Gradient Random Fields – Banff workshop 2011

Introduction

Gradient random fields is a class of model systems arising in the studies of random interfaces, field theory and elasticity theory. Although a lot is known about the subset of these systems with interactions obeying specific convexity requirements, for the more physically relevant non-convex interactions progress started only recently. New ideas have nonetheless emerged also for convex interactions; e.g., the connection between the level lines of the Gaussian Free Field and Schramm's SLE. Applied analysts, motivated by problems from material science, have been traditionally interested in the variational problems for suitable, albeit mostly phenomenological, free-energy functionals. An approach based on microscopic modeling began to be pursued only recently and even the most basic problems (e.g., the verification of the so-called Cauchy-Born rule) remains open at positive temperatures.

A diverse range of goals and technical means involved makes this subject a natural meeting ground of three mathematical disciplines: analysis, probability, and mathematical physics. Indeed, the existing body of work employs methods ranging from homogenization theory to renormalization group/multiscale analysis to conformal mappings to variational analysis of quasi-convex functionals, etc. It is well understood that a key technical obstacle is the long range decay of correlations (a.k.a. zero mass). A unifying role of the continuum Gaussian Free Field as the scaling limit of these systems is widely appreciated. However, there has so far been little crossfertilization between the various subareas and/or technical approaches.

The meeting attempted to address these issues by bringing together participants with backgrounds in the subject areas mentioned above. Specifically, the audience included numerous experts on variational analysis (Andrea Braides, Stephan Luckhaus, Felix Otto, Florian Theil, and Joseph Conlon), multiscale analysis and rigorous renormalisation group methods (Abdelmalek Abdesselam, Margherita Disertori, Peirluigi Falco, and Jean Bernard Bru) and probability theory and mathematical statistical mechanics (Codina Cotar, Takao Nishikawa, Christof Kuelske, Miloš Zahradník, and Noemi Kurt). The program was arranged to give a fair share of time to each of the represented subject areas; speakers were urged to spend time on introducing their field to the non-experts in the audience.

Participants and titles of the talks

Abdelmalek Abdesselam
The rigorous renormalization group via explicit combinatorial expansions

Margherita Disertori
Anderson localization/delocalization transition for a supersymmetric sigma model

Peirluigi Falco
Interacting Fermions Approach to 2D Critical Models

Jean Bernard Bru
Diagonalization of Quadratic Operators via Non-Autonomous Evolution Equations

Joseph Conlon
Strong Central Limit Theorems in PDE with random coefficients and Euclidean Field Theory

Andrea Braides
Variational problems with percolation

Stephan Luckhaus
Interpolation and large deviations for elasticity hamiltonians

Felix Otto
Optimal error estimates in stochastic homogenization

Takao Nishikawa
Hydrodynamic limit for the Ginzburg-Landau grad-phi interface model with a conservation law and the Dirichlet boundary condition

Florian Theil
Crystallization in three dimensions: The FCC case

Tadahisa Funaki
Scaling limits for dynamic models of Young diagrams

Christof Külske
Discrete approximations to massless models

Codina Cotar
Uniqueness of random gradient states

Nicholas Crawford
Localization via Randomness in Classical Statistical Mechanics

Giambattista Giacomin
Transitions in active rotator systems

Ron Peled
High-dimensional Lipschitz functions are typically flat

Richard Kenyon
Conformal invariance of double-dimer loops

Oren Louidor
Fixation for distributed clustering processes

Miloš Zahradník
Cluster expansion around massless gaussian, for a special class of perturbing potential wells

Noemi Kurt
Pinning of a Laplacian interface model

Patrick Dondl
Pinning and depinning in random media

SUMMARY OF THE PRESENTATIONS

As already mentioned, the workshop has been built around three principal areas: gradient random fields, stochastic geometry and Gaussian free field, and mathematical approaches to elasticity theory. The talks were aimed to introduce the essential sets of ideas underlying these fields and thus initiate and nurture interdisciplinary discussions and formulation of projects. Below we give a breakdown of the topics into further, more specific, categories.

Basic properties of gradient fields: The gradient random fields have been discussed by Codina Cotar. She introduced an easy and efficient one-step decimation technique to obtain the strict convexity of the surface tension for high temperature when the potential is non-convex. The same method has been extended to deal with random interactions. For these models, the existence of gradient Gibbs measures can be shown for dimensions two and higher. This extends earlier work by Van Enter and Külske on non-existence of such measures in dimension two. Uniqueness has been established as well.

