Inductive Constructions for Rigidity, 2 day workshop, July 2012

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1 Overview of the Field

The rigidity and flexibility of a structure, either man-made in buildings, linkages, and lightweight deployable forms, or found in nature ranging from crystals to proteins, is critical to the form, function, and stability of the structure. The mathematical theory of 'rigidity and flexibility' is developing methods for the analysis and design of man-made structures, as well as predictions of the behavior of natural structures such as proteins. We live in 3-dimensions, and a fundamental problem is to develop results for 3-dimensions which are as good and as efficient as the recently developed theory for structures in 2-dimensions.

One of the key ways to build examples and prove general results is an inductive construction: a sequence of local steps that build all possible structures from a few simple starting examples. Since at least the classic book of Henneberg [7], inductive constructions for infinitesimal rigidity of structures have played a key role in combinatorial characterizations of graphs supporting infinitesimal rigidity and independence of structures [24], [6]. More recently key results in global rigidity of structures were proved using key inductive constructions [1, 4].

2 Recent Developments and Open Problems

Recent work from a new generation of contributors to rigidity theory have developed new, refined inductive constructions for rigidity and independence of new classes of frameworks, under symmetry and periodicity conditions, for specialized families of examples, as well as for broader problems in CAD constraints.

Some examples are:

- the recent Ph.D. Thesis of Tony Nixon on inductive construction of frameworks on surfaces [16] and the ongoing work on inductive classes (including current investigations of Nixon [17, 18], Ross, Theran and Malestin);
- the thesis of Elissa Ross on inductive construction of fixed lattice isostatic periodic frameworks in the plane [19],

- Ross's more recent complete characterization of which gains yield generically isostatic bar-body periodic frameworks on any fixed lattice in 3-space;
- some recent work of Schulze on classes of symmetric frameworks [21, 20];
- the recent work of Lee-St.John on CAD Constraints in 3-D [13];
- the paper of Connelly, Jordán and Whiteley on global rigidity of redundant body-bar frameworks in all dimensions [4];
- ongoing work of Connelly, Whiteley, and others, on when vertex splitting in 3-D (and higher) preserves global rigidity;
- the inductive proof of the molecular conjecture and extensions by Katoh and Tanigawa [14];
- work of Finbow-Singh and Whiteley on inductive constructions of Block and Hole polyhedra, and possible extensions [5];
- recent and ongoing work of Cheng and Sitharam related to 3-D bar-and-joint rigidity; on constructing graphs (or locating maximal subgraphs) satisfying weaker notions of independence [2];
- paper and ongoing work of Cheng, Sitharam and Streinu related to 3D bar-and-joint rigidity; on constructions of specialized families of independent graphs [3]
- work of Berg, Jackson and Jordán on global rigidity of frameworks in 2-D [1, 9, 10];
- work of Jackson, Jordán, Whiteley, Servatius and Nguyen on global rigidity of mixed direction/length frameworks in 2-D [1, 11, 12, 22, 15].

As mentioned above, recent results for symmetry generic frameworks, and periodically generic frameworks have generated extended inductive techniques, and some of these results have potential extensions and applications to the study of protein structures with symmetry (such as dimers) or repetitive structures such as beta sheets or crystals.

For these generic results (and some geometric results), inductive constructions have provided full combinatorial characterizations, sometimes as alternatives to non-inductive techniques, and sometimes as the initial proofs which may later find non-inductive proofs. Recently, inductive proofs have been particularly prominent in the study of 3D body-bar frameworks, and the solution of the long standing molecular conjecture, proving key results in all dimensions.

This leaves the central, 100 year old problem of extensions to the bar and joint frameworks in 3-D. Some of the above results are steps in this direction.

3 Presentation Highlights

There was a survey talk by Tony Nixon `http://www.birs.ca/events/2012/2-day-workshops/12w2181/videos/watch/201207201407-Nixon.mp4` There was another long talk by Bill Jackson about the obstacles to proving that X-replacement is an inductive construction step in the 3-D bar-joint case. In addition, there were 5-7 minute talks by all participants, effectively moderated by one of the organizers, Tibor Jord'an. These talks offered a glimpse of the upcoming presentations, or posed an open problem. These can be found at: `http://www.birs.ca/events/2012/2-day-workshops/12w2181/videos/watch/201207201535-Whiteley.mp4`

- Oleg Karpenkov posed questions about inductive constructions for tensegrity structures, useful since the number of geometric conditions for the different regions of the configuration space explode.
- Meera Sitharam asked for inductive construction of $(3, 6)$-sparse graphs, which can be shown to be equivalent to construction of 3D generalized body-hingestructures of appropriate sparsity, where many bodies could share a hinge and hinges could share points. It has been shown that for maintaining sparsity, in d = 2,3 dimensions, the average number of hinges per body is strictly smaller than $d + 1$.

This was used to show that all maximal $(3, 6)$-sparse subgraphs provide an upper bound on the rank of the bar-joint rigidity matroid. For $d = 2$ this bound is tight, but the conjecture is that it is far from tight for higher dimensions, i.e, the average number of hinges is no more than 3, in 3D (Sitharam) and there is at least 1 body with at most 2 hinges in any dimension (Jackson). Inductive constructions will help.

