New Trends and Directions in Combinatorics*

Organizers

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1 Overview

Combinatorics, or Discrete Mathematics, is a fundamental mathematical discipline, focusing on the study of discrete objects and their properties. Although Combinatorics is probably as old as the human ability to count, the field has experienced tremendous growth in the last fifty years, partially spurred by its tight connections with other growing branches of science, most notably Theoretical Computer Science. While in the past many of the basic combinatorial results were obtained mainly through ingenuity and detailed reasoning, the modern theory has grown out of this early stage, and often relies on deep, well-developed and sophisticated tools. Faithfully reflecting the mature status of the field are deep and prolific interconnections between its various branches – one frequently uses probabilistic considerations to attack a problem in Extremal Graph Theory, or relies on extremal results to make advances in a discrete geometric problem; examples of such fruitful cooperation abound.

In recent years Combinatorics appears to have made a qualitative leap forward, with sizeable or even astonishing progress having been achieved in a variety of directions. Some of the key areas of Combinatorics to have developed greatly in these years are Extremal Graph Theory, Extremal Set Theory, Ramsey Theory, and Probabilistic Combinatorics. In what follows we provide a brief outline of these fields, which were the focus of attention over the course of this workshop.

Extremal Graph Theory explores the question of how large a graph can be if a certain substructure is forbidden. This is one of the most important branches of modern Graph Theory, with a variety of methods and arguments applied, including linear algebraic arguments, analytic tools, probabilistic considerations, and the regularity method. A concrete example of an extremal problem in which there has been striking progress in recent years is the Turán problem for which the forbidden subgraph is bipartite. This is a notoriously difficult problem that has eluded researchers for many years, in sharp contrast to the case in which the chromatic number is at least three (which by now is very well understood). However, recent new advances involving sophisticated algebraic and geometric constructions have given fresh insight on the topic, and promise that much more can be achieved.

Extremal Set Theory problems are usually formulated and studied for families of sets satisfying given restrictions. Problems of this type are especially appealing, in part due to the fact that they frequently arise in a variety of applications in diverse fields of Mathematics, Computer Science, and Coding and Information Theory. Recent advances have resulted from several new sophisticated methods such as hypergraph regularity, the stability approach, and Fourier analytic techniques. These approaches have been developed over the past decade, and are now reaching the point of maturity that can make them standard general (and very powerful) tools for

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researchers. Again to give just one example, significant new progress on Turán problems for hypergraphs has been achieved using some of these methods.

Ramsey Theory is undoubtedly one of the most central branches of modern mathematics, studying quantitatively the phenomenon that every sufficiently large object, chaotic as it may be, must contain a well-structured sub-object. Quite a few questions from Ramsey theory, including estimates on the so called Ramsey numbers, can be cast and viewed as problems in Extremal Graph or Set Theory. Probabilistic arguments are essential here, and their importance and applicability cannot be overestimated. The recent development of analytic tools, used in conjunction with the probabilistic methods, has inspired a great deal of progress in this area in the last few years, leading to the resolution of a number of long-standing conjectures.

Probabilistic Combinatorics studies probability spaces of discrete structures. In quite a few cases these probability spaces are deep and complex and are studied entirely for their own sake, yet probabilistic considerations are indispensable in many other areas of Combinatorics, most notably for extremal problems and Ramsey Theory. They also form a mathematical foundation for the design and analysis of algorithms involving randomness, and more generally for addressing the fundamental role of randomness in Theoretical Computer Science, especially in Algorithmics and Complexity. Recently, there has been an impressive stream of novel and exciting results and concepts in the field. Some of them include development of sophisticated techniques, deepening our understanding of the phase transition in various models of random graphs; important advances in extremal properties (Turán and Ramsey-type problems) of random graphs; analysis of the so-called controlled random processes; and applications of sophisticated tools from Extremal Graph Theory (such as the Sparse Regularity Lemma) to random graphs, to mention just a few.

It should be stressed that the above mentioned main fields of Combinatorics, as well as several of its other branches, are tightly intertwined, and in quite a few cases progress in one of the directions soon results in some equally exciting progress in another. One aim of the workshop was to encourage interaction between researchers in different fields, to facilitate the exchange of ideas and enable the introduction of novel techniques to different areas of Combinatorics. Moreover, certain new directions of combinatorial research were highlighted, most notably the study of combinatorial problems in an algebraic setting.

In the remainder of this report we present in detail some of the advances presented at the workshop.

2 Extremal Graph Theory

Rainbow Turán problems

Shagnik Das *joint with C. Lee and B. Sudakov*

The rainbow Turán problem, first introduced by Keevash, Mubayi, Sudakov and Verstraëte in 2007, beautifully combines two central and oft-studied branches of Extremal Graph Theory, namely Turán Theory and Graph Colouring. We say an edge-coloured graph is coloured *properly* if no two adjacent edges share a colour, and we say it is *rainbow* if every edge receives a unique colour. The rainbow Turán problem asks, for a given graph H, how many edges a properly edge-coloured graph on n vertices can have if it does not contain a rainbow copy of H. In their paper, Keevash, Mubayi, Sudakov and Verstraëte determine asymptotically the rainbow Turán number for any non-bipartite graph. However, as is the case in classical Turán Theory, the problem appears to be much more difficult for bipartite graphs. The original authors remark that the problem is most interesting in the case of even cycles, due to a connection to a problem in Additive Number Theory. As observed by Mubayi and Verstraëte during this workshop, this case also has implications for the Turán number of linear cycles in 3-uniform hypergraphs.