Related results on "localisation via randomness" were discussed by Nicholas Crawford. A characteristic example is the $O(2)$-model in d dimensions with an additional local field term in the Hamiltonian given by i.i.d. unit vectors pointing in the North/South direction with equal probability. The question is whether, for any sufficiently small local field strength, there is a temperature low enough so that sufficiently large block averages of spin variables typically have preferred orientations under the corresponding Gibbs measure. If so, what direction(s) are preferred? This question may be viewed as a classical, many body analog of the well known phenomenon of Anderson Localization. The talks outlined an answer in the context of models with Kac interactions.

A completely different approach to gradient random fields (i.e. massless systems) has been outlined by Christoph Külske. His talk was focused on a certain discrete approximation scheme for (the spin-state space in) these models and the accuracy with which they approximate the continuum system. The talk touched upon the subjects of continuous symmetry breaking and Kosterlitz-Thouless behavior in the discretized rotor-model.

Pinning in interface models: Noemi Kurt discussed the centered Gaussian field on the d-dimensional integer lattice whose covariance matrix is given by the inverse of the discrete bi-Laplacian (as opposed to the discrete Laplacian in the case of the classical gradient interface model). In addition, the interface gets a reward when it touches the 0-hyperplane. A technical obstacle in analyzing this system is the unavailability of the random walk representation; notwithstanding, some of the quantities can be related to intersections of random walks. The talk described progress towards proving an exponential decay of correlations of the pinned field in the supercritical dimensions.

Patrick Dondl analyzed a similar (pinning) phenomenon on the background of a random media. Specifically, he discussed a parabolic model for the evolution of an interface and the connection to the recently analyzed Lipschitz-percolation problem.

Dynamics: A dynamical aspect of gradient models was addressed by Takao Nishikawa. Specifically, he discussed the hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with a conservation law and the Dirichlet boundary condition and a conservation law for the area under the interface. For the system with periodic boundary conditions, a nonlinear fourth-order partial differential equation is derived as the macroscopic equation.

Another problem with dynamic has been outlined by Giambattista Giacomin. His talk focused on active rotator systems which are ubiquitous in biological applications. The large scale behavior of a system of many finite dimensional stochastic dynamical systems may be very difficult to predict. A novel input was to consider active rotators with mean field gradient interactions. It is well known that the large scale interacting dynamics is accurately described by a Fokker-Planck equation and, in the limit of "inactive dynamics" the model reduces to a particular case of the Kuramoto synchronization PDE. The approach yields a complete description of the phase diagram of the active rotators model.

Fluctuations in grad models: A stimulating insight from analysis has been provided by Joe Conlon in his talk on strong central limit theorems in PDE with random coefficients and Euclidean Field Theory. It was shown in 1997 by Naddaf and Spencer that the two point correlation function of a gradient field with uniformly convex action is the expectation of the Green's function for a parabolic PDE with random coefficients. Using this identity and homogenization theorems for PDE with random coefficients, they were able to prove that large scale averages of the correlation function converge those of a Gaussian field field. Joe Conlon showed how to prove the pointwise convergence. The main tools used in the proof are the Helffer-Sjöstrand formula and the continuity of the norms of Calderon-Zygmund operators on L^p and weighted L^p spaces.

A very interesting other aspect of gradient random fields has been described by Felix Otto. His talked discussed optimal estimates on errors for the simplest possible model problem: a discrete elliptic equation on the d-dimensional lattice $\mathbb{Z}^d$. The basic ideas are again Calderon-Zygmund theory and estimates on the Green's functions, and the corrector, in the random conductance model. As a bonus, the tightness of the corrector (as opposed to gradients thereof) is shown to hold in systems with i.i.d. conductances in three and higher dimensions. This solves a long-standing open problem in homogenization literature.

Multiscale analyses: Multiscale analysis and renormalization group methods played a key role in Abdelmalek Abdesselam's talk whose focus was on the p-adic field theory. Instead following the standard approach to these systems by Brydges and coworkers, he based his approach on explicit combinatorial expansions and multiscale cluster expansion in phase space. As a case study, the Brydges-Mitter-Scoppola Φ^4-model with modified propagator in three dimensions has been discussed in detail.

