Higher order spreading models

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Introduction

- The context of this talk is part of a joint work with S.A. Argyros and V. Kanellopoulos.
- The main objective of this talk is a generalization of the classical notion of the spreading model invented by A. Brunel and L. Sucheston in the middle of 70's.

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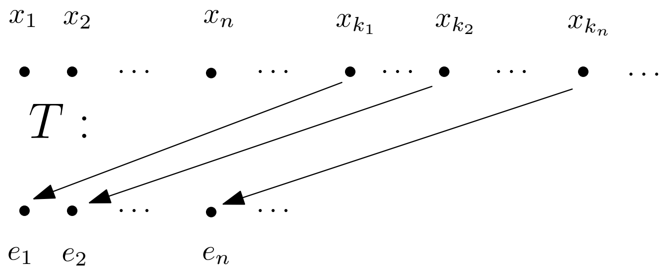
The classical spreading models

A sequence $(x_n)_n$ in a Banach space X generates a sequence $(e_n)_n$ as a spreading model if there exists a null sequence $(\delta_n)_n$ of positive reals, such that
for every $n \leq k_1 < \ldots < k_n$ in $\mathbb{N}$, the spaces $< x_{k_1}, \ldots, x_{k_n} >$ and $< e_1, \ldots, e_n >$, through the linear operator sending each x_{k_j} to e_j, are $1 + \delta_n$ isomorphic.



Motivation

Theorem (E. Odell and Th. Schlumprecht)

There exists a reflexive space X such that every space generated by a spreading model of X does not contain any isomorphic copy of ℓ^p, for $p \in [1, \infty)$, or c_0.

In the same paper they ask the following concerning the k-iterated spreading models.

Problem

Does for every Banach space X exist a natural number k such that X admits a k-iterated spreading model equivalent to the usual basis of ℓ^p, for some $p \in [1, \infty)$, or c_0?

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The extended definition of a spreading model

- For every Banach space X and every countable ordinal ξ we assign to X the family of ξ-order spreading models denoted by $\mathcal{SM}_\xi(X)$.

- The transfinite hierarchy $(\mathcal{SM}_\xi(X))_{\xi<\omega_1}$ is increasing and the ξ-spreading models of X have a weaker asymptotic connection to X as ξ tends to ω_1. Moreover, the Brunel-Sucheston spreading models coincide with the order one spreading models $(\mathcal{SM}_1(X))$.

- The definition of the ξ-order spreading models pass through the notion of the $\mathcal{F}$-spreading models.

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- The definition of the ξ-order spreading models pass through the notion of the $\mathcal{F}$-spreading models.

The $\mathcal{F}$-spreading models

In order to define the $\mathcal{F}$-spreading models we introduce the following two concepts.

- The first one is the $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$, where $\mathcal{F}$ is a family of finite subsets of $\mathbb{N}$ satisfying certain properties. These sequences will replace the common sequences $(x_n)_{n\in\mathbb{N}}$ in a Banach space.
- The second one is the concept of plegma families. These families specify the finite subsequences of an $\mathcal{F}$-sequence which determine the spreading model.

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On finite subsets of $\mathbb{N}$

The $\mathcal{F}$-sequences are an extension of the notion of the sequence. First we need to recall some terminology concerning families of finite subsets of $\mathbb{N}$.

- A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called
- *hereditary* if for every $s \in \mathcal{F}$ and $t \subseteq s$ we have that $t \in \mathcal{F}$.
- *spreading* if for every $s \in \mathcal{F}$ and $t \in [\mathbb{N}]^{<\infty}$ such that
  1. $|s| = |t|$
  2. $s(i) \leq t(i)$, for all $1 \leq i \leq |s|$,

  we have that $t \in \mathcal{F}$.
- *compact*, if it is closed in $\{0, 1\}^{\mathbb{N}}$ and
- *thin* if there are no $s, t \in \mathcal{F}$ with $s \sqsubset t$.
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An important feature of a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$, is the *order* of $\mathcal{F}$, denoted by $o(\mathcal{F})$. We consider the set

$$\widehat{\mathcal{F}} = \{t \in [\mathbb{N}]^{<\infty} : \exists s \in \mathcal{F} \text{ with } t \sqsubseteq s\}.$$

If $\widehat{\mathcal{F}}$ is compact, we set $o(\mathcal{F})$ to be the rank of $\emptyset$ in the well founded partial ordered set $\widehat{\mathcal{F}}$ endowed with the inverse initial segment inclusion.

