Cycles on modular varieties in char. p

We really talk about Shimura varieties of PEL-type. (What of ~~abelian ty~~ Hodge type?. $\mathcal{M}_g$? $\mathcal{H}_g$? Hilbert schemes?)

Sources of cycles:

(1) Images of other Shimura varieties.

(2) Tautological classes: Chern classes of automorphic vector bundles.

(3) "Accidental" classes.

(4) "Frobenius classes" (in char. p).

SOME early papers: HZ (1976), Oort (1974), Koblitz (1975)
After 35 yrs know a lot but still don't understand how all the data is related and organized.

<u>Example</u>: The Siegel modular variety $\mathcal{A}_{g,N}$.

$N \geq 3$. The scheme $\mathcal{A}_{g,N} / \mathbb{Z}[\zeta_N, N^{-1}]$ param.

(A, λ, α) It's a fine moduli scheme.

dim g, pr. pol'n, full sympl. level N.

Dimension $g(g+1)/2$, $\mathcal{A}_{g,N}(\mathbb{C}) = \Gamma(N) \backslash \mathfrak{h}_g$.

Let L = tot. real field, $[L:\mathbb{Q}] = g$.

$\mathcal{M}_{L,N} / \mathbb{Z}[\zeta_N, 1/N, 1/\mathrm{disc}_{L/\mathbb{Q}}]$ – smooth scheme, fine moduli space for $(A, \iota, \lambda, \alpha)$.

level N

$\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}(A)$

<u>Functor</u>: $(A, \iota, \lambda, \alpha) \longmapsto (A, \lambda, \alpha)$ gives

$$\underbrace{\mathcal{M}_{L,N}}_{\text{dim'n } g} \longrightarrow \underbrace{\mathcal{A}_{g,N}}_{\text{dim'n } \frac{g(g+1)}{2}}.$$

Seems useful for $g = 2, 3$.

(Used in general by C.L. Chai to show that the Hecke orbit of any ordinary pt. is dense in $\mathcal{A}_g \otimes \overline{\mathbb{F}}_p$)

K quad. imag., $m+n=g$,

$$\underset{\substack{\text{unitary Shimura}\\ \text{variety, dim } mn}}{U(m,n)} \longrightarrow \mathcal{A}_{g,N}$$

$[K:\mathbb{Q}]=2g$, K CM, get K-CM points in $\mathcal{A}_{g,N}$.

All these and more are examples of Shimura varieties.

Accidental cycles

$\mathcal{M}_g$ = moduli space of curves $\xrightarrow{\text{Torelli map}} \mathcal{A}_g$

$\mathcal{H}_g$ = hyperelliptic curves $\longrightarrow \mathcal{A}_g$

X = hypergeometric curve arising from a triangle group $\longrightarrow \mathcal{A}_g$

Andreotti–Mayer loci (singularities of theta divisor)

Tautological classes:

The automorphic v. bundles are "derived" from

$\mathbb{H}^1_{dR}(X^{univ}/\mathcal{A}_g)$. For example $\mathbb{E} = R^0\pi_* \Omega^1_{X^{univ}/\mathcal{A}_g}$
$= s^* \Omega^1_{X^{univ}/\mathcal{A}_g}$

$$0 \longrightarrow \mathbb{E} \longrightarrow \mathbb{H}^1_{dR}(X^{univ}/\mathcal{A}_g) \longrightarrow \mathbb{E}^\vee \longrightarrow 0.$$

Sections of $(\det \mathbb{E})^k$ = Siegel modular forms of wt. k.

<u>Kodaira–Spencer</u>: $\mathrm{Sym}^2(\mathbb{E}) = \Omega^1_{\mathcal{A}_g/\mathbb{Z}}$.

Let

$$\lambda_i = \lambda_i(\mathbb{E}) \in Ch^i(\mathcal{A}_g \otimes \mathbb{C})$$

be the Chern class (over $\mathbb{C}$) of $\mathbb{E}$.

$\mathbb{Q}(\lambda_1, \dots, \lambda_g) \subseteq Ch^*(\mathcal{A}_g \otimes \mathbb{C}) \otimes \mathbb{Q}$ is the tautological subring.

<u>Remark</u>: By adding level we can get a relative divisor Θ on X^{univ} (defining $2\times$ pol'n).