The approach by Brydges and coworkers appeared also in the talk by Pierluigi Falco who discussed interacting Fermions approach to two-dimensional critical models. It is well known in the physics literature that many of these models can be re-cast as a system of interacting fermions: Ising, 6-vertex, 8-vertex, Ashkin-Teller, Interacting Dimers, q-state Potts, Completely Packed Loops, $(1+1)$-dimensional Hubbard, XYZ quantum chain, etc. The scaling limit of these systems is described by the (non-Gaussian) field called Thirring Model. In his talk Pierluigi outlined some recent works in which the physicists' ansatz for the exact solution of the Thirring model has been proved.

Gradiend fields and Anderson localization: A seemingly remote subject has been brought up by Margherita Disertori. She outlined a model of Anderson localization/delocalization transition that can be reduced to a supersymmetric sigma model. A key point of this rewrite is that the decay of the Green's function in the original system is now related to the Green's function of an associated gradient field. Using methods developed by Naddaf and Spencer, these can then be shown to exhibit diffusive behavior — which implies delocalized spectrum in the associated Anderson model.

Scaling limits: A combinatorial approach to gradient random fields has been presented by Ron Peled in the case of high-dimensional random Lipschitz and homomorphism height functions. With a novel counting techniques he showed that, in high enough dimensions, a typical uniform sample from these classes exhibits long range order and is, in fact, more or less constant on either the even or the odd sublattice. Henceforth the random function is very flat, having a bounded variance at each point and (for integer-valued functions) taking few values overall. Replacing the homomorphism height function f by $f \mod 3$ we get a perfect 3-coloring and thus a phase coexistence in high dimensions for the anti-ferromagnetic 3-state Potts model.

Another model of combinatorial nature showing interesting scaling properties has been presented by Tadahisa Funaki. He discussed scaling limits for dynamic models of Young diagrams. In particular he established the hydrodynamic limits and the fluctuation limits for non-conservative dynamics of $2D$ Young diagrams associated with the uniform or restricted uniform grandcanonical ensembles.

The class of dimer models is a close relative of the gradient random fields. Rick Kenyon presented results on conformal invariance of double-dimer loops. Here, a double-dimer configuration on a planar graph G is a union of two statistically independent dimer covers of G. Introducing quaternion weights in the dimer model he showed how they can be used to study the homotopy classes (relative to a fixed set of faces) of loops in the double dimer model on a planar graph. As an application in the scaling limit of the "uniform" double-dimer model on $\mathbb{Z}^2$ the loops are conformally invariant. This is a major step towards proving that the double-dimer paths are described by SLE_4.

Variational problems: As an introduction to the material and variational analysis aspect of gradient random fields, Florian Theil gave a talk on crystallization in three dimensions and the Cauchy-Born hypothesis. The talk outlined in detail the variational analysis viewpoint and familiarized the audience with standard problems and techniques in that field. In particular, the asymptotic behavior of the minimizers of atomistic pair interaction systems of Lennard-Jones type was characterized in the limit when the number of particles tends to infinity. For a large class of pair interaction potential it can be shown rigorously that the minimizers converge to a rigid fcc lattice.

Variational problems have been discussed also by Andrea Braides. In his talk he introduced a connection of variational problems with percolation. In the passage from discrete systems to continuous variational problems, percolation offers a perfect environment to model problems with a random dependence (e.g., of the type of interaction between neighboring points on a lattice on a random variable). As a result, the discrete energy functional can be approximated by by a continuous deterministic energy whose properties may be different above or below a percolation threshold. The continuous surface tension is described by using homogenization formulae of percolation type, which show a very interesting interaction between variational techniques and percolation arguments.

Other topics: An old problem of mathematical physics is proper diagonalization. Jean Bernard Bru introduced a diagonalisation of quadratic operators via non-autonomous evolution equations. Here, the quadratic Hamiltonians are acting on a Boson Fock space. Based on the Brocket-Wegner flow, the mathematical novelty lies in the use of the theory of non-autonomous evolution equations as a key ingredient to diagonalize such operators.

Oren Louidor discussed a discrete-time resource flow in $\mathbb{Z}^d$, where wealthy (in terms of a weight) vertices attract the resources of their less affluent neighbors. For any translation-invariant probability distribution of initial resource quantities, he proved that the flow at each vertex terminates after finitely many steps. The outlined result answers (a generalised version of) a question posed by van den Berg and Meester in 1991.