Pierre Colmez

Shintani's method

Totally real case:

$V \subset (\mathbb{R}_+^*)^d$ discrete, rank $d-1$

$V \subset \{z : Nz = 1\}$

Notation: $\operatorname{Tr} z = z_1 + \dots + z_d$, $Nz = z_1 \cdots z_d$.

Assume $v = (v_1, \dots, v_d) \longmapsto v_d$ is injective.

Question: Find a nice fundamental domain mod V.

Example: $[F:\mathbb{Q}] = d$, totally real
$F \hookrightarrow \mathbb{R}^d$
$V \subset \mathcal{O}_{F,+}^*$ finite index

Answer of Shintani: There is a finite polyhedral cone ~~decomposition~~ as a fundamental domain.

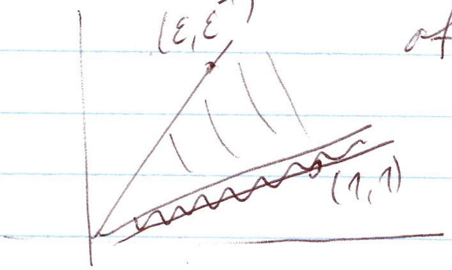
$e_1, \dots, e_d$ std basis of $\mathbb{R}^d$.

$B = (f_{B,1}, \dots, f_{B,d})$ other basis, $f_{B,i} \in \mathbb{R}^d$.

$$C_B = \{x_1 f_{B,1} + \dots + x_d f_{B,d} : x_i \geq 0 \text{ if } e_d \ \& \ f_{B,i} \text{ are } \langle f_{B,1}, \dots, \widehat{f_{B,i}}, \dots, f_{B,d}\rangle,\ x_i > 0 \text{ otherwise}\}.$$

②

<u>Thm</u> (Shintani) ~~...~~ There is a finite set of B st $\coprod C_B$ is a fundamental domain of $\mathbb{R}_+^{*}$ mod V.



<u>Rk</u>. $\mathcal{C} = \{ z : \mathrm{Tr}\, z \leq \mathrm{Tr}\, vz \ \forall v \in V \}$

<u>Show</u>: $\mathcal{C}$ is a closed polyhedral cone.

- It's almost a fundamental domain, sort out the overlapping of translates on the boundary by using ~~...~~ recipe from definition of C_B.
- Can achieve $f_{B,i} \in \mathbb{Q}V$

Application: F totally real, $\eta : (\mathbb{R}_+^{*})^d \longrightarrow \{\pm 1\}$.
$A \subset \mathbb{C}$ subring

~~...~~ $\mathcal{S}(F, \eta, A) =$ ~~...~~ Schwarz functions
$= \{ \varphi : F \longrightarrow A$ loc. const of compact support, ~~...~~ $\eta(u)\varphi(ux) = \varphi(x) \ \forall u \in \mathcal{O}_F^{*} \}$.

Choose $V \subset \mathcal{O}_{F,+}^{*}$ finite index.

$$L(\varphi, s) = \frac{1}{[\mathcal{O}_F^{*} : V]} \sum_{\alpha \in F^{*}/V} \frac{\eta(\alpha)\varphi(\alpha)}{|N\alpha|^{s}}$$

(3)

Let $\coprod C_B$ be a Shintani decomposition of $(\mathbb{R}_+^*)^d/V$. Then

$$L(\varphi, \alpha) = \frac{1}{\Gamma(s)^d} \int_{\mathbb{R}_+^d} \Phi(x) \prod_{i=1}^{d} x_i^s \frac{dx_i}{x_i}, \text{ where}$$

$$\Phi(x) = \frac{1}{[O_F^*:V]} \sum_B \sum_{\alpha \in X_B} \frac{\varphi(\alpha) e^{-\operatorname{Tr} \alpha x}}{\prod_{i=1}^{d} 1 - e^{-\operatorname{Tr} f_{B,i} x}}$$

finite set

Fourier transform.

If $\operatorname{Tr} f_{B,i} \alpha \notin \mathbb{Z}$ whenever $\check{\varphi}(\alpha) \neq$, then Φ is C^∞ on $\mathbb{R}^d$. Moreover,

$$L(\varphi, -k) = (-1)^{kd} \prod_{i=1}^{d} \left(\frac{d}{dx_i}\right)^k \Phi(x) \Big|_{x=0}.$$

<u>Variants</u>: $V \subset (\mathbb{C}^*)^d$ free of rank $d-1$, $V \subset \{z : Nz = 1\}$

What to do?

~~Baa~~

Back to $V \subset (\mathbb{R}_+^*)^d$ for motivation, $\coprod C_B$ decomp.

$$\frac{1}{z_1 \cdots z_d} = \int_{\mathbb{R}_+^d} e^{-\operatorname{Tr}(xz)} dx = \sum_B \sum_{v \in V} \int_{vC_B} e^{-\operatorname{Tr}(xz)} dx$$

$$= \sum_B \sum_{v \in V} \frac{|\det B|}{\prod_{i=1}^{d} (\operatorname{Tr} f_{B,i} vz)}$$

– we now abstract this –

We say that $\sum \alpha_B B$ is a Shintani decomp of $(\mathbb{C}^*)^d$ mod V if

$$\frac{1}{z_1\cdots z_d} = \sum_{v\in V} \sum_{B} \frac{\alpha_B \det B}{\prod_{i=1}^{d} \mathrm{Tr}(f_{B,i}\, z)}.$$

Thm (Colmez's thesis). Shintani decompositions exist, can take $f_{B,i} \in V$.

