A new isomorphic ℓ_1 predual which is not isomorphic to a complemented subspace of a $C(K)$ space

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In 1972 Benyamini and Lindenstrauss constructed an isometric ℓ_1 predual E which is not isomorphic to a complemented subspace of a $C(K)$ space, answering a question of Pelczynski.

In 1988 Alspach and Benyamini showed, with a different proof, that a variation of E had the same property. Alspach's quotient of $C(\omega^\omega)$ which does not embed into any $C(\alpha)$, $\alpha < \omega_1$, is an isometric ℓ_1 predual which contains a complemented copy of E and so it is also not isomorphic to a complemented subspace of a $C(K)$ space.

We remark that E is isometric to a subspace of $C(\omega^\omega)$ and that all aforementioned examples contain a copy of $C(\omega^\omega)$.

The preceding examples are related to the problem of the isomorphic classification of the complemented subspaces of $C(K)$ spaces. By a result of Rosenthal, any such subspace with non-separable dual is isomorphic to $C(K)$. It is still open if a complemented subspace of $C(K)$ with separable dual must be isomorphic to some $C(L)$ space. By combining results of Alspach, Benyamini, Johnson and Zippin the following holds:

Theorem Let Y be a complemented subspace of $C(K)$. Then either Y is isomorphic to c_0, or $C(\omega^\omega)$ embeds into Y.

Indeed, if the Szlenk index $\eta(Y)$ of Y exceeds ω, then a result of Alspach implies that $C(\omega^\omega)$ embeds into Y. When $\eta(Y) = \omega$, a result of Benyamini yields that Y is isomorphic to a quotient of c_0 and thus Y is isomorphic to c_0 by Johnson-Zippin's result.

It follows from the preceding theorem that any isomorphic ℓ_1 predual not isomorphic to c_0 and not containing $C(\omega^\omega)$ isomorphically, is not isomorphic to a complemented subspace of a $C(K)$ space.

Question: Does there exist a subspace X of $C(\omega^\omega)$ with X^* isomorphic to ℓ_1 and such that neither X is isomorphic to c_0, nor $C(\omega^\omega)$ embeds into X ?

Theorem. There exists an isomorphic ℓ_1 predual X with a normalized, shrinking basis (e_n) satisfying the following properties:

1. X is isomorphic to a subspace of $C(\omega^\omega)$.

2. Every normalized weakly null sequence in X admits a subsequence which is either equivalent to the c_0 basis, or equivalent to a subsequence of the natural basis of Schreier's space. In particular, (e_n) satisfies the second alterrnative.

Recall that Schreier's space is the completion of c_{00} under the norm

$$\|x\| = \mathrm{sup}\{\sum_{i\in F} |x(i)| : |F| \leq \min F\}$$

It is known that every normalized weakly null sequence in $C(\omega^\omega)$ which admits ℓ_1 as a spreading model, has a subsequence equivalent to a subsequence of the natural basis of Schreier's space.

The proof of this result uses a dual version of the Bourgain-Delbaen method of constructing $\mathcal{L}_\infty$ spaces.

Notation. Let X be a Banach space with a normalized Schauder basis (e_n) and let $D \subset B_{X^*}$ be a norming set for X so that $D \subset < e_n^* : n \in \mathbb{N} > \setminus \{0\}$. Assume that $e_n^* \in D$ for all $n \in \mathbb{N}$ and that $|d^*(e_n)| \leq 1$ for all $d^* \in D$ and all $n \in \mathbb{N}$.

We also consider a sequence $\Delta_1 < \Delta_2 < \cdots < \Delta_n < \cdots$ of successive finite intervals of $\mathbb{N}$ whose union is $\mathbb{N}$. Assume that $|\text{supp} d^* \cap \Delta_n| \leq 1$ for all $d^* \in D$ and all $n \in \mathbb{N}$.

We set $D_n = \{d^* \in D : \max \text{supp} d^* \in \Delta_n\}$ for all $n \in \mathbb{N}$. Thus, $D = \cup_n D_n$.

We finally let P_n denote the basis projection onto $[e_i : i \in \cup_{k=1}^n \Delta_k]$.

