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Introduction: A result of Bertolini-Darmon-Prasanna

$E/\mathbb{Q}$ elliptic curve of conductor N
$f = f_E$ associated modular form
$p \nmid N$ good, ordinary prime
$K = \mathbb{Q}(\sqrt{-D})$ class # 1, satisfying hypothesis $\ell | N \Rightarrow \left(\frac{K}{\ell}\right) = +1$
$\Rightarrow$ sign $L(E/K, s) = -1 \Rightarrow L(E/K, 0) = 0$
$\Rightarrow$ Heegner pt. $P \in X_0(N)$

Assume p splits in K, $\vartheta = (\vartheta_k)$ Hida family of Θ-series attached to K.
- on $\Gamma_0(D), \varepsilon_D$ if k is odd, on $\Gamma_0(D^2)$ otherwise.

$\exists L_p(f, \underline{\vartheta}, j)$ interpolating $L(f \otimes \vartheta_{1+2j}, 1+j)$, $j \geq 1$.

note: sign $+1$, so it makes sense to interpolate.

Thm. $L^*(f, \underline{\Theta}, 0) \doteq AJ_p(P)(\omega_E)$, where

equality up to explicit nonzero constants.

$$AJ_p : X_0(N)(\mathbb{Q}_p) \longrightarrow E(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p \quad \text{eval on } \omega_E$$

$$H^1(\mathbb{Q}_p, V_pE) \longrightarrow (\Omega^1_{E/\mathbb{Q}_p})^\vee \cong \mathbb{Q}_p$$

"Proof:" Waldspurger. $L(f \otimes \vartheta_{1+2j}, 1+j)^{1/2} \doteq \delta_2^{j-1} f(P)$,

where δ_k = Shimura-Maass differential operator $= \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{k}{z - \bar{z}}\right)$

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$$L_p(f, \underline{\vartheta}, j) \doteq d^{j-1} f^{[p]}(P), \quad d = q\frac{d}{dq}, \quad f^{[p]} = \sum_{p \nmid n} a_n(f) q^n.$$

$$L_p(f, \underline{\Theta}, 0) = \lim_{j \to 0} L_p(f, \underline{\Theta}, j) \doteq \underbrace{d^{-1} f^{[p]}(P)}_{\text{p-adic integral}}.$$

§2 The p-adic Beilinson formula. (jt work-in-progress w/ Darmon)

f, N, p as before.

Brunault's formula: $L_p(f, \chi, 2) \dfrac{L(f,1)}{\Omega^{+}} = \langle \eta_f, \mathrm{reg}_p\{u_F, u_G\} \rangle$,

where: u_F, u_G are the modular units associated via dlog to wt 2 Eisenstein series F, G of character χ^{-1}, χ, respectively.

$$- \mathrm{reg}_p : K_2(Y_1(N)) \otimes \mathbb{Q} \xrightarrow[\mathrm{reg}_{\acute{e}t}]{} H^1(\mathbb{Q}_p, V_f(2))$$

$$\Big\downarrow \log \ \wr\!\wr$$

$$H^1_{dR}(Y_1(N)/\mathbb{Q}_p)^f \xleftarrow[\text{comparison}]{} D_{dR}(V_f(2))$$

$$V_f = H^1_{\acute{e}t}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) / \langle T_n - a_n(f) \rangle$$

$- \eta_f \in H^1_{dR}(Y_1(N)/\mathbb{Q}_p)^f$ unique class st (i) $\varphi \eta_f = \alpha \eta_f$

(ii) $\langle \eta_f, \omega_f \rangle = 1$.

Remark: Brunault deduces this kind of formula from Kato's reciprocity law

Goal: Give a direct approach based on p-adic families & p-adic integration.

Simple-minded setting: E, f, N, p ($p \nmid N$ ordinary)

$\underline{f} = (f_k)$ Hida family through f, $f_k \in S_k(\Gamma_0(N))$

χ even prim. dirichlet character mod N.

$F = Eis_2(1, \chi^{-1})$ wt 2 Eisenstein series $a_\ell(F) = 1 + \chi(\ell)\ell$.