Regular thin families

We will consider a special class of thin families, which we will call regular thin families.

- A family $\mathcal{F}$ is called *regular thin* if
  - $\mathcal{F}$ is thin and
  - $\widehat{\mathcal{F}}$ is regular, i.e. $\widehat{\mathcal{F}}$ is hereditary, spreading and compact.
- Typical examples of low order regular thin families are the families of k-subsets of $\mathbb{N}$, $[\mathbb{N}]^k$ with $o([\mathbb{N}]^k) = k$ as well as the maximal elements of the Schreier family, $\mathcal{F}_\omega = \{s \subset \mathbb{N} : \min s = |s|\}$ with $o(\mathcal{F}_\omega) = \omega$.
- By recursion on ordinals one can define regular thin families of order ξ for every $\xi < \omega_1$.

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$\mathcal{F}$-sequences

- Given a regular thin family $\mathcal{F}$ and a set X, by the term *$\mathcal{F}$-sequence in X* we will mean a map $\varphi : \mathcal{F} \to X$. Setting for each $s \in \mathcal{F}$, $x_s = \varphi(s)$ an $\mathcal{F}$-sequence will be denoted by $(x_s)_{s\in\mathcal{F}}$.
- Also by taking restrictions of $\mathcal{F}$ to infinite subsets L of $\mathbb{N}$, we define the *$\mathcal{F}$-subsequences*, denoted by $(x_s)_{s\in\mathcal{F}\upharpoonright L}$, where $\mathcal{F}\upharpoonright L = \{s \in \mathcal{F} : s \subset L\}$.

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Plegma families

Rather than giving the explicit definition of the *plegma families*, we will try to describe them.

- Roughly speaking the plegma families are tuples $(s_1, \ldots, s_l)$ of *pairwise disjoint* finite subsets of $\mathbb{N}$ satisfying the following property.
- The first elements of s_i, $1 \leq i \leq l$ are in increasing order and they lie before their second elements which are also in increasing order and so on.
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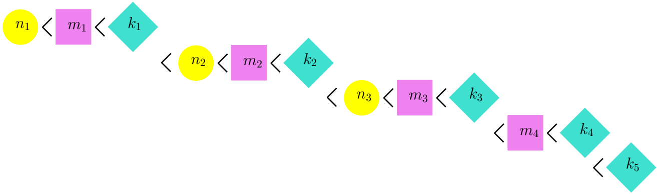
An example of a plegma family

For instance, let
$s_1 = \{n_1 < n_2 < n_3\}$,
$s_2 = \{m_1 < m_2 < m_3 < m_4\}$ and
$s_3 = \{k_1 < k_2 < k_3 < k_4 < k_5\}$. The 3-tuple (s_1, s_2, s_3) is plegma if it has the following form.



There are several combinatorial properties concerning the plegma families consisting of elements belonging to a regular thin families.

- Given a regular thin family $\mathcal{F}$, $M \in [\mathbb{N}]^\infty$ and $l \in \mathbb{N}$, let $Plm_l(\mathcal{F} \upharpoonright M)$ be the set of all plegma families $(s_1, \ldots, s_l)$ with each $s_i \in \mathcal{F} \upharpoonright M$. Moreover we set $Plm(\mathcal{F} \upharpoonright M) = \cup_{l=1}^{\infty} Plm_l(\mathcal{F} \upharpoonright M)$.
- The crucial property of the plegma families is the following.

Proposition

Let $M \in [\mathbb{N}]^\infty$, $l \in \mathbb{N}$ and $\mathcal{F}$ be a regular thin family. Then for every finite coloring of $Plm_l(\mathcal{F} \upharpoonright M)$ there exists $L \in [M]^\infty$ such that $Plm_l(\mathcal{F} \upharpoonright L)$ is monochromatic.

The Ramsey property of the plegma families

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Plegma compatibility from higher to lower order families

Concerning the maps between regular thin families we have the following results.

- The first one allows the plegma preserving embeddings of regular thin families into ones with lower order.