Proposition. Let X, (e_n), D and (Δ_n) be as above. Let $0 < b < 1/4$. Assume that the following properties hold for all $n \geq 3$

1. For each $i \in \Delta_n$ there exists a unique $\gamma_i^* \in D_n$ with $|\text{supp}\gamma_i^*| > 1$ and $\max \text{supp}\gamma_i^* = i$.

2. Each $d^* \in D_n$ admits a representation of the form

$$d^* = \rho_1 d_1^* + \rho_2 (d_2^* | \cup_{j=k+1}^{l} \Delta_j) + e_i^*$$

where, $d_1^* \in D_k$ and $d_2^* \in D_l$ for some $1 \leq k < l \leq n-1$, $i \in \Delta_n$ and $|\rho_1| \leq 1$, $|\rho_2| \leq b$.

Then, X is an $\mathcal{L}_\infty$ space.

Moreover, letting $b_i^* = e_i^*$ for $i \in \Delta_1 \cup \Delta_2$, while if $n \geq 3$, $b_i^* = (1/2)\gamma_i^* P_{n-1} + e_i^*$ for $i \in \Delta_n$, then (b_i^*) is equivalent to the ℓ_1 basis and $[(b_i^*)] = [(e_i^*)]$. Hence, if (e_n) is shrinking then X^* is isomorphic to ℓ_1.

Proof. For each $n \geq 2$ we define linear maps $T_n : \ell_\infty(\cup_{k=1}^n \Delta_k) \to [e_i : i \in \cup_{k=1}^n \Delta_k]$ as follows:

$$T_2 x = \sum_{i=1}^{m} x(i) e_i$$

where $m = \max \Delta_2$ and inductively,

$$T_{n+1} x = T_n \pi_n x + \sum_{i \in \Delta_{n+1}} [x(i) - (1/2)\gamma_i^* T_n \pi_n x] e_i$$

where $\pi_n : \ell_\infty \to \ell_\infty$ is the restriction operator to the first $\cup_{k=1}^n \Delta_k$ coordinates and γ_i^* is the unique element of D whose support contains at least two points and i is the maximum of this support. It is clear that

$$P_m T_n x = T_m \pi_m x$$

whenever $m \leq n$ and $x \in \ell_\infty(\cup_{k=1}^n \Delta_k)$.

It will suffice showing that there exist absolute constants $0 < A < B$ so that

$$A \|x\|_\infty \leq \|T_n x\| \leq B \|x\|_\infty$$

for all $x \in \ell_\infty(\cup_{k=1}^n \Delta_k)$ and $n \geq 2$.

Let $\rho = (1/2)[1+3b/(1-b)]$. Then $1/2 < \rho < 1$ as $0 < b < 1/4$.

We choose $\lambda > 0$ such that $\|T_2\| \leq 1 + \lambda/2$, $\|(I - P_1)T_2\| \leq 1 + 3\lambda/2$ and

$$\lambda > (1-\rho)^{-1}(1-b)^{-1}$$

We show by induction on $n \geq 2$ that for all $x \in \ell_\infty(\cup_{k=1}^n \Delta_k)$, $\|x\|_\infty = 1$, there exist $d^* \in \cup_{k=1}^n D_k$ and an initial interval I of $\mathbb{N}$ so that

$$|(d^*|I)(T_n x)| \geq 1/2$$

This clearly implies the lower estimate.

We then show by induction on $n \geq 2$ that the following properties hold:

1. $\|d^* T_n\| \leq 1 + \lambda/2$, for all $d^* \in \cup_{k=1}^n D_k$.

2. $\|d^*(I - P_m)T_n\| \leq 1 + 3\lambda/2$, for all $d^* \in \cup_{k=1}^n D_k$ and $m \leq n$.

3. For every $d^* \in D$ and all $m \leq n$ there exists $l > 0$ so that

$$\|d^*(I - P_m)T_n\| \leq (1 + 3\lambda/2) \sum_{k=0}^{l} b^k$$

4. $\|T_n\| \leq \lambda$.

The last property implies the upper estimate. For the inductive step we assume that all four properties hold for n and then prove they are also valid for $n + 1$.