In this talk, we discuss some recent results on the rainbow Turán number for even cycles. We show that any properly edge-coloured graph on n vertices with $O\left(n^{1+(1+\varepsilon_k)\ln k/k}\right)$ edges contains a rainbow cycle of length $2k$, where $\varepsilon_k \to 0$ as $k \to \infty$. This improves the previous best-known bound of $O\left(n^{3/2}\right)$, and is significantly closer to the lower bound of $\Omega\left(n^{1+1/k}\right)$. We also narrow the gap between the upper and lower bounds on the size of properly edge-coloured graphs without rainbow cycles of any length.

Chromatic number, clique subdivisions, and the conjectures of Hajos and Erdős-Fajtlowicz
Jacob Fox *joint with C. Lee and B. Sudakov*

A *subdivision* of a graph H is any graph formed by replacing edges of H by internally vertex disjoint paths. This is an important notion in graph theory, e.g., the celebrated theorem of Kuratowski uses it to characterize planar graphs. For a graph G, we let $\sigma(G)$ denote the largest integer p such that G contains a subdivision of a complete graph of order p. Clique subdivisions in graphs have been extensively studied and there are many results which give sufficient conditions for a graph G to have large $\sigma(G)$. For a given graph G, let $\chi(G)$ denote its chromatic number. A famous conjecture made by Hajós in 1961 states that $\sigma(G) \geq \chi(G)$. Dirac proved that this conjecture is true for all $\chi(G) \leq 4$, but in 1979, Catlin disproved the conjecture for all $\chi(G) \geq 7$. By considering random graphs, Erdős and Fajtlowicz in 1981 showed that the conjecture actually fails for almost all graphs.

We revisit Hajós' conjecture and study to what extent the chromatic number of a graph can exceed the order of its largest clique subdivision. Let $H(n)$ denote the maximum of $\chi(G)/\sigma(G)$ over all n-vertex graphs G. Erdős and Fajtlowicz showed that almost all graphs on n vertices satisfy $\sigma(G) = O(n^{1/2})$ and $\chi(G) = \Theta(n/\log n)$. Thus it implies that $H(n) = \Omega(n^{1/2}/\log n)$. Erdős and Fajtlowicz conjectured that this bound is tight up to a constant factor so that $H(n) = \Theta(n^{1/2}/\log n)$. This conjecture says that the random graph is essentially the strongest counterexample to the Hajós' conjecture. We verify the Erdős-Fajtlowicz conjecture.

Extremal problems in Eulerian digraphs
Hao Huang *joint with J. Ma, A. Shapira, B. Sudakov and R. Yuster*

One of the central themes in graph theory is to study the extremal graphs which satisfy certain properties. There are many classical results in this area. For example, any undirected graph G with n vertices and m edges has a subgraph with minimum degree at least m/n, and thus G also contains a cycle of length at least $m/n + 1$. It is natural to ask whether such results can be extended to digraphs. However, it turns out that graphs and digraphs behave quite differently, and many classical results for graphs are often trivially false when extended to general digraphs. Therefore it is usually necessary to restrict to a smaller family of digraphs to obtain meaningful results. One such very natural family is Eulerian digraphs, in which the in-degree equals out-degree at every vertex. In this talk, we discuss several natural parameters for Eulerian digraphs and study their connections. In particular, we prove the following result.

Theorem 1. *For any Eulerian digraph G with n vertices and m edges, the minimum feedback arc set (the smallest set of edges whose removal makes G acyclic) has size at least $m^2/2n^2 + m/2n$, and this bound is tight.*

Using this result, we show how to find subgraphs of high minimum degrees, and also the existence of long cycles in Eulerian digraphs. In particular, we prove that every Eulerian digraphs with n vertices and m edges contains a cycle of length at least $\Omega(m^2/n^3)$. This verifies a conjecture of Bollobás and Scott for dense digraphs.

On k-color-critical n-vertex graphs with fewest edges
Alexander Kostochka *joint with M. Yancey*

A graph G is *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$–colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. We give a lower bound, $f_k(n) \geq F(k,n)$, that is sharp for every $n = 1 \pmod{k-1}$. It is also sharp for $k = 4$ and every $n \geq 6$. The result improves the classical bounds by Gallai and Dirac and subsequent bounds by Krivelevich and Kostochka and Stiebitz. It establishes the asymptotics of $f_k(n)$ for every fixed k. It also proves that the conjecture by Ore from 1967 that for every $k \geq 4$ and $n \geq k+2$, $f_k(n+k-1) = f(n) + \frac{k-1}{2}(k - \frac{2}{k-1})$ holds for each $k \geq 4$ for all but at most $k^3/12$ values of n. We give a polynomial-time algorithm for $(k-1)$-coloring a graph G that satisfies $|E(G[W])| < F_k(|W|)$ for all $W \subseteq V(G)$, $|W| \geq k$. We also present some applications of the result.

One of the corollaries of our theorem is a half-page proof of the theorem due to Grötzsch that every triangle-free planar graph is 3-colorable.