$\mathcal{L} := \mathcal{O}(\Theta)$, $\theta = c_1(\pi_* \mathcal{L})$. Then

$$2\theta = -\lambda_1 \quad \text{in } Ch^1 \otimes \mathbb{Q}.$$

So unlike the case of curves, enough to consider the λ_i's.

<u>Characteristic</u> p $\bar{\mathcal{A}}_g := \mathcal{A}_g \otimes \bar{\mathbb{F}}_p$.

$x \in \mathcal{A}_g(\bar{\mathbb{F}}_p) \rightsquigarrow (A_x, \lambda_x) \rightsquigarrow \mathbb{D}(A_x, \lambda_x) =$ covariant Dieudonné module

$\mathbb{D}(A_x, \lambda_x)$ rank $2g$ module over $W(\bar{\mathbb{F}}_p)$

$\circlearrowleft_F \; \circlearrowleft V$, $F \lambda v = \sigma(\lambda) Fv$, $F \circ V = V \circ F = [p]$,

Determines completely $A_x(p) = \varinjlim A_x[p^n]$.

$\langle , \rangle : \mathbb{D} \times \mathbb{D} \longrightarrow W(\bar{\mathbb{F}}_p)$ alternating, perfect bilinear

$\langle Fx, y \rangle = \langle x, Vy \rangle^{\sigma}$

F, V "come from" Fr, Ver on A.

<u>3 types of stratifications</u> (2 for $\mathcal{A}_g$)

((0) Singularities)

(1) <u>EO strata</u> (1999): group $x \in \mathcal{A}_g(\bar{\mathbb{F}}_p)$ into the same locally closed set if $A_x[p^n] \otimes \bar{\mathbb{F}}_p$ belongs to same isomorphism type.

<u>EO is the case $n=1$:</u>

There are 2^g strata W_φ^{EO}.

Originally: $\varphi : \{1, \dots, g\} \to \mathbb{Z}$, $\varphi(0) = 0$ and

$\forall i \quad \varphi(i) \le \varphi(i+1) \le \varphi(i) + 1$.

Then: $\dim(W_\varphi^{EO}) = \sum_{i=1}^{g} \varphi(i)$

W_φ^{EO} quasi-affine, $\overbrace{\text{equi-dim'l}}^{\text{non-singular}}$,

this ~~is~~ a stratification.

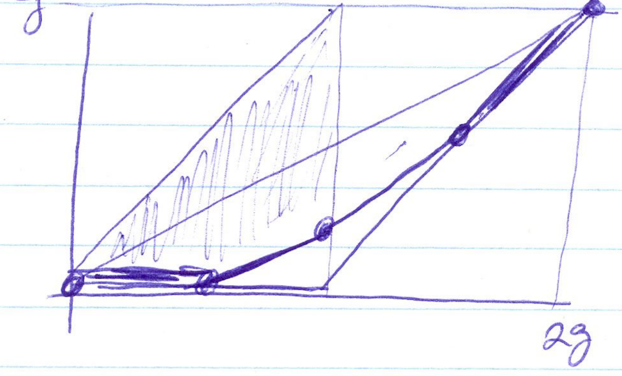
~~(Later: Moonen, Wedhorn, $\varphi \in$ Weyl gp of Sp_{2g})~~.

The analogues for Shimura Varieties are quite well understood (Moonen, Wedhorn, Viehmann-Wedhorn, G-Ort)

(2) Newton polygon stratification:

$A_x(p)$ up to isogeny encoded by one of f. many polygons

- $\lambda \longleftrightarrow 1-\lambda$ (w. mult.)
- convex
- integral break pts.



We group $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ together according to N. polygons.

- A stratification.
- $\dim W_{\mathcal{P}}^{N} = \#\{(x,y) \in \mathbb{Z}^2 \mid y < x \leq g,\ (x,y) \text{ is on or above } \mathcal{P}\}$
- Non-singular.
- ~~Order relation~~
- $W_{\mathcal{P}_2} \subseteq \overline{W_{\mathcal{P}_1}} \iff \mathcal{P}_2 \geq \mathcal{P}_1$.

(all due to Oort).

Studied first by Koblitz: if $x \in \mathcal{A}_g(\mathbb{F}_q)$ the

$$NP(A_x(p)) = \frac{1}{n} NP(\mathrm{Frob}_q \curvearrowright H^1(\overline{A}_x, \mathbb{Q}_\ell)),\ q = p^n.$$

So related to variation of Zeta functions in families.