Theorem

Let $\mathcal{F}, \mathcal{G}$ be regular thin families. If $o(\mathcal{F}) \leq o(\mathcal{G})$ then there exist $N \in [\mathbb{N}]^\infty$ and a map $\varphi : \mathcal{G} \upharpoonright N \to \mathcal{F}$ such that for every $(s_i)_{i=1}^l \in Plm(\mathcal{G} \upharpoonright N)$, we have that $(\varphi(s_i))_{i=1}^l \in Plm(\mathcal{F})$.

- The above theorem is based on the following proposition.

Proposition

Let $\mathcal{H}_1, \mathcal{H}_2$ be regular families of finite subsets of $\mathbb{N}$ with $o(\mathcal{H}_1) \leq o(\mathcal{H}_2)$. Then there exists $L \in [\mathbb{N}]^\infty$ such that $\mathcal{H}_1(L) \subseteq \mathcal{H}_2$.

Plegma compatibility from higher to lower order families

Concerning the maps between regular thin families we have the following results.

- The first one allows the plegma preserving embeddings of regular thin families into ones with lower order.

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Let $\mathcal{F},\mathcal{G}$ be regular thin families. If $o(\mathcal{F})\leq o(\mathcal{G})$ then there exist $N\in[\mathbb{N}]^\infty$ and a map $\varphi:\mathcal{G}\upharpoonright N\to\mathcal{F}$ such that for every $(s_i)_{i=1}^l\in Plm(\mathcal{G}\upharpoonright N)$, we have that $(\varphi(s_i))_{i=1}^l\in Plm(\mathcal{F})$.

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Plegma incompatibility from lower to higher order families

- The second one forbids the plegma preserving embeddings of regular thin families into ones with higher order.

Theorem

Let $\mathcal{F}, \mathcal{G}$ be regular thin families. If $o(\mathcal{F}) < o(\mathcal{G})$ then for every $\varphi : \mathcal{F} \to \mathcal{G}$ and $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F} \upharpoonright L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is a plegma pair.

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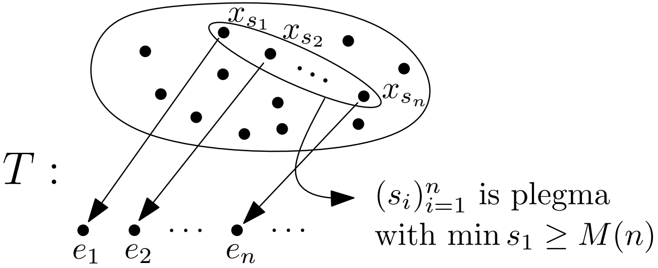
Definition of the $\mathcal{F}$-spreading models

Let X be a Banach space, $\mathcal{F}$ be a regular thin family, $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates a sequence $(e_n)_n$ as an $\mathcal{F}$-spreading model, if there exists a null sequence of positive reals $(\delta_n)_n$ satisfying the following. For every $n \in \mathbb{N}$ and every plegma family $(s_i)_{i=1}^n$ of length n in $\mathcal{F} \upharpoonright M$ with $\min s_1 \geq M(n)$, the spaces $< x_{s_1}, \dots, x_{s_n} >$ and $< e_1, \dots, e_n >$, through the linear operator sending each x_{s_i} to e_i, are $1 + \delta_n$ isomorphic.



Existence of the $\mathcal{F}$-spreading models

Theorem

Let X be a Banach space and $\mathcal{F}$ be a regular thin family. Then every bounded $\mathcal{F}$-sequence in X contains an $\mathcal{F}$-subsequence generating an $\mathcal{F}$-spreading model.

- A consequence of the plegma compatibility from higher to lower order families is the following.

Proposition

Let X be a Banach space. If $o(\mathcal{F}) = o(\mathcal{G})$ then $(e_n)_n$ is an $\mathcal{F}$-spreading model of X if and only if $(e_n)_n$ is a $\mathcal{G}$-spreading model of X. More generally, if $o(\mathcal{F}) \leq o(\mathcal{G})$ and $(e_n)_n$ is an $\mathcal{F}$-spreading model of X then $(e_n)_n$ is a $\mathcal{G}$-spreading model of X.

Independence of the family $\mathcal{F}$

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The hierarchy of spreading models

- This proposition permits us to classify the spreading models in a transfinite hierarchy as follows.