Asymptotic structure of graphs with the minimum number of triangles

Oleg Pikhurko *joint with A. Razborov*

Let $g(m,n)$ be the smallest number of triangles in a graph with n vertices and m edges. Let us consider the asymptotic question, that is, what is the limit

$$g(a) = \lim_{n\to\infty} \frac{g\big(\lfloor a\binom{n}{2}\rfloor, n\big)}{\binom{n}{r}}?$$

While it is not difficult to show that the limit exists, determining $g(a)$ is a much harder task that was accomplished only recently by Razborov (with previous partial results obtained by Bollobás, Erdős, Lovász, Mantel, Simonovits, and others).

The following construction gives the value of $g(a)$. Choose t and $c \in \left[\frac{1}{t+1}, \frac{1}{t}\right]$ such that the complete $(t+1)$-partite graph of order $n \to \infty$ with t largest parts each of size $(c+o(1))n$ has edge density $a + o(1)$. Partition the vertex set $[n] = \{1, \dots, n\}$ into $t+1$ parts $V_1, \dots, V_{t+1}$ with $|V_1| = \cdots = |V_t| = \lfloor cn \rfloor$ for $i \in [t]$. Let G be obtained from the complete t-partite graph $K(V_1, \dots, V_{t-1}, U)$, where $U = V_t \cup V_{t+1}$, by adding an arbitrary triangle-free graph on U with $|V_t|\,|V_{t+1}|$ edges. Let $\mathcal{H}$ consist of all graphs obtained this way.

We prove the following result.

Theorem 2. *For every $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that every graph G with $n \geq n_0$ vertices and at most $(g(a)+\delta)\binom{n}{3}$ triangles, where $a = e(G)/\binom{n}{2}$, can be made isomorphic to some graph in $\mathcal{H}$ by changing at most $\varepsilon\binom{n}{2}$ adjacencies.*

A proof of Ohba's conjecture

Bruce Reed *joint with J. Noel and H. Wu*

List colouring is a variation on classical graph colouring. An instance of list colouring is obtained by assigning to each vertex v of a graph G a list $L(v)$ of *available colours.* An *acceptable colouring* for L is a proper colouring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. When an acceptable colouring for L exists, we say that G is *L-colourable.* The *list chromatic number* χ_ℓ is defined in analogy to the chromatic number:

$$\chi_\ell(G) = \min\{k : G \text{ is } L\text{-colourable whenever } |L(v)| \geq k \text{ for all } v \in V(G)\}.$$

List colouring was introduced by Vizing, and independently by Erdős, Rubin, and Taylor, and researchers have devoted a considerable amount of energy towards its study ever since.

A graph G has an ordinary k-colouring precisely if it has an acceptable colouring for L where $L(v) = \{1, 2, \dots, k\}$ for all $v \in V(G)$. Therefore, the following bound is immediate:

$$\chi \leq \chi_\ell.$$

At first glance, one might expect the reverse inequality to hold as well. It would seem that having smaller intersection between colour lists could only make it easier to find an acceptable colouring. However, this intuition is misleading, there are bipartite graphs with arbitrary high list-chromatic number.

The problem of determining which graphs satisfy $\chi_\ell = \chi$ is well studied; such graphs are said to be *chromatic-choosable.*

For example, the famous List Colouring Conjecture claims that every line graph is chromatic-choosable. It first appeared in print in a paper of Bollobás and Harris, but had also been formulated independently by Albertson and Collins, Gupta, and Vizing. Galvin showed that the List Colouring Conjecture is true for line graphs of bipartite graphs. Kahn proved that the List Colouring Conjecture is asymptotically correct.

We will present a proof of the following conjecture of Ohba:

Conjecture 1 (Ohba)**.** *If $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic-choosable.*

On a conjecture of Erdős and Simonovits on bipartite Turán numbers

Jacques Verstraëte *joint with P. Allen, P. Keevash and B. Sudakov*

Let $\mathcal{C}_k = \{C_3, C_5, \ldots, C_k\}$ for an odd integer k denote the family of all odd cycles of length at most k and let $\mathcal{C}$ denote the family of all odd cycles. Erdős and Simonovits conjectured that for every family $\mathcal{F}$ of bipartite graphs, there exists k such that $\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \mathrm{ex}(n, \mathcal{F} \cup \mathcal{C})$ as $n \to \infty$. This conjecture was proved by Erdős and Simonovits when $\mathcal{F} = \{C_4\}$, a result that has since been extended to certain families of even cycles. In this paper, we give a general approach to the conjecture using Scott's sparse regularity lemma. Our approach proves the conjecture for complete bipartite graphs $\mathcal{F} = \{K_{2,t}\}$ and $\mathcal{F} = \{K_{3,3}\}$: we obtain more strongly that for any odd $k \geq 5$,

$$\mathrm{ex}(n, \mathcal{F} \cup \{C_k\}) \sim \mathrm{ex}(n, \mathcal{F} \cup \mathcal{C})$$

and we show further that the extremal graphs can be made bipartite by deleting very few edges. It is natural to ask whether for large enough n, extremal $\mathcal{F} \cup \{C_k\}$-free graphs are exactly bipartite. We prove that this is true for those of large enough minimum degree, for instance, if $k \geq 5$ is odd and G is a $\{C_4, C_k\}$-free n-vertex non-bipartite graph, then G has a vertex v such that $d(v) \leq \sqrt{2n/5} + o(\sqrt{n})$. In contrast, these results do not extend to triangles – the case $k = 3$ – and we give an algebraic construction for odd $t \geq 3$ of $K_{2,t}$-free C_3-free graphs with substantially more edges than an extremal $K_{2,t}$-free bipartite graph on n vertices. Our general approach to the Erdős-Simonovits conjecture is effective based on some reasonable assumptions on the maximum number of edges in an m by n bipartite $\mathcal{F}$-free graph.