Stratifications of $A_x[p^n]$, $n = 1, 2, 3, \ldots$

$\longleftrightarrow$ Newton Stratification

(Katz), Nicole-Vasiu-Wedhorn give precise bounds on n s.t. $A_x[p^n]$ determines $A_x(p)$.

What do we know about relations b/w ~~[illegible]~~ this data?

Geometry

(a) λ_1 is ample; it represents the zero locus of the Hasse invariants.

($\Rightarrow$ Every 1-dim'l family of ordinary AV in char. p is isotrivial).

Define $R_g = \mathbb{Q}[u_1, \ldots, u_g] /$ ~~[illegible]~~

$$((1+u_1+\cdots+u_g)(1-u_1+\cdots+(-1)^g u_g)-1).$$

(Note: $R_g/(u_g) \simeq R_{g-1}$).

Van der Geer: The tauto' ring of $Ch^*_{\mathbb{Q}}(\mathcal{A}_g) \cong R_{g-1}$

The tauto' ring of $Ch^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g) \cong R_g$ $\qquad \lambda_i \mapsto u_i$

($\tilde{\mathcal{A}}_g$ any sm. toroidal cptn).

Further, the EO strata $Z^{EO}_\varphi := \overline{W^{EO}_\varphi} \in$ Tauto' ring(s) + given by explicit formulas.

E.g.:

$V^f = \{x \mid \# A_x[p](\overline{\mathbb{F}}_p) \leq p^f\}$ has class in $\tilde{\mathcal{A}}_g$

$$(p-1)(p^2-1)\cdots(p^{g-f}-1)\cdot\lambda_{g-f}.$$

$\mathcal{A}_g^{ssp}$ $\quad (p-1)(p^2+1)\cdots\cdots(p^g+(-1)^g)\lambda_1\cdots\cdots\lambda_g$

Questions:

* ~~[illegible]~~ $\mathcal{M}_g, \mathcal{H}_g, W^N_I \in$ tauto' ring??

$\exists$? larger but explicit ring containing those.
($\rightsquigarrow$ finding curves w. sp. properties.
$\rightsquigarrow$ relation by Newton and EO strata)

* What about $\mathcal{A}_g \times \mathcal{A}_g$ and Hecke correspondences?
* How does the prime-to-p Hecke algebra act on Z^{EO}_φ?

Superspecial locus — in $\mathcal{A}_g$ A. Ghitza
— in $\mathcal{M}_L$ M.-H. Nicole

Geometric avatar of JL — $GU(B_{p,\infty})$
— $B_{p,\infty} \otimes_{\mathbb{Q}} L$.

Arithmetic

* HZ (1976), ~~[illegible]~~ GKZ ('87), vdG ('82) showed that a generating series with coef. HZ divisors / Heegner pts on $X_0(N)$ / Humbert surfaces ("=" images of $\mathcal{M}_2$'s) are modular forms on SL_2 of wt. $2/\tfrac{3}{2}/\tfrac{5}{2}$..

Borcherds (1997) gives a vast generalization:

$$SL_2 \approx SO(2,1),\ \operatorname{Res}_{L/\mathbb{Q}} SL_2 \approx SO(2,2),\ Sp_4 \approx SO(2,3)$$

He studies arith. quot. of $X = SO(2,n)(\mathbb{R})/K$ and defines Heegner divisors $\begin{cases} \rightsquigarrow \text{Heegner pts} \\ \rightsquigarrow \text{HZ divisors} \\ \rightsquigarrow \text{Humbert surfaces} \end{cases}$

Λ = lattice $(2,n)$, $\Gamma(\Lambda) \doteq \{g \in \operatorname{Aut}(\Lambda) \mid g|_{\Lambda^\vee/\Lambda} \equiv 1\}$

$$y_{-n,\mu} = \text{divisor of } \Gamma(\Lambda)\backslash X$$

$(n \in \mathbb{Q}_{>0},\ \mu \in \Lambda^\vee/\Lambda)$

$$\sum_{n,\mu} y_{-n,\mu}\, q^n e_\mu$$

is a vector valued wt. $1+\frac{n}{2}$ modular form valued in $\mathbb{C}[\Lambda^\vee/\Lambda] \otimes \mathrm{Ch}^1_{\mathbb{Q}}(\Gamma(\Lambda)\backslash X)/_\sim$ and $\overline{\rho}_L$.

