On a class of operators on $C(K)$

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Definition

Let $\mathcal{X}$ be a Banach space. An operator $T \in \mathcal{L}(\mathcal{X})$ is called a commutator if $T = AB - BA (= [A, B])$ for some bounded operators A and B in $\mathcal{L}(\mathcal{X})$.

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- The identity element in a unital, normed algebra is not a commutator. (Wintner, 1947)
  $(AB - BA = I \Rightarrow A^n B - BA^n = nA^{n-1} \Rightarrow n\|A^{n-1}\| \leq 2\|A\|\|B\|\|A^{n-1}\|)$ (Wielandt, 1949)

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- $\lambda I + K$ can not be a commutator for any K- compact and $\lambda \neq 0$.
- An operator T on a separable Hilbert space is a commutator if and only if T is not of the form $\lambda I + K$ for some compact operator K and some $\lambda \neq 0$.
  (Brown - Pearcy, 1965)

- If $T \in \mathcal{L}(\ell_p)$, $(1 < p < \infty)$, or $T \in \mathcal{L}(c_0)$ then T is a commutator if and only if T is not of the form $T = \lambda I + K$, where K - compact operator, $\lambda \neq 0$. (Apostol, 1972-1973)

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- If $T \in \mathcal{L}(\ell_\infty)$ then T is a commutator if and only if T is not of the form $T = \lambda I + S$, where S - strictly singular, $\lambda \neq 0$. (Dosev & Johnson, 2010)
- Let $\mathcal{M}$ be the largest ideal in $\mathcal{L}(L^p)$, $1 \leq p < \infty$. An operator $T \in \mathcal{L}(L^p)$ is a commutator if and only if $T - \lambda I \notin \mathcal{M}$ for any $\lambda \neq 0$. (Dosev, Johnson & Schechtman, 2011)

- Let $\mathcal{M}$ be the largest ideal in $\mathcal{L}((\sum \ell_q)_p)$ for $1 \leq q < \infty$ and $1 < p < \infty$. An operator $T \in \mathcal{L}((\sum \ell_q)_p)$ is a commutator if and only if $T - \lambda I \notin \mathcal{M}$ for any $\lambda \neq 0$. (Chen, Johnson & Zheng, 2011)

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For all aforementioned examples the largest ideal $\mathcal{M}$ is defined as follows:

$$\mathcal{M}_{\mathcal{X}} = \{T \in \mathcal{L}(\mathcal{X}) \,:\, I_{\mathcal{X}} \text{ does not factor through } T\}.$$

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For a general Banach space $\mathcal{X}$, $\mathcal{M}_{\mathcal{X}}$ may not be an ideal but if it is in fact an ideal (equivalently, closed under addition) it is the largest ideal on $\mathcal{L}(\mathcal{X})$.

Conjecture

Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ or $p = 0$. Assume that $\mathcal{L}(\mathcal{X})$ has a largest ideal $\mathcal{M}$. Then every non-commutator on $\mathcal{X}$ has the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$.

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To verify the conjecture for a given Banach space $\mathcal{X}$, one must prove two statements:

Step 1. Every operator $T \in \mathcal{M}$ is a commutator.

Step 2. If $T \in \mathcal{L}(\mathcal{X})$ is not of the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$, then T is a commutator.

Theorem (Used in Step 1)

Let $\mathcal{X}$ be a Banach space for which $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X} \right)_p$ for some $1 \leq p < \infty$ or $p = 0$. In the case $p = 1$ we will assume that $\mathcal{X} = L_1$ or $\mathcal{X} = \ell_1$. Then every compact operator on $\mathcal{X}$ is a commutator.

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Theorem (Used in Step 1)

Let $\mathcal{X}$ be a Banach space for which $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$ for some $1 < p < \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ and suppose that P is a projection on $\mathcal{X}$ such that $P\mathcal{X} \simeq \mathcal{X} \simeq (I - P)\mathcal{X}$ and that either TP or PT is a compact operator. Then T is a commutator.

Theorem (Used in Step 2)

Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $X \subset \mathcal{X}$ such that $X \simeq \mathcal{X}$, $T_{|X}$ is an isomorphism, $X + T(X)$ is complemented in $\mathcal{X}$, and $d(X, T(X)) > 0$. Then T is a commutator.

