L-spaces and left-orderability

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(joint with Steve Boyer and Liam Watson)

BIRS Workshop

Banff, February 2012

Left Orderability

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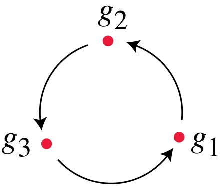
- G (countable) LO $\iff \exists$ embedding $G \subset \mathrm{Homeo}_+(\mathbb{R})$
- G LO $\iff$ every finitely generated $H < G$ has an LO quotient (Burns-Hale, 1972)

G is left circularly orderable (LCO) if $\exists$ strict circular order on G, $T \subset G^3$, such that $(g_1, g_2, g_3) \in T \Rightarrow (fg_1, fg_2, fg_3) \in T \; \forall f \in G$



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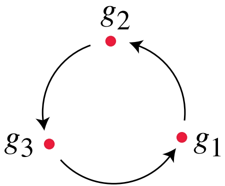

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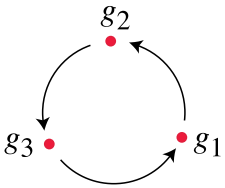

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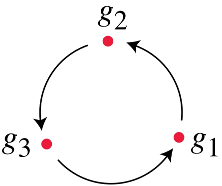

- S^1 is LCO
- G (countable) LCO $\iff$ $\exists$ embedding $G \subset \mathrm{Homeo}_+(S^1)$
- G LO $\Rightarrow$ G LCO

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H finite index: $\varphi : \pi_1(M) \twoheadrightarrow Q,\ Q$ LO

Then $\varphi(H) < Q$ finite index $\therefore\ \varphi(H) \neq 1$ □

Hence $\beta_1(M) > 0 \Rightarrow \pi_1(M)$ LO

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So interesting case is when

$$H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$$

M is a $\mathbb{Q}$-homology 3-sphere ($\mathbb{Q}$HS)

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$\exists$ complete arithmetic characterization of those $S^2(a_1, \ldots, a_n)$'s that admit horizontal foliations (Jankins-Neumann, 1985, Naimi, 1994).

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Central extension

$$\begin{array}{ccccccccc} 1 \to & \mathbb{Z} & \to & \widetilde{\text{Homeo}_+(S^1)} & \to \text{Homeo}_+(S^1) \to 1 \\ & \| & & \| & \\ \end{array}$$

$$\left\{\text{integer translations}\right\} \subset \left\{f \in \text{Homeo}_+(\mathbb{R}) : f(x+1) = f(x) + 1 \ \forall\, x \in \mathbb{R}\right\}$$

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Restriction of ρ to $\pi_1(\widetilde{M})$ lifts to $\widetilde{\mathrm{Homeo}}_+(S^1) \subset \mathrm{Homeo}_+(\mathbb{R})$

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Conjecture

M a prime $\mathbb{Q}$HS. Then

$$M \text{ is an } L\text{-space} \iff \pi_1(M) \text{ is not LO}$$

E.g.

$$\pi_1(M) \text{ finite } \begin{array}{l} \nearrow \quad M \text{ is an } L\text{-space} \\ \\ \searrow \quad \pi_1(M) \text{ not LO} \end{array}$$

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By Calegari-Dunfield we do have (if M atoroidal)

(1) $\pi_1(M)$ virtually LO

(2) $\pi_1(M)$ LCO

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M Seifert fibered $\mathbb{Q}$HS $\quad S^2(p_1/q_1,\, p_2/q_2,\, p_3/q_3),\ \sum 1/q_i < 1$

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$\exists$ central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \Delta \longrightarrow 1$$

$$\Delta = (q_1, q_2, q_3)\text{-triangle group}$$
$$\Delta \subset \mathrm{Isom}_+(\mathbb{H}^2) \cong PSL_2(\mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$$

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$$\Delta \ \text{LCO}, \ \mathbb{Z}\ \text{LO} \Longrightarrow \pi_1(M) \ \text{LCO}$$

But $\pi_1(M)$ LO $\Longleftrightarrow$ M admits a horizontal foliation, and this doesn't always hold.

