Exercise 1. In this exercise, you will solve

$$x^{2p} + y^{2p} = z^5, \qquad x,\, y,\, z \text{ coprime},\ p \text{ prime},\ p \geq 7.$$

(i) Show z is odd. **Without loss of generality x is even and y is odd.**

(ii) Show that

$$x^p + iy^p = (u + iv)^5$$

for some integers u, v.

(iii) Deduce that

$$x^p = u(u^4 - 10u^2v^2 + 5v^4), \qquad y^p = v(5u^4 - 10u^2v^2 + v^4).$$

(iv) Show that u, v are coprime, with u even.

(v) **Case I:** Suppose that $5 \nmid uv$.

- Show that

$$u = A^p, \qquad u^4 - 10u^2v^2 + 5v^4 = B^p,$$
$$v = C^p, \qquad 5u^4 - 10u^2v^2 + v^4 = D^p.$$

- Deduce

$$D^p + 20A^{4p} = w^2$$

for an appropriate integer w.

- Use an appropriate Frey curve to deduce a contradiction.

(vii) **Case II:** Repeat for $5 \mid uv$.