Suppose $\varepsilon_1, \dots, \varepsilon_{d-1}$ is a basis of V over $\mathbb{Z}$. $\sigma \in S_{d-1}$, $B_\sigma := B(f_{\sigma,1}, \dots, f_{\sigma,d})$, where

$$f_{\sigma,i} := \prod_{j<i} \varepsilon_{\sigma(j)}.$$

then $\sum_\sigma (\operatorname{sgn}\sigma) B_\sigma$ is a Shintani decomposition.

<u>Remark</u>: Apply $\prod \left(\frac{d}{dz}\right)^{k-1}$ & get a formula for $\left(\frac{1}{z_1\cdots z_d}\right)^k$.

Eg. $K \subset \mathbb{C}$ imag. quadratic, $[F:K]=d$, $F \hookrightarrow \mathbb{C} \otimes_K F \cong \mathbb{C}^d$.

$[\mathcal{O}_F^* : V] < \infty$, $\eta : \mathcal{O}_K^* \to \mathbb{C}^*$ finite order character.

If $\varphi \in \mathcal{S}(F, \eta_k, A) = \{\varphi : \eta(u) N u^k \varphi(ux) = \varphi(x) \ \forall u \in \mathcal{O}_K^*\}$.

set
$$L(\varphi, k) = \frac{1}{[\mathcal{O}_F^* : V]} \sum_{\alpha \in F^*/V} \frac{\eta(\alpha)\varphi(\alpha)}{(N\alpha)^k}$$

spse $\sum \alpha_B B$ is a Shintani decomp. mod V

$$= \frac{1}{(k!)^d} \sum_{\alpha \in F^*} \eta(\alpha)\varphi(\alpha) \sum_B \alpha_B \det B \prod_{i=1}^{d} \left(\frac{d}{dz_i}\right)^k \left(\frac{1}{\prod \mathrm{Tr}\, f_{B,i}\, z}\right)$$

5

$$\alpha \longmapsto \mathrm{Tr}(f_{B,1}\alpha, \dots, \mathrm{Tr} f_{B,d}\alpha)$$

$$F \longrightarrow K^n.$$

~~The Elliptic functions~~ the above can be expressed in terms of elliptic functions at torsion points. an $\mathbb{C}/\mathfrak{a}\mathcal{O}_K$ (Still need to sort out convergence problems)

$$\Longrightarrow L(\varphi, k) \in \Omega^{kd}\overline{\mathbb{Q}}$$

Would like to do the same thing with

$$L(\varphi, k, j) = \sum \frac{\varphi(\alpha)\overline{N\alpha}^j}{N\alpha^k} \in \Omega^{(k+j)d}\ \pi^{\overset{?}{(\pm)}jd}\ \overline{\mathbb{Q}}.$$

Would like a Shintani decomp. of the form:

$$c\,\frac{\overline{(z_1 \cdots z_d)}^j}{(z_1 \cdots z_d)^k} = \sum_{v \in V} \sum_B \alpha_B \prod_{i=1}^d \frac{\overline{\mathrm{Tr} f_{B_i} v z}^j}{(\mathrm{Tr} f_{B_i} v z)^k}$$

Colmez can find such α_B. Problem: Is $c \neq 0$?

Back to totally real situation: (V not assumed $\subset (\hat{\mathbb{R}}_+^*)^d$)

$$\frac{1}{z_1 \cdots z_d} \sum_{v \in V} \sum_B \frac{\alpha_B \det B}{\prod \mathrm{Tr} f_{B_i} v z} \qquad \text{Apply Fourier Transform}$$

$$\sum_{v \in V} \mathrm{sgn}\, Nv \sum_B \alpha_B\, \chi_{vB}(x,y) = \chi_{(e_1 \cdots e_d)}(x,y)$$

6

$\chi_{\sigma B}(x,y) = \pm 1_{C(B,y)}(x)$, or $C(B,y)$ = cone generated by the $(\operatorname{Tr} f_{B,j} \cdot y) f_{B,j}$

$[F:\mathbb{Q}] = d$, totally real, $[\mathcal{O}_F^* : V] = 2$, $L(\varphi,\alpha)$ as before.

$$\Phi = \sum_{\alpha \in F^*} \sum_B \alpha_B \chi_B(\alpha, y) e^{-\operatorname{Tr}\alpha y}$$ (simple rational functions in the $e^{-\operatorname{Tr}\alpha y}$)

$$L(\varphi,\alpha) = \frac{1}{2}\frac{1}{\Gamma(s)^d}\int_{\mathbb{R}^d} \Phi(y) \underbrace{\operatorname{sgn}}_{\eta(y)}(Ny)\, |Ny|^{s-1}\, dy$$

$\rightarrow L(\varphi,\alpha)$ has an analytic continuation

$\rightarrow \eta(y) = \prod_{i \in I} \operatorname{sign} y_i$

$(I \subset \{1,\dots,d\})$

$L(\varphi,\alpha)$ has a zero of order at least $|I_k|$ at $s=-k$, when $I_k = I$ if k odd & $\{1,\dots,d\} - I$ if k even.

$$\circledast \quad \lim_{s \to -k} (s+k)^{-|I_k|} L(\varphi,\alpha) = 2^{d-1-|I_k|} \int_{\mathbb{R}^{I_k}} \left(\prod_{i=1}^{d}\right) \frac{dy_i}{y_i}$$

$\circledast$ $I_k = \phi$: 2-adic divisibility of Deligne-Ribet.

$\circledast$ $I = \{1,\dots,d\}$, $\varphi(0) \neq 0$, $k=0 \Rightarrow$ zero of order $\geq d-1$, class # formula at $s=0$ without functional eq^n.