At least for $\mathcal{M}_L$, $[L:\mathbb{Q}] \leq 2$, $\exists$ extn using $\widehat{\mathrm{Ch}}^1$ (KRY 2006, BBK 2007).

K, KR, KRY, Howard-Y, Howard: certain generating series for cycles supported on special fibers of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{M}_L, \ldots$, Unitary groups, ... are related to E. series and their derivatives.

<u>For $g=1$ (KRY):</u>

B = indef. quat. algebra $/\mathbb{Q}$, O_B max'l order.

$\mathcal{M}_B$ Shimura curve param. (A, ι).

$$V(A,\iota) = \{ x \in \mathrm{Cent}_{\mathrm{End}^0(A)}(O_B),\ \mathrm{tr}(x)=0 \}$$

$x \longmapsto -x^2$ = quad. form on $V(A,\iota)$.

$Z(t) \subseteq \mathcal{M}_B$ param. (A, ι, x) s.t. $-x^2 = t$.

$t>0$, $Z(t)(\mathbb{C}) \longleftrightarrow$ CM pts. for $\mathbb{Z}[\sqrt{-t}]$

$t<0$ $\quad Z(t) = \phi$

$t=0$ $\quad$ ad hoc.

One lifts $Z(t)$ to classes in $\widehat{Ch}^1(\mathcal{M}_B)$

$$\hat{\phi}_1(\tau) := \sum_{t \in \mathbb{Z}} \hat{Z}(t) \cdot q^t \in \widehat{Ch}^1_{\mathbb{C}}[\![q]\!].$$

<u>Thm</u>: This is a modular form of wt. 3/2 for some $\Gamma \subseteq SL_2(\mathbb{Z})$. Moreover, $\exists$ Eisenstein series $\mathcal{E}(\tau, s)$ s.t.:

(1) $\mathcal{E}(\tau, 1/2) = \deg_{\mathbb{Q}}(\hat{\phi}_1(\tau)) = \sum_t \deg_{\mathbb{Q}}(\hat{Z}(t))\, q^t.$

(2) $\mathcal{E}'(\tau, 1/2) = \langle \hat{\phi}_1(\tau), \hat{\omega} \rangle = \sum_t \langle \hat{Z}(t), \hat{\omega} \rangle q^t.$

Further $\hat{\phi}_1(\tau) - \dfrac{\mathcal{E}(\tau, 1/2)}{\deg \hat{\omega}} \cdot \hat{\omega}$ in $Ch^1(\mathcal{M}_B \otimes \mathbb{Q})$ = special case of Borcherds.

<u>Also</u>: $\hat{\phi}_2(\tau) = \sum_{T \in M_2(\mathbb{Z})^{sym}} \hat{Z}(T) q^T$, $\tau \in \mathfrak{h}_2$, $q^T = e^{2\pi i \, Tr(T\tau)}$

$\hat{Z}(T) \in \widehat{Ch}^2_{\mathbb{R}}(\mathcal{M}_B) \cong \mathbb{R}$ with geometric part

- if T pos. def. $Z(T)$ param. (A, ι, x_1, x_2)

s.t. $x_1, x_2 \in V(A, \iota)$, $T = \frac{1}{2}(\langle x_i, x_j \rangle)$

Note that $\mathcal{Z}(T)_{\mathbb{Q}} = \phi$. So conceptually this ties to mod p strata.

<u>Thm</u> $\hat{\phi}_2$ is a Siegel modular form of wt. $5/2$.

Apologies for not discussing S-W. Zhang, YZZ.

—o—

- We don't know the analogues of most of the results we have discussed for general Shimura varieties. Emerging work on $U(n,1)$ — Rapoport, Kudla, Terstiege, ... Howard, ...
- For $\mathcal{M}_L$ - geometry of $\overline{\mathcal{M}}_L \longleftrightarrow$ p-adic Hilbert modular forms, canonical subgps (G w. Andreatta, Kassaei, Oort).
- HZ generalized by Getz-Goresky.
- Further open questions on $\mathcal{A}_g$:
  - $\mathcal{A}_g^{ssp} \longrightarrow$ good expanders??
  - $g=3$, $\exists$ connection b/w generating series of $\mathcal{M}_L$'s + modular form on G_2 ??