To accomplish this we make use of the splitting property of the elements of D which yield estimates of the following kind

$$\|d^*T_n\| \leq 1 + \lambda/2 + b(1 + 3\lambda/2) \sum_{k=0}^{l} b^k$$

$$< 1 + \lambda/2 + (1 + 3\lambda/2)b(1 - b)^{-1}$$

$$= (1 - b)^{-1} + \rho\lambda < \lambda$$

Definition of X. Let $\mathcal{F} = \{F \subset \mathbb{N} : |F| \leq \min F + 2\} \cup \{\emptyset\}$. This is a pointwise compact family of finite subsets of $\mathbb{N}$.

We inductively construct a sequence $\Delta_1 < \Delta_2 < \ldots$ of successive intervals of $\mathbb{N}$ whose union is $\mathbb{N}$ and a sequence (D_n) of subsets of c_{00} so that for all n

1. $e_i^* \in D_n$ for all $i \in \Delta_n$.

2. $\mathrm{supp} d^* \subset \cup_{k=1}^n \Delta_k$ and $\max \mathrm{supp} d^* \in \Delta_n$ for all $d^* \in D_n$.

3. $|\mathrm{supp} d^* \cap \Delta_k| \leq 1$ for all $d^* \in D_n$ and $k \leq n$.

4. $|d^*(i)| \leq 1$ for every $i \in \mathbb{N}$ and all $d^* \in D_n$.

5. $\mathrm{supp} d^* \in \mathcal{F}$ for all $d^* \in D_n$.

Indeed, define $\Delta_k = \{k\}$ and $D_k = \{e_k^*\}$ for $k \leq 2$. Assume that Δ_k and D_k have been defined for all $k \leq n$.

If ξ^* and η^* are elements of $\cup_{k=1}^n D_k$, then we say that (ξ^*, η^*) is a linked pair provided that there exist integers $1 \leq k < l \leq n$ with $\xi^* \in D_k$, $\eta^* \in D_l$ and supp$\xi^* \cup$ supp$[\eta^*|(\cup_{i=k+1}^l \Delta_i)] \in \mathcal{F}$ without being a maximal element. Denote by Σ_n the set of all possible linked pairs formed by elements of $\cup_{k=1}^n D_k$.

Let Δ_{n+1} be the interval adjacent to Δ_n having $|\Sigma_n|$ elements. Let $\sigma_n \colon \Sigma_n \to \Delta_{n+1}$ be an injection. Define

$$D_{n+1} = \{\xi^* + b\eta^*|(\cup_{i=k+1}^l \Delta_i) + e^*_{\sigma_n(\xi^*,\eta^*)} : (\xi^*, \eta^*) \in \Sigma_n\} \cup \{e_i^* : i \in \Delta_{n+1}\}$$

Let $D = \cup_n D_n$ and define a norm on c_{00} by

$$\|x\| = \text{sup}\{|\sum_i d^*(i)x(i)| : d^* \in D\}$$

X is the completion of c_{00} under this norm.

(e_n) is a normalized basis for X since $|d^*(i)| \leq 1$ for all $i \in \mathbb{N}$ and $d^* \in D$ and $\|d^*|I\| \leq 2$ for all $d^* \in D$ and initial intervals I (use induction and the fact that $b < 1/4$).

Evidently the construction of X implies that the assumptions of the Proposition are fulfilled and thus X is an $\mathcal{L}_\infty$ space. Moreover, since the supports of the elements of D lie within the compact family $\mathcal{F}$, we deduce that (e_n) is shrinking and so X^* is isomorphic to ℓ_1. We also have that X embeds into $C(\mathcal{F})$ which is isomorphic to $C(\omega^\omega)$.

Finally, consider a normalized block basis (u_n) of (e_n). If $\lim_n \|u_n\|_{c_0} = 0$, then standard arguments yield that some subsequence of (u_n) is equivalent to the c_0 basis.

In case there is some $\delta > 0$ so that $\|u_n\|_{c_0} > \delta$ for all $n \in \mathbb{N}$, then for all $k \in \mathbb{N}$ there exist $J \subset \mathbb{N}$ with $|J| = k$ and $d^* \in D$ so that $|d^*(u_i)| \geq \delta b$, for all $i \in J$. It follows now that some subsequence of (u_n) admits ℓ_1 as a spreading model. It follows that $C(\omega^\omega)$ does not embed into X.