The Turán number of sparse spanning graphs

Raphael Yuster *joint with N. Alon*

For a graph H, the *extremal number* $ex(n, H)$ is the maximum number of edges in a graph of order n not containing a subgraph isomorphic to H. Let $\delta(H) > 0$ and $\Delta(H)$ denote the minimum degree and maximum degree of H, respectively. We prove that for all n sufficiently large, if H is any graph of order n with $\Delta(H) \leq \sqrt{n}/40$, then $ex(n, H) = \binom{n-1}{2} + \delta(H) - 1$. The condition on the maximum degree is tight up to a constant factor. This generalizes a classical result of Ore for the case $H = C_n$, and resolves, in a strong form, a conjecture of Glebov, Person, and Weps for the case of graphs. A counter-example to their more general conjecture concerning the extremal number of bounded degree spanning hypergraphs is also given.

The proof of the main result is constructive. We consider the equivalent packing version of the problem where G is a graph with $n - \delta - 1$ edges, where $\delta = \delta(H)$. It suffices to prove that G and H pack. The proof constructs a bijection $f : V(G) \to V(H)$ such that for all $(u, v) \in E(G)$, $(f(u), f(v)) \notin E(H)$ (the packing property). The initial step consists of packing the large degree vertices of G (in decreasing order), and some random subsets of independent non-neighbors of them, with appropriate vertices of H. This, however, can only be done iteratively until some point, where one has to switch to a more "global-approach" of packing the remaining vertices all at once. We make sure, by maintaining certain invariants (whp), that when arriving at this final stage, the global packing can indeed succeed as it satisfies (whp) sufficient conditions for a perfect matching between the yet-unpacked vertices of G and the yet-unpacked vertices of H that does not violate the packing property.

3 Extremal Set Theory

2-cancellative codes, an algebraic construction for a hypergraph Turán problem
Zoltan Furedi

There are many instances in Coding Theory when codewords must be restored from partial information, like defected data (error correcting codes), or some superposition of the strings. These lead to superimposed codes, a close relative of group testing problems.

There are lots of versions and related problems, like Sidon sets, sum-free sets, union-free families, locally thin families, cover-free codes and families, etc. We discuss here *cancellative* codes, esp. 2-cancellative uniform hypergraphs.

A family $\mathcal{F}$ is called 2-*cancellative* if for all distinct four members A, B, C, D we have

$$A \cup B \cup C \neq A \cup B \cup D.$$

Our main result is to determine the order of magnitude of the largest $2r$-uniform 2-cancellative hypergraph on n vertices. The proof of the lower bound is an almost explicit construction obtained by a combination of the probabilistic and the algebraic methods.

On the construction of 3-chromatic hypergraphs with few edges
Heidi Gebauer

A *hypergraph* is a pair (V, E), where V is a finite set whose elements are called *vertices* and E is a family of subsets of V, called *hyperedges*. A hypergraph is n-*uniform* if every hyperedge contains exactly n vertices. An r-coloring of a hypergraph (V, E) is a mapping $c : V \to \{1, \dots, r\}$. An r-coloring c is proper if no edge in E is monochromatic under c. The *chromatic number* $\chi(H)$ of a hypergraph H is the minimum r such that H admits a proper r-coloring. A hypergraph H is r-*chromatic* if $\chi(H) = r$.

The minimum number $m(n)$ of hyperedges in a 3-chromatic n-uniform hypergraph has been widely studied in the literature. Erdős found that $2^{n-1} \leq m(n) \leq O(n^2 2^n)$. The lower bound was improved in a sequence of publications: Schmidt showed that $m(n) \geq (1 - \frac{2}{n})2^n$, then Beck increased the factor $(1 - \frac{2}{n})$ to $n^{\frac{1}{3}-o(1)}2^n$. A simpler proof of Beck's result was given by Spencer. The best known lower bound is due to Radhakrishnan and Srinivasan, who proved that $m(n) \geq 0.7 \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} 2^n$. In 2009 Pluhár gave a very short and elegant proof that $m(n)$ is at least $2^n f(n)$ where $f(n)$ goes to infinity. (In his paper, $f(n)$ is roughly $n^{\frac{1}{4}}$.)

The upper bound of Erdős is still the best we know. The proof uses the probabilistic method and does not indicate how the corresponding hypergraph can be constructed. We investigate a constructive upper bound on $m(n)$. The only previous result we are aware of is the trivial bound $m(n) \leq \binom{2n-1}{n}$, which is achieved by the hypergraph on the vertex set $V = \{v_1, \dots, v_{2n-1}\}$ where the edge set consists of all subsets of V of cardinality n. In this talk we give an explicit construction of a 3-chromatic n-uniform hypergraph with at most $2^{(1+o(1))n}$ hyperedges. Our technique can also be used to describe n-uniform hypergraphs with chromatic number at least $r + 1$ and at most $r^{(1+o(1))n}$ hyperedges, for every $r \geq 3$.