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Definition

Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. We say that a subspace $Z \subseteq \mathcal{X}$ is "nice" for T if $Z \simeq \mathcal{X}$, Z is complemented in $\mathcal{X}$, $T_{|Z}$ is an isomorphism, $d(TZ, Z) > 0$, and $Z + TZ$ is a subspace isomorphic to $\mathcal{X}$ and complemented in $\mathcal{X}$.

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Proposition

Let $T \in \mathcal{L}(C(K))$ and $Z \subset C(K)$ be subspace which is "nice" for T. Then for every $\lambda \in \mathbb{C}$, there exists a subspace Y_λ which is "nice" for $(T - \lambda I)_{|Y}$.

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Proposition

Let $Z \subseteq C(K)$ be a "nice" subspace for an operator $T \in \mathcal{L}(C(K))$ and let $S \in \mathcal{M}_{C(K)}$. Then there exists a subspace $Y \subseteq Z$ which is "nice" for $T + S$.

Proposition (J. Lindenstrauss & A. Pelczynski, 1971)

Let $T : C(\Delta) \to Y$ *be an operator such that for every* $\varepsilon > 0$ *and for every clopen nonempty subset* Δ_1 *of* Δ*, there is an* $f \in C(\Delta_1)$ *such that* $\|f\| = 1$ *and* $\|Tf\| < \varepsilon$*. Then for each* $\varepsilon > 0$ *there is a sequence* $\{g_i\}_{i=1}^{\infty}$ *in* $C(\Delta)$ *which is isometrically equivalent to the Haar system and such that* $\sum_{i=1}^{\infty} \|Tg_i\| < \varepsilon$*.*

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All $C(K)$ - strictly singular operators satisfy the conditions of the above Proposition and combining this with the fact that all $C(K)$ spaces are isomorphic we have that there exists a subspace Y of X such that $Y \simeq C(K)$ and $T_{|Y}$ is a compact operator.

Setup: K be a compact metric space, μ be a probability measure on K, $T : C(K) \to C(K)$ and let $\{\mu_s\}_{s\in K}$ be the representing kernel of T (the family of Borel measures on K defined by $\mu_s = T^*\delta_s, s \in K$).

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Using a theorem of Kalton, we decompose each μ_s into their atomic and diffuse parts as follows

$$\mu_s = \sum_{n=1}^{\infty} a_n(s) \delta_{\sigma_n(s)} + \nu_s$$

where by δ_x we denote point evaluation at the point x and

- Each $a_n : K \to \mathbb{R}$ is measurable for the completion $\sum_\mu$ of the Borel sets of K with respect to μ
- Each $\sigma_n : K \to K$ is $\sum_\mu$-Borel measurable
- Each ν_s is diffuse, and $s \to |\nu_s|$ is $\sum_\mu$-Borel measurable with respect to the weak* Borel sets of the unit ball of $C(K)^*$
- If $i \neq j$ then $\sigma_i(s) \neq \sigma_j(s)$ for all $s \in K$
- $|a_j(s)| \geq |a_{j+1}(s)|$ for all $s \in K$ and all $j \geq 1$
- $\sum_{n=1}^{\infty} |a_n(s)| \leq \|T\|$ for all $s \in K$

Theorem

Let $T \in \mathcal{L}(C(K))$ *be an operator such that*

- $Tf(x) = \sum_{n=1}^{N} a_n(x) f(\sigma_n(x))$ *for every* $f \in C(K)$ *and* $x \in K$
- $a_n : K \to \mathbb{R}$ *and* $\sigma_n : K \to K$ *are continuous functions for* $n = 1, 2, \ldots N$
- *For all* $i \neq j$, $\sigma_i(x) \neq \sigma_j(x)$ *for all* $x \in K$
- $T - \lambda I \notin \mathcal{M}$ *for any* $\lambda \in \mathbb{C}$.

Then there exists a subspace $Y \subset C(K)$ *which is "nice" for* T.

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2. Are the compact operators (on any Banach space) always commutators ?
3. In which spaces every compact operator is a commutator of two compacts?