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Show M is an L-space by induction on n; surgery argument using:

X compact, orientable 3-manifold, ∂X a torus; α essential scc $\subset \partial X$,

$X(\alpha) = \alpha$-Dehn filling on X

Suppose $\alpha, \beta \subset \partial X, \quad \alpha \cdot \beta = 1$, and

$$|H_1\big(X(\alpha+\beta)\big) = |H_1\big(X(\alpha)\big)| + |H_1\big(X(\beta)\big)|$$

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Then $\qquad X(\alpha), X(\beta)$ L-spaces $\Rightarrow X(\alpha+\beta)$ L-space $\qquad (*)$

(OS, 2005)

(uses $\widehat{HF}$ surgery exact sequence of a triad)

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(Also work by Boyer-Clay-Watson)

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M Seifert $\Leftrightarrow f(\varphi_i) = \pm\varphi_j$ (some $i, j \in \{0, 1\}$)

Otherwise, M is a Sol manifold

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Theorem

M is an L-space

$$f_* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (c \neq 0) \text{ with respect to basis } \varphi_0, \varphi_1$$

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where $t_0 : \partial N \longrightarrow \partial N$ is Dehn twist along φ_0

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(3) In general, induct on $|c|$: do surgery on suitable simple closed curves $\subset \partial N$ and use $(*)$

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Theorem (Roberts, 1995)

K an alternating knot.

(1) *If K is not special alternating then $K(r)$ has a taut foliation $\forall\, r \in \mathbb{Q}$.*

(2) *If K is special alternating then $K(r)$ has a taut foliation either $\forall\, r > 0$ or $\forall\, r < 0$.*

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Pretzel knot $K(-2, 3, n)$, n odd ≥ 5, is an L-space knot.

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$\pi_1(K(-2,3,n)(r))$ *is not* LO *if* $r \geq n + 10$.

(D) 2-fold branched covers

L a link in S^3

$\Sigma(L)$ =2-fold branched cover of L

Theorem (OS, 2005)

If L is a non-split alternating link then $\Sigma(L)$ is an L-space.

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Theorem

If L is a non-split alternating link then $\pi_1(\Sigma(L))$ is not LO.

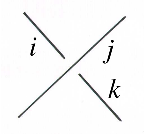
(Also proofs by Greene, Ito)

L a link in S^3, D a diagram of L

Define group $\pi(D)$:

$$\text{generators } a_1, \ldots, a_n \longleftrightarrow \text{arcs of } D$$

$$\text{relations} \longleftrightarrow \text{crossings of } D$$



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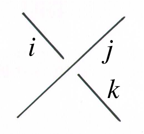

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Theorem (Wada, 1992)

$\pi(D) \cong \pi_1(\Sigma(L)) * \mathbb{Z}$

Suppose $\pi(D)$ LO

$$a_j^{-1} a_i a_j^{-1} a_k = 1 \iff a_j^{-1} a_i = a_k^{-1} a_j$$

$$a_i < a_j \iff a_j^{-1} a_i < 1$$

$\therefore$ at each crossing either

$$a_i < a_j < a_k$$

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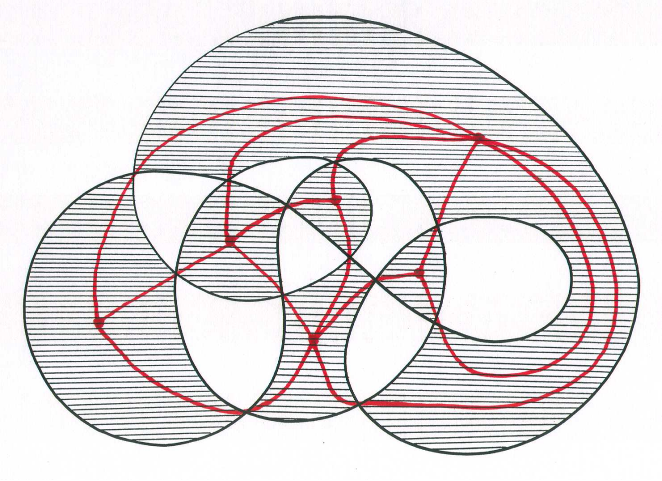
Shade complementary regions of D alternately Black/White