Lagrangians of intersecting families and Turán numbers of hypergraphs
Dan Hefetz *joint with P. Keevash*

It is a central problem in Extremal Combinatorics to determine the Turán number of an r-graph F, that is, the maximum possible number of edges in an F-free r-graph on n vertices (at least for sufficiently large n). The special case of graphs (that is, $r = 2$) is mostly solved. However, to date very little is known if $r \geq 3$, even asymptotically. Recent years have witnessed increasing interest and the development of novel methods leading to new results.

In this talk we prove a Turán type theorem for the 3-graph $\mathcal{K}^3_{3,3}$ whose vertices are $\{x_i, y_i : 1 \leq i \leq 3\}$ and $\{z_{ij} : 1 \leq i, j \leq 3\}$ and whose edges are $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$ and $\{\{x_i, y_j, z_{ij}\} : 1 \leq i, j \leq 3\}$. We prove that for large n, the unique largest $\mathcal{K}^3_{3,3}$-free 3-graph on n vertices is a balanced blowup of the complete 3-graph on 5 vertices. Our proof consists of three stages. First we determine the Turán density of $\mathcal{K}^3_{3,3}$ using lagrangians of intersecting families. We then use the stability method to prove that every $\mathcal{K}^3_{3,3}$-free 3-graph with sufficiently many edges is close to a balanced blowup of the complete 3-graph on 5 vertices. Finally, we use this stability result to determine exactly the Turán number of $\mathcal{K}^3_{3,3}$ and to prove uniqueness.

4 Ramsey Theory

The critical window for the classical Ramsey-Turán problem
Po-Shen Loh *joint with J. Fox and Y. Zhao*

The first application of Szemerédi's powerful regularity method was the following celebrated Ramsey-Turán result proved by Szemerédi in 1972: any K_4-free graph on n vertices with independence number $o(n)$ has at most $(\frac{1}{8}+o(1))n^2$ edges. Four years later, Bollobás and Erdős gave a surprising geometric construction, utilizing the isoperimetric inequality for the high dimensional sphere, of a K_4-free graph on n vertices with independence number $o(n)$ and $(\frac{1}{8}-o(1))n^2$ edges. Starting with Bollobás and Erdős in 1976, several problems have been asked on estimating the minimum possible independence number in the critical window, when the number of edges is about $n^2/8$. These problems have received considerable attention and remained one of the main open problems in this area. In this paper, we give nearly best-possible bounds, solving the various open problems concerning this critical window.

More generally, it remains an important problem to determine if, for certain applications of the regularity method, alternative proofs exist which avoid using the regularity lemma and give better quantitative estimates. In their survey on the regularity method, Komlós, Shokoufandeh, Simonovits, and Szemerédi surmised that the regularity method is likely unavoidable for applications where the extremal graph has densities in the regular partition bounded away from 0 and 1. In particular, they thought this should be the case for Szemerédi's result on the Ramsey-Turán problem. Contrary to this philosophy, we develop new regularity-free methods which give a linear dependence, which is tight, between the parameters in Szemerédi's result on the Ramsey-Turán problem.

A problem of Erdős on the minimum number of k-cliques
Jie Ma *joint with S. Das, H. Huang, H. Naves and B. Sudakov*

Fifty years ago Erdős asked to determine the minimum number of k-cliques in a graph on n vertices with independence number less than l (we will refer this as (k,l)-problem). He conjectured that this minimum is achieved by the disjoint union of $l-1$ complete graphs of size $\frac{n}{l-1}$. This conjecture was disproved by Nikiforov who showed that Erdős' conjecture can be true only for finite many pairs of (k,l). For $(4,3)$-problem, Nikiforov further conjectured that the balanced blow-up of a 5-cycle, which has fewer 4-cliques than the union of 2 complete graphs of size $\frac{n}{2}$, achieves the minimum number of 4-cliques.

Using a combination of explicit and random counterexamples, we first sharpen Nikiforov's result and show that Erdős' conjecture is false whenever $k \geq 4$ or $k=3, l \geq 2074$. After introducing tools (including Flag Algebra) used in our proofs, we state our main theorems, which characterize the precise structure of extremal examples for $(3,4)$-problem and $(4,3)$-problem, confirming Erdős' conjecture for $(k,l)=(3,4)$ and Nikiforov's conjecture for $(k,l)=(4,3)$. We then focus on $(4,3)$-problem and sketch the proof how we use stability arguments to get the extremal graphs, the balanced blow-ups of 5-cycle.

Densities of cliques and independent sets in graphs
Humberto Naves *joint with H. Huang, N. Linial, Y. Peled and B. Sudakov*

A variety of problems in extremal combinatorics can be stated as: For two given graphs H_1 and H_2, if the number of induced copies of H_1 in a n-vertex graph G is known, what is the maximum or minimum number of induced copies of H_2 in G? Numerous cases of this question were studied by Turán, Erdős, Kruskal and Katona, and several others. Turán proved that the maximal edge density in any graph with no cliques of size r is attained by an $r-1$ partite graph. Kruskal and Katona found that cliques, among all graphs, maximize the number of induced copies of K_s when $r < s$ and the number of induced copies of K_r is fixed. In this talk, we discuss the following analogue of the Kruskal-Katona theorem: If the complement of a graph has fixed induced K_r-density, when is the induced K_s-density maximized? Using the technique of shifting borrowed from extremal set theory and some powerful analytical methods, one can demonstrate that the extremal graph for our proposed question is either a clique or the complement of a clique.