$G = Eis_2(\chi, 1)$ $\qquad a_\ell(G) = \chi(\ell) + \ell$

Study p-adic interpolation of $L(f_k \otimes G, \frac{k}{2}+1)$

$$k \equiv 2 \ (p-1), \quad k \geq 4$$

Rankin's method: the above special value

$$\doteq \left(f_k, \delta^{\frac{k-4}{2}} F \times G\right)_{\text{wt } k} \longleftarrow \text{period } \langle f_k, f_k \rangle$$

Factorization: $L(f_k \otimes G, \frac{k}{2}+1) = L(f_k \otimes \chi, \frac{k}{2}+1) \times \underbrace{L(f_k, k/2)}_{\text{central critical value.}}$

$$\exists L_p(\underline{f}, G)(k) \underset{\uparrow}{=} \langle \eta_{f_k}, e_{f_k} e_{\text{ord}}(d^{\frac{k}{2}-2} F^{[p]} \times G) \rangle$$

k classical

$k=2$ outside range of p-adic interpolation.

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$$L_p(\underline{f}, G, 2) = \lim_{\substack{k \to 2 \\ p\text{-adically}}} L_p(\underline{f}, G, k)$$

$$= \langle \eta_f, e_f\, e_{\mathrm{ord}}(d^{-1} F^{[p]} \times G) \rangle$$

Factorization of p-adic L-functions: $L_p(f, G, k) \doteq L_p(f \otimes \chi_j, k, \frac{k}{2}+1) \times L_p(f, k, k/2)$,

where $L_p(f, k, s)$ is the Mazur-Kitagawa p-adic L-fn. interpolating $L(f_k, j)$, $1 \le j \le k-1$, k classical.

$$\Longrightarrow L_p(\underline{f}, G, 2) = L_p(f \otimes \chi, 2, 2)\, L_p(\underline{f}, 2, 1)$$

$$= L_p(f \otimes \chi, 2) \frac{L(f, 1)}{\Omega^+}$$

Besser: $\langle \eta_f, e_f\, e_{\mathrm{ord}}(d^{-1} F^{[p]} \times G) \rangle$ is the p-adic regulator: cf. Coleman, de Shalit

$$\int_{\mathrm{div}\, u_G} d^{-1} F^{[p]} \omega_f$$

Remarks:

1) p-adic Beilinson at $m \ge 2$?
Using the same method for $L(f_k \otimes G_m, \frac{k+2m-2}{2})$, get an expression

$$L_p(f \otimes \chi, m) \frac{L(f, 1)}{\Omega^+} = \langle \eta_f, d^{1-m} \overline{F_m}^{[p]} \times G_m \rangle.$$

2) Try to use the same method replacing χ by $\chi\psi$, cond$^r\psi = p$? to obtain similar expressions for

$$L_p(f\otimes\chi\psi, 2)\frac{L(E,1)}{\Omega^+}.$$

They (i.e. varying ψ) $\leadsto L_p(f\otimes\chi, \alpha)$.
You can derive Kato reciprocity from this.

3) κ Kato's class in $H^1(\mathbb{Q}, V_pE)$

Reciprocity law:

$$\begin{array}{ccc} \partial_p \kappa \in H^1(\mathbb{Q}_p, V_pE)/H^1_f(\mathbb{Q}_p, V_pE) \\ \downarrow \qquad\qquad \downarrow \exp^* \\ \frac{L(E,1)}{\Omega^+}\omega_E \in \Omega^1(E/\mathbb{Q}) \end{array}$$

If sign $L(E,s) = -1$, then $\mathrm{res}_p \kappa \in H^1_f(\mathbb{Q}_p, V_pE)$

$$\updownarrow \delta$$

$$E(\mathbb{Q}_p)\otimes\mathbb{Q}.$$

So $\mathrm{res}_p \kappa = \delta(P_p)$

Perrin-Riou's conj: $\log P_p = (\log P)^2$, $P \in E(\mathbb{Q})\otimes\mathbb{Q}$.

They can prove this!!