Define graph $\Gamma(D) \subset S^2$:

$$\text{vertices} \longleftrightarrow B\text{-regions}$$

$$\text{edges} \longleftrightarrow \text{crossings}$$



Assume D connected, alternating

We want to show $\pi_1(\Sigma(L))$ not LO

True if $L =$ unknot; so assume $L \neq$ unknot

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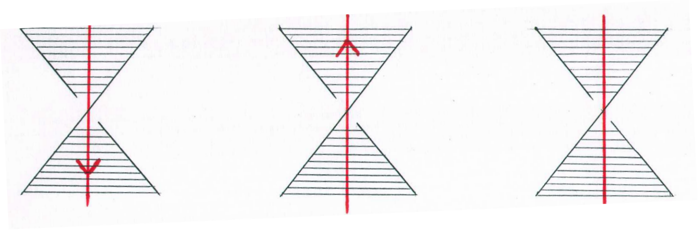
Orient edges of $\Gamma(D)$



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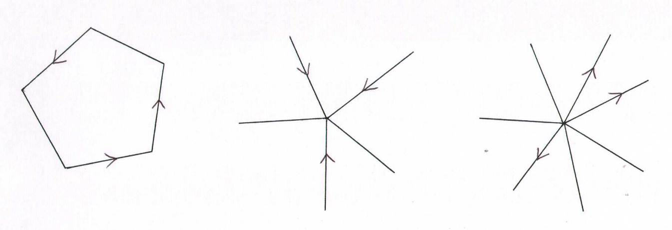

cycle **sink** **source**

where, in each case, there is at least one oriented edge

Lemma

Let $\Gamma \subset S^2$ *be a connected semi-oriented graph with at least one oriented edge. Then* Γ *has a sink, source or cycle.*

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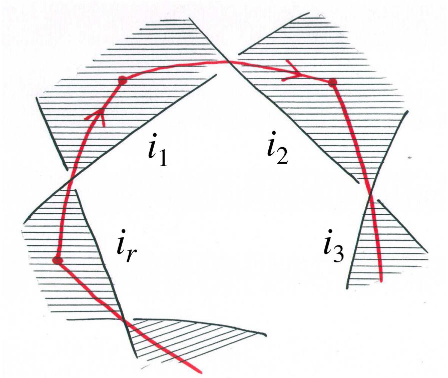
Let $\quad \Gamma = \Gamma(D)$

cycle:

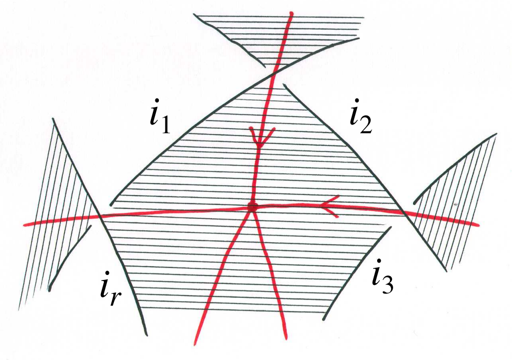


$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_r} \leq a_{i_1}$$

$$\therefore \quad a_{i_1} = a_{i_2} = \cdots = a_{i_r}$$

a contradiction, since at least one oriented edge

sink:



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$\therefore$ by Lemma, all edges of $\Gamma(D)$ are unoriented

$\therefore$ (since D connected) $a_1 = a_2 = \cdots = a_n$

$$\therefore \quad \pi(D) \cong \mathbb{Z}$$

$$\therefore \quad \pi_1(\Sigma(L)) = 1$$

$$\therefore \quad L = \text{ unknot,} \quad \text{contradiction}$$

(E) Questions

Question 1

If M is a $\mathbb{Q}$HS with a co-orientable taut foliation, is $\pi_1(M)$ LO?

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Conjecture $\implies$ Q's 1, 2 and 3 have answer "yes"

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Question 6

M a hyperbolic $\mathbb{Z}$HS. Is $\pi_1(M)$ LO?