Ramsey theory, integer partitions and a new proof of the Erdős-Szekeres theorem
Asaf Shapira *joint with G. Moshkovitz*

Let H be a k-uniform hypergraph whose vertices are the integers $1,\ldots,N$. We say that H contains a monotone path of length n if there are $x_1 < x_2 < \cdots < x_{n+k-1}$ so that H contains all n edges of the form $\{x_i, x_{i+1}, \ldots, x_{i+k-1}\}$. Let $N_k(q,n)$ be the smallest integer N so that every q-coloring of the edges of the complete k-uniform hypergraph on N vertices contains a monochromatic monotone path of length n.

While the study of $N_k(q,n)$ for specific values of k and q goes back (implicitly) to the seminal 1935 paper of Erdős and Szekeres, the problem of bounding $N_k(q,n)$ for arbitrary k and q was studied by Fox, Pach, Sudakov and Suk.

Our main contribution here is a novel approach for bounding the Ramsey-type numbers $N_k(q,n)$, based on establishing a surprisingly tight connection between them and the enumerative problem of counting high-dimensional integer partitions. Some of the concrete results we obtain using this approach are the following:

- We show that for every fixed q we have $N_3(q,n) = 2^{\Theta(n^{q-1})}$, thus resolving an open problem raised by Fox et al.
- We show that for every $k \geq 3$, $N_k(2,n) = 2^{\cdot^{\cdot^{2^{(2-o(1))n}}}}$ where the height of the tower is $k-2$, thus resolving an open problem raised by Eliáš and Matoušek.
- We give a new pigeonhole proof of the Erdős-Szekeres Theorem on cups-vs-caps, similar to Seidenberg's proof of the Erdős-Szekeres Lemma on increasing/decreasing subsequences.

5 Probabilistic Combinatorics

Independent sets in hypergraphs
Joszef Balogh *joint with R. Morris and W. Samotij*

Many important theorems and conjectures in combinatorics, such as the theorem of Szemerédi on arithmetic progressions and the Erdős-Stone Theorem in extremal graph theory, can be phrased as statements about families of independent sets in certain uniform hypergraphs. In recent years, an important trend in the area has been to extend such classical results to the so-called 'sparse random setting'. This line of research has recently culminated in the breakthroughs of Conlon and Gowers and of Schacht, who developed general tools for solving problems of this type. Although these two papers solved very similar sets of longstanding open problems, the methods used are very different from one another and have different strengths and weaknesses.

In this talk, we explain a third, completely different approach to proving extremal and structural results in sparse random sets that also yields their natural 'counting' counterparts. We give a structural characterization of the independent sets in a large class of uniform hypergraphs by showing that every independent set is almost contained in one of a small number of relatively sparse sets. We then derive many interesting results as fairly straightforward consequences of this abstract theorem. In particular, we prove the well-known conjecture of Kohayakawa, Łuczak, and Rödl, a probabilistic embedding lemma for sparse graphs, for all 2-balanced graphs. We also give alternative proofs of many of the results of Conlon and Gowers and of Schacht, such as sparse random versions of Szemerédi's theorem, the Erdős-Stone Theorem and the Erdős-Simonovits Stability Theorem, and obtain their natural 'counting' versions, which in some cases are considerably stronger. We also obtain new results, such as a sparse version of the Erdős-Frankl-Rödl Theorem on the number of H-free graphs and, as a consequence of the KŁR conjecture, we extend a result of Rödl and Ruciński on Ramsey properties in sparse random graphs to the general, non-symmetric setting. Similar results have been discovered independently by Saxton and Thomason.

Self correcting estimates for the triangle-free process

Tom Bohman *joint with P. Keevash*

Consider the triangle-free process. We start with $G(0)$ which is the empty graph on n vertices. Given $G(i)$, let $O(i)$ be the set of pairs xy such that xy is not an edge in $G(i)$ and $G(i)+xy$ does not contain a triangle. The edge e_{i+1} is chosen uniformly at random from $O(i)$, and we set $G(i+1) = G(i) + e_{i+1}$. This process terminates at a maximal triangle-free graph $G(M)$.

In previous work, the speaker showed that with high probability we have $M = \Theta\left(n^{3/2}\sqrt{\log n}\right)$. In this talk we present a refinement of that argument that gives a more precise result. With high probability we have

$$M = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}\sqrt{\log n}.$$

The new argument makes use of the fact that many keep statistics of the process exhibit self-correcting behavior; that is, when these variables deviate substantially from their expected trajectory they are subject to a drift that brings them back toward the trajectory.

We also establish a high probability upper bound on the independence number of $G(M)$. It follows from this bound that we have

$$R(3,k) \geq \left(\frac{1}{4} + o(1)\right)\frac{k^2}{\log k},$$

where the Ramsey number $R(3,k)$ is the smallest n such that every graph on n vertices has a triangle or an independent set on k vertices.

