistent**:

$$\bigcap \mathcal{C}' \neq \emptyset \qquad \text{for all } \mathcal{C}' \subseteq \mathcal{C} \text{ with } |\mathcal{C}'| \leqslant n+1.$$

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S. Starchenko pointed out that in the case of an o-minimal $\boldsymbol{R}$, our theorem follows from an analysis of forking in o-minimal theories due to A. Dolich.

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Let $\boldsymbol{R}$ be an o-minimal expansion of a real closed field, and let $\mathcal{C} = \{C_a\}_{a \in A}$ be a definable family of closed and bounded subsets of R^n parameterized by a subset A of R^m.

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$$N(m,n) = (1+2^m)\cdot(1+2^{2^m})\cdots \qquad (n \text{ factors}),$$

then $\mathcal{C}$ has a finite transversal.

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Corollary

Let $\mathcal{C} = \{C_a\}_{a \in A}$ be a family of compact subsets of $\mathbb{R}^n$ definable in an o-minimal structure on $\mathbb{R}$. If $\mathcal{C}$ is $(n+1)$-consistent, then $\mathcal{C}$ has a finite transversal.

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If there is some d_1 such that $\mathrm{vc}(\varphi) \leqslant d_1$ for each $\varphi(x;y)$ with $|y| = 1$, is there is some d_m such that $\mathrm{vc}(\varphi) \leqslant d_m$ for each $\varphi(x;y)$ with $|y| = m$?