- Bill Jackson talked about free submechanisms of mechanisms. Specifically he conjectured that for any submechanism K of a mechanism G, there is a generic framework (G, p) where K is free.
- John Owen talked about inductive operations for graphs that ensure that the Galois group of the original graph is a subgroup of the Galois group of the augmented graph (after applying the inductive operation). Specifically, he argued that this does not seem to hold for the Henneberg 2 and X-replacement moves.
- Jialong Cheng gave a new inductive construction maintaining independence (and isostaticity) of 3D bar-joint graphs, and also for constructing 3D bar-joint circuits. These inductive constructions have the additional feature that they maintain "nucleation-free" property, which is the second obstacle in obtaining a combinatorial characterization of 3D-rigidity: this method helps prove, understand and extend some of the construction schemes given by Tay [23].
- Bernd Schulze talked about the need for inductive constructions to control the number of cases for proving characterizations of isostatic incidentally symmetric frameworks. Specifically he gave the example of his theorem on C_S (single reflection) incidentally symmetric isostatic frameworks in 2D.
- Audrey Lee-St John asked for inductive constructions on bi-colored graphs that are expressed as specific types of unions of spanning trees.
- Steve Power talked about Generalized Periodic Rigidity Matrices with function entries. Starting with motif edges, he was interested in inductive constructions to control the determinant and the RUM spectrum.
- Viktoria Kazanitzky asked about characterizing absolutely 2-rigid graphs after defining them and showing some basic properties and counterexamples to various attempts at characterization.
- Herman Servatius talked about 2-sums of matroids, frameworks and circuits. This led to matroid decompositions of rigidity matroids into non-graphic, non-rigidity matroids. He also talked about such decompositons for universally rigid graphs.
- Csaba Király talked about balanced generic circuits without long paths and gave interesting ways of constructing them, together with many examples.
- Wendy Finbow-Singh and Walter Whiteley described inductive constructions on modified triangulated surfaces [5].
- Laura Chavez Lomeli talked about tree partictions for 2D circuits and inductive constructions of circuits by splitting and gluing.
- Walter Whiteley talked about inductive constructions for periodic structures on fixed lattices, specifically "coatings" on "substrates." He talked about inductive constructions both for the plane case and for body bar.
- Shinichi Tanigawa talked about symmetry-forced generic rigidity for the dihedral group. He had a characterization using inductive constructions and one case that was the minimal counterexample to the inductive constructions, namely the double cycle. He asked for the correct sparsity condition.
- Viet Han Nguyen talked about operations preserving rigidity and global rigidity of direction-length frameworks. She gave results and posed open questions concerning specific operations such 1 extensions on direction edges.

4 Scientific Progress Made and Outcome of the Meeting

Many collaborative groups were formed during the workshop. Some groups have reported progress on projects whose origin, direction or momentum can be traced back to the workshop.

- Steve Power reports: ”the workshop facilitated collaboration with John Owen and Tony Nixon and we have just submitted a joint paper to a research journal, with improved results and inductive techniques. The paper was recently put on the ArXiv:

  A. Nixon, J.C. Owen and S.C. Power, A Laman theorem for frameworks on surfaces of revolution, `http://arxiv.org/abs/1210.7073`, (2012).

  Also I learnt of a number of directions from the international researchers which I expect to impact on my future work (such as global rigidity and universal rigidity) and my joint work with my new postdoc, Dr. Derek Kitson.”

- Louis Theran reports: ”Tony Nixon and I have been working on the question of generic rigidity of frameworks supported by surfaces with no isometries. An inductive approach seems promising.

  Also, Audrey Lee-St John and I have been working on some things relating to body-CAD and matroids. Bernd Schulze and I have also been looking a bit at generic incidental symmetry in the plane.”

- Bill Jackson reports: ” Viet Hang Nguyen and myself began discussing the rigidity of d-dimensional body-direction-length frameworks in Banff. Hang is currently visiting me in london to continue this research. we have used a recursive construction to characterise rigidity in the cases when the bodies are either rigid or direction rigid and are now working on the case when the bodies are length rigid.”

- Brigitte Servatius reports: ”Bill Jackson asked a question related to this one: It is true that a 1-extension preserves the degree of freedom of a bar-and joint framework, infinitesimally and generically, but is it true that for a generic realization of a mechanism the operation of 1-extension may be performed without restricting the motion?

  Brigitte and Herman Servatius worked out a counterexample to this question. This example is mentioned in a recent article of Jackson and Jordán [8]. Brigitte and Herman Servatius are writing up a short paper giving not just a counterexample but a more general answer to the question.”

- Walter Whiteley reports: ”Wendy Finbow-Singh and I have applied the inductive constructions (see Presentation Highlights) to extend results resolving Kuiper’s Conjecture [25] to give the proof that any triangulated sphere with one added edge which forms a 4-connected graph, then the graph is a generic circuit in 3-space.”

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