BIRS 2012 – Nils Bruin: 4. Workshop Problems

N1. Solve the Thue equation

$$x^3 - 17y^3 = 1 \text{ with } x, y \in \mathbb{Z}$$

using Skolem's method.

N2. Solve the Thue equation

$$x^3 - 63y^3 = 1 \text{ with } x, y \in \mathbb{Z}$$

using Skolem's method. You probably need to combine information at several primes for this equation.

N3. If a Thue equation has no solutions then Skolem's method need not come into play at all. Consider the equation:

$$x^3 - 2y^3 = 11 \text{ with } x, y \in \mathbb{Z}$$

(a) Verify that this equation has solutions with $x, y \in \mathbb{Z}_p$ for all p (i.e., there are no local obstructions)
(b) Prove there are no solutions.

N4. (Ljunggren's equation) Ljunggren considers the equation

$$x^2 + 1 = 2y^4 \text{ with } x, y \in \mathbb{Z}.$$

(a) While this equation is not a Thue equation itself, it is easily reduced to quartic Thue equations. Do this and show why Skolem's method does not immediately apply to them.
(b) Solving a Thue equation $f(u, v) = 1$ essentially boils down to determining integral points on the variety $C = \mathbb{P}^1 \setminus \{f(u, v) = 0\}$. The descent principle tells us that if $D \to C$ is an unramified cover (for instance, a cover $\mathbb{P}^1 \to \mathbb{P}^1$ only ramified above $f(u, v) = 0$), then the integral points on C are covered by the integral points on finitely many twists of D. Can you find such a cover for which the corresponding Thue equations are amenable to Skolem's method?
(c) One way to obtain such covers is by rewriting the equation as

$$\begin{aligned} x^2 - 2(y^2)^2 &= -z^3 \\ z &= 1 \end{aligned}$$

Analogous to the parametrization of Pythagorean triples, the first equation gives rise to a number of parametrizations of the form

$$\begin{aligned} y^2 &= Y(s, t) \\ x &= X(s, t) \\ z &= Z(s, t) \end{aligned}$$

where $Y(s, t), X(s, t)$ are degree 3 forms and $Z(s, t)$ is a degree 2 form. We can take several routes from here

(i) Rewrite the equation $Z(s, t) = 1$ as $Z(s, t) = u^2 = 1$. This gives rise to parametrizations where $s = S(p, q)$, $t = T(p, q)$ are given by degree 2 forms. The equation

$$y^2 = Y(S(p, q), T(p, q))$$

is now of the form quartic equals a square, which is amenable to various methods.

(ii) The equation $Y(s,t) = y^2$ (you can check that we are only interested in solutions with $\gcd(s,t) = 1$) can be parametrized in a similar way (See some work by Euler). This leads to parametrizations with $s = S(p,q)$, $t = T(p,q)$ quartic forms. Then the equation

$$Z(S(p,q), T(p,q)) = 1$$

leads to a degree 8 Thue equation. It is worthwhile to check if this leads to the same equations that Ljunggren originally considered in his 1942 paper.

N5. Sieving methods are not limited to curve-related problems. Provided the surface you are interested in can be mapped sufficiently non-trivially into a torus or an abelian variety, sieving techniques may be applied. For instance, let ζ be a primitive fifth root of unity. Then the equation

$$N(x + y\zeta + z\zeta^2) = 29921$$

does not appear to have solutions, and this should be provable via sieving methods. This amounts to finding the integral solutions to the equation

$$x^4 - x^3y - x^3z + x^2y^2 + 2x^2yz + x^2z^2 - xy^3 - 3xy^2z + 2xyz^2 - xz^3 + \\ y^4 - y^3z + y^2z^2 - yz^3 + z^4 = 29921.$$

Replacing 29921 with much smaller primes (congruent to 1 modulo 5 of course) quickly gets you several solutions. However, the unit rank should still allow Skolem's method to apply.

Can you find other methods that are capable of handling this kind of equation?

N6. Sieving is sensitive to unit rank for its running time, but not for expected success. So, for equations where there are no solutions, sieving heuristically should be able to prove so even if the unit rank is too high to apply Skolem's method. For instance, totally real Th ue equations without solutions should be solvable using sieving. Can you find examples? Can you find examples where other known methods are unable to show there are no solutions?