Definition

Let X be a Banach space and ξ be a countable ordinal. We will say that $(e_n)_n$ is *a ξ-order spreading model of X* if there exists a regular thin family $\mathcal{F}$ with $o(\mathcal{F}) = \xi$ such that $(e_n)_n$ is an $\mathcal{F}$-spreading model of X.

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- The preceding proposition yields that the above defined transfinite hierarchy of spreading models is *increasing*, i.e. for every Banach space X and $1 \leq \zeta < \xi < \omega_1$ we have that

$$\mathcal{SM}_\zeta(X) \subseteq \mathcal{SM}_\xi(X)$$

An open problem

Problem

Is it true that for every separable Banach space X there is a countable ordinal ξ such that for every $\zeta > \xi$, $\mathcal{SM}_\zeta(X) = \mathcal{SM}_\xi(X)$?

- Another natural question is whether for every $\xi < \omega_1$ there exists a Banach space X such that $\mathcal{SM}_\zeta(X) \neq \mathcal{SM}_\xi(X)$, for all $\zeta < \xi$. Towards this direction we have the following result.

Theorem

Let ξ be a finite or a limit countable ordinal. Then there exists a reflexive space X with an unconditional basis satisfying the following properties:

1. *The space X admits ℓ^1 as a ξ-order spreading model.*
2. *For every ordinal $\zeta < \xi$, the space X does not admit ℓ^1 as a ζ-order spreading model.*

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A reflexive space not admitting ℓ^p or c_0 as a spreading model

- Odell and Schlumprecht asked if there exist a Banach space X such that for every $k \in \mathbb{N}$, X does not admit ℓ^p or c_0, $1 \leq p < \infty$ as a k-iterated spreading model.
- The answer to the above problem is affirmative. Actually the following more general result holds.

Theorem

There exists a reflexive space X with an unconditional basis such that for every $\xi < \omega_1$ and every $(e_n)_n \in \mathcal{SM}_\xi(X)$, the space $E = \overline{<(e_n)_n>}$ is reflexive and does not contain any isomorphic copy of c_0 or ℓ^p, for all $1 \leq p < \infty$.

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Some comments

- The space X is similar to the one constructed by Odell and Schlumprecht. However the proof requires a systematic analysis of the generic form of the $\mathcal{F}$-sequences with weakly relatively compact range.
- The above theorem shows that the finite representability of ℓ^p, $1 \leq p \leq \infty$, asserted by Krivine's theorem, cannot be recovered by the higher order spreading models that we introduced here.

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Cesàro summability vs ℓ^1-spreading models

- We define the k-Cesàro summability as follows.

Definition

Let X be a Banach space, $x_0 \in X$, $k \in \mathbb{N}$, $(x_s)_{s\in[\mathbb{N}]^k}$ be a $[\mathbb{N}]^k$-sequence in X and $M \in [\mathbb{N}]^\infty$. We will say that the $[\mathbb{N}]^k$-subsequence $(x_s)_{s\in[M]^k}$ is *k-Cesàro summable to x_0* if

$$\frac{1}{\#[M|n]^k} \sum_{s\in[M|n]^k} x_s \xrightarrow[n\to\infty]{\|\cdot\|} x_0$$

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- We prove the following lemma.

Lemma

Let $\delta > 0$ and $k, l \in \mathbb{N}$. Then there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ and every subset A of the set $[\{1, \ldots, N\}]^k$ of all k-subsets of $\{1, \ldots, N\}$ of size at least $\delta\binom{N}{k}$, there is a plegma l-tuple $(s_j)_{j=1}^{l}$ in A.

- While for the case $k = 1$ the above is immediate, for $k \geq 2$ the proof seems to require the multidimensional Szemeredi's theorem of H. Furstenberg and Y. Katznelson (1978).

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- Using the above lemma we get the following extension of a well known result of H. P. Rosenthal (which corresponds to the case $k = 1$).

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Let X be a Banach space, $k \in \mathbb{N}$ and $(x_s)_{s\in[\mathbb{N}]^k}$ be a wrc $[\mathbb{N}]^k$-sequence in X. Then there exists $M \in [\mathbb{N}]^\infty$ such that at least one of the following holds:

- *The subsequence $(x_s)_{s\in[M]^k}$ generates a k-order spreading model equivalent to the standard basis of ℓ^1.*
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