Similar results have been obtained independently by Gonzalo Fiz Pontiveros, Simon Griffiths, Robert Morris and Roberto Imbuzeiro Oliveira.

On the K ŁR conjecture in random graphs

David Conlon *joint with T. Gowers, W. Samotij and M. Schacht*

The KŁR conjecture of Kohayakawa, Łuczak, and Rödl is a statement that allows one to prove that asymptotically almost surely all subgraphs of the random graph $G_{n,p}$, for sufficiently large $p := p(n)$, satisfy an embedding lemma which complements the sparse regularity lemma of Kohayakawa and Rödl. We prove a variant of this conjecture which is sufficient for most applications to random graphs. In particular, our result implies a number of recent probabilistic versions of classical extremal combinatorial theorems due to Conlon, Gowers, and Schacht. We also discuss several further applications.

Cycles in random subgraphs of graphs

Choongbum Lee *joint with M. Krivelevich and B. Sudakov*

Many theorems in graph theory establishes the fact that if a graph G satisfies certain property $\mathcal{C}$, then it has some property $\mathcal{P}$. One example is Dirac's theorem, which asserts that every graph on n vertices of minimum at least $\frac{n}{2}$ is Hamiltonian. Once such a result is established, it is natural to ask, "How strongly does G satisfy $\mathcal{P}$"? Recently, there has been a series of work answering this question in various ways for several theroems. In this context, we further study the following classical theorem in more depth: "every graph G of minimum degree at least k contains a path of length at least k and a cycle of length at least $k+1$".

For a given finite graph G of minimum degree at least k, let G_p be a random subgraph of G obtained by taking each edge independently with probability p. We show that (i) if $p \geq \omega/k$ for a function $\omega = \omega(k)$ that tends to infinity as k does, then G_p asymptotically almost surely contains a cycle (and thus a path) of length at least $(1-o(1))k$, and (ii) if $p \geq (1+o(1))\ln k/k$, then G_p asymptotically almost surely contains a path of length at least k.

The binomial random graph $G(n,p)$ is G_p for $G = K_n$. It is known that $G(n,p)$ contains a path of length $(1-o(1))n$ if $p \gg \frac{1}{n}$, and is Hamiltonian if $p \gg (1+o(1))\ln n/n$. Our theroems can also be viewed as extensions of these theorems, by taking the graph G to be the complete graph K_k.

Eigenvalues and quasirandom hypergraphs

Dhruv Mubayi *joint with J. Lenz*

Chung and Graham began the systematic study of hypergraph quasirandom properties soon after the foundational results of Thomason and Chung-Graham-Wilson on quasirandom graphs. One feature that became apparent in the early work on hypergraph quasirandomness is that properties that are equivalent for graphs are not equivalent for hypergraphs, and thus hypergraphs enjoy a variety of inequivalent quasirandom properties. In the past two decades, there has been an intensive study of these disparate notions of quasirandomness for hypergraphs, and a fundamental open problem that has emerged is to determine the relationship between these quasirandom properties. Our first result completely determines the poset of implications between essentially all hypergraph quasirandom properties that have been studied in the literature. This answers a recent question of Chung, and in some sense completes the project begun by Chung and Graham in their first paper on hypergraph quasirandomness in the early 1990's.

Let $p(k)$ denote the partition function of k. For each $k \geq 2$, our second result describes a list of $p(k)-1$ quasirandom properties that a k-uniform hypergraph can have. These new properties connect previous notions on hypergraph quasirandomness, beginning with the early work of Chung and Graham and Frankl-Rödl related to strong hypergraph regularity, the spectral approach of Friedman-Wigderson, and more recent results of Kohayakawa-Rödl-Skokan and Conlon-Hàn-Person-Schacht on weak hypergraph regularity and its relation to counting linear hypergraphs.

For each of the quasirandom properties that are described, we define a hypergraph eigenvalue analogous to the graph case (thereby answering a question of Conlon et. al.) and a hypergraph extension of a graph cycle of even length whose count determines if a hypergraph satisfies the property. Our work can therefore be viewed as an extension to hypergraphs of the seminal results of Chung-Graham-Wilson for graphs.

The evolution of the Achlioptas processes

Lutz Warnke *joint with O. Riordan*

In the *Erdős–Rényi random graph process*, starting from an empty graph, in each step a new random edge is added to the evolving graph. One of its most interesting features, both mathematically and in terms of applications, is the 'percolation phase transition': as the ratio of the number of edges to vertices increases past a certain critical point, the global structure changes radically, from only small components to a single macroscopic ('giant') component plus small ones.

We study *Achlioptas processes*, which are widely studied variations of the classical Erdős–Rényi process. Starting from an empty graph these proceed as follows: in each step *two* potential edges are chosen uniformly at random, and using some rule *one* of them is selected and added to the evolving graph. These processes were introduced by Dimitris Achlioptas in 2000, and he originally asked whether there is a rule which substantially delays the appearance of the linear size 'giant' component compared to the classical case. In fact, many simulations suggested that for certain rules the percolation phase transition is particularly radical: more or less as soon as the macroscopic component appears, it is already extremely large; this phenomenon is known as 'explosive percolation'. In particular, Achlioptas, D'Souza and Spencer recently presented 'conclusive numerical evidence' for the conjecture that the largest component can grow from size at most $\sqrt{n}$ to size at least $n/2$ in at most $2n^{2/3}$ steps.

We first disprove this conjecture in a strong form: no matter which rule is used, we show it is impossible to obtain explosive percolation. Afterwards we discuss some recent progress in our mathematical understanding of Achlioptas processes, which is based on a new approach for proving convergence to the solution of a system of differential equations.

6 Extremal Combinatorics in Algebra

STABLE DICTATORS AND JUNTAS IN THE SYMMETRIC GROUP
Ehud Friedgut *joint with D. Ellis and Y. Filmus*

A useful theme in the application of analytical methods in combinatorics is the attempt to characterize combinatorial structures that depend on a single coordinate (a dictatorship), or on few coordinates (a junta). An important feature of this method is the ability to prove combinatorial stability results via analytical stability results. In this talk I present such a result in the symmetric group. The result here is complemented by the results that David Ellis presented in his talk.

Some definitions and background to the main result. Let V_1 denote the space of all functions on S_n that have their Fourier transform concentrated on the irreducible representations contained in the permutation representation (i.e. the trivial, and the $(n-1)$-dimensional one). A result of Ellis, Friedgut and Pilpel shows that the only Boolean functions in V_1 are the indicators of disjoint unions of cosets of the stabilizer of a point. Here I present a stability version of this.

Theorem: Let $f : S_n \to \{-1, 1\}$, and let $E[f] = 0$. Let f_1 be the orthogonal projection of f onto V_1, and let $E[(f - f_1)^2] \leq \epsilon$. Then there exists a Boolean function g in V_1 such that $E[(f - g)^2] = O(\epsilon^{1/7})$.

STABLE JUNTAS IN THE SYMMETRIC GROUP, II
David Ellis *joint with Y. Filmus and E. Friedgut*

We prove that a Boolean function on S_n of expectation $O(1/n)$, whose Fourier transform is highly concentrated on the first two irreducible representations of S_n, is close in structure to a union of *cosets of point-stabilizers* (or *1-cosets*, for short).

This phenomenon is not 'stability' in the strongest sense. Indeed, a Boolean function f has its Fourier transform completely supported on the first two irreducible representations of S_n, if and only if it is a *dictatorship* (a function determined by the image or the preimage of just one element.) A union of two 1-cosets which are *not* disjoint is not close in structure to any of these, and yet its characteristic function *does* have its Fourier transform highly concentrated on the first two irreducible representations of S_n. We may call our result a 'quasi-stability' result.

We use our result to give a natural proof of a stability result on intersecting families of permutations, originally conjectured by Cameron and Ku, and first proved by the first author. We also use it to prove a 'quasi-stability' result for an edge-isoperimetric inequality in the transposition graph on S_n, namely that subsets of

S_n with small edge-boundary in the transposition graph are close to being unions of cosets of point-stabilizers.

In the talk, we sketch the proof of our quasi-stability result. This relies on analysing the second and third moments of a (non-negative) affine shift of the relevant projection of the Boolean function. We choose a useful representation of this affine shift, in terms of the expectations of the function restricted to 1-cosets.

We are also able to prove a generalization of our quasi-stability result, dealing with Boolean functions on S_n whose Fourier transform is close to being of 'degree t', for fixed t. Namely, we show that prove that a Boolean function on S_n of expectation $O(1/n^t)$, whose Fourier transform is highly concentrated on irreducible representations of S_n corresponding to Young diagrams with at most t cells below the first row, must be close in structure to a union of cosets of stabilizers of ordered t-tuples. The proof of this uses some technical lemmas on the representations of the symmetric group, and is considerably more involved.

We use the general result to prove an exact edge-isoperimetric inequality for the symmetric group: namely, that if n is large depending on t, and $\mathcal{A} \subset S_n$ with $|\mathcal{A}| = (n-t)!$, then the edge-boundary of $\mathcal{A}$ (in the transposition graph on S_n) is minimized when $\mathcal{A}$ is a coset of a stabilizer of t points. This verifies a conjecture of Ben-Efraim in these cases.

What are high-dimensional permutations and how many are they?

Nathan Linial *joint with Z. Luria*

What is the higher-dimensional analog of a permutation? If we think of a permutation as given by a permutation matrix, then the following definition suggests itself: A d-dimensional permutation of order n is an $n \times n \times \ldots n = [n]^{d+1}$ array of zeros and ones in which every *line* contains a unique 1 entry. A line here is a set of entries of the form $\{(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{d+1}) | n \geq y \geq 1\}$ for some index $d+1 \geq i \geq 1$ and some choice of $x_j \in [n]$ for all $j \neq i$. It is easy to observe that a one-dimensional permutation is simply a permutation matrix and that a two-dimensional permutation is synonymous with an order-n Latin square. We seek an estimate for the number of d-dimensional permutations. Our main result is the following upper bound on their number

$$\left((1+o(1))\frac{n}{e^d}\right)^{n^d}.$$

We tend to believe that this is actually the correct number, but the problem of proving the complementary lower bound remains open. Our main tool is an adaptation of Brègman's proof of the Minc conjecture on permanents. More concretely, our approach is very close in spirit to Schrijver's and Radhakrishnan's proofs of Brègman's theorem.

This result is part of our ongoing effort to seek higher-dimensional counterparts for many basic concepts in combinatorics.