Mean-preserving stochastic renormalization of differential equations

Sergey Lototsky
Department of Mathematics
University of Southern California

Collaborators: Sivaditya Kaligotla, Remis Mikulevicius, Boris Rozovskii, Dora Seleši, Xiaoliang Wan

Renormalization

Objective: To make sense of things that don't make sense.

Some examples:

$u_t = \mathbf{\Delta} u + u\dot{W}(t,x)$, $d>1$;

$u_t = u_{xx} + (u\cdot\nabla)u + \dot{W}(t,x)$, $d>1$;

$\nabla\cdot\big((1+\dot{W}(x))\nabla u\big) = 0$;

$u_t = \big(1+\dot{W}(t,x)\big)u_{xx}$.

The plan:

- Re-thinking multiplication and stochastic integration;
- Re-thinking integrability;
- Dealing with the consequences.

The mean value of stochastic equation

Sometimes it is preserved:

if $\dot{x} = ax$, $dX = aX\,dt + \sigma(X)dw(t)$, and $\mathbb{E}X(0) = x(0)$

then $\mathbb{E}X(t) = x(t)$.

Most of the time, it is not.

This is especially problematic for SPDEs:

$u_t = \mathbf{\Delta} u + (u \cdot \nabla)u + \dot{W}$.

A toy example

The equation: $v = 1 + v\xi$, $\xi \sim \mathcal{N}(0,1)$. $v = \frac{1}{1-\xi} = \sum_{k\geq 0} \xi^k$; $\mathbb{E}v \neq 1$.

Hermite polynomials: $e^{z\xi-(z^2/2)} = \sum_{k\geq 0} \frac{z^k}{k!}\mathrm{H}_k(\xi)$

How about $u = \sum_{k\geq 0} \mathrm{H}_k(\xi)$? At least $\mathbb{E}u = 1$.

The problem: $\xi \mathrm{H}_k(\xi) \neq \mathrm{H}_{k+1}(\xi)$.
The solution: $\xi \diamond \mathrm{H}_k(\xi) := \mathrm{H}_{k+1}(\xi)$.

The **renormalized equation**: $u = 1 + u \diamond \xi$; $u = \sum_{k\geq 0} \mathrm{H}_k(\xi) = (1-\xi)^{\diamond(-1)}$.

More generally: $f(\xi) = \sum_{k\geq 0} f_k \xi^k$; $f^{\diamond}(\xi) = \sum_{k\geq 0} f_k \mathrm{H}_k(\xi)$.

Mean-preserving: $\mathbb{E}f^{\diamond}(\xi) = f_0 = f(0) = f(\mathbb{E}\xi)$.

Hida-Kondratiev spaces

Motivation 1: $\mathbb{E}\left(\sum_{k\geq 0}\mathrm{H}_k(\xi)\right)^2=\sum_{k\geq 0}k!$.

Motivation 2: $\varphi_z(\xi)=e^{\diamond(z\xi)}=\sum_{k\geq 0}\frac{z^k}{k!}\mathrm{H}_k(\xi)$.

The construction:

$f(\xi)\in L_2(\xi)\Longleftrightarrow f=\sum_{k\geq 0}f_k\mathrm{H}_k(\xi),\ \sum_{k\geq 0}f_k^2k!<\infty.$

$f(\xi)\in(\mathcal{S})_{\rho,\ell}\Longleftrightarrow f=\sum_{k\geq 0}f_k\mathrm{H}_k(\xi),\ \sum_{k\geq 0}f_k^2(k!)^{1+\rho}\,2^{\ell k}<\infty.$

- $\rho\geq 0$: $(\mathcal{S})_\rho=\bigcap_\ell(\mathcal{S})_{\rho,\ell},\ (\mathcal{S})_{-\rho}=\bigcup_\ell(\mathcal{S})_{-\rho,\ell}$; $\mathbb{E}f:=f_0$
- $\rho\leq 1\iff\varphi_z\in(\mathcal{S})_\rho$.
- $f\in(\mathcal{S})_{-\rho},\ \psi\in(\mathcal{S})_\rho\implies\langle f,\psi\rangle=\sum_k f_k\psi_k\in\mathbb{R}$.
- S **transform:** $f\in(\mathcal{S})_{-\rho}\Leftrightarrow\widetilde{f}(z)=\langle f,\varphi_z\rangle$ is analytic; entire if $\rho<1$.

Wick product $f\diamond g$**:** $(f\diamond g)_k=\sum_{i=0}^k f_{k-i}g_i\Longleftrightarrow\widetilde{f\diamond g}(z)=\widetilde{f}(z)\,\widetilde{g}(z)$

$\mathbb{E}f\diamond g=\big(\mathbb{E}f\big)\big(\mathbb{E}g\big)$

1. **The original**: $u = 1 + u \diamond \xi$;

$\widetilde{u}(z) = 1 + \widetilde{u}(z)\, z$, $\widetilde{u}(z) = 1/(1-z)$.

$u \in (\mathcal{S})_{-1}$.

2. **One more**: $u^{\diamond 2} - u + \xi = 0$.

$\big(\widetilde{u}(z)\big)^2 - \widetilde{u}(z) + z = 0$.

$u^{(1)} = 1 + \sum_{k\geq 1} C_{k-1}\mathrm{H}_k(\xi),\ u^{(0)} = -\sum_{k\geq 1} C_{k-1}\mathrm{H}_k(\xi)$, both $u \in (\mathcal{S})_{-1}$.

$$C_n = \frac{1}{n+1}\binom{2n}{n} \leq 4^n$$

Note: $v^2 - v + \xi = 0, v = (1 \pm \sqrt{1-4\xi^2})/2$, does not look good.

But...

$v^2 - 2v - \xi^2 = 0$ is perfectly fine: $v = 1 \pm \sqrt{1+\xi^2}$

Meanwhile, solutions of the renormalized equation

$u^{\diamond 2} - 2u - \mathrm{H}_2(\xi) = 0$ still live in $(\mathcal{S})_{-1}$:

$\big(\widetilde{u}(z)\big)^2 - 2\widetilde{u}(z) - z^2 = 0$;

$\widetilde{u}(z) = 1 \pm \sqrt{1+z^2}$, so $\widetilde{u}(z)$ is not an entire function.

A generalized Gaussian chaos space

It is $(\mathbb{F}, \boldsymbol{\xi}, H, \mathrm{Q})$, where

- $\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P})$ is a probability space;
- $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots)$ are iid $\mathcal{N}(0,1)$, $\mathcal{F}$ is generated by $\boldsymbol{\xi}$;
- H is a separable Hilbert space;
- Q is an unbounded, self-adjoint positive-definite operator on H such that Q has a pure point spectrum: $\mathrm{Q}\mathfrak{h}_k = q_k \mathfrak{h}_k,\ k \geq 1,$ $\{\mathfrak{h}_k,\ k \geq 1\}$— CONS in H; $\sum_{k\geq 1} \frac{1}{q_k^{\gamma}} < \infty, \ \ \gamma > 0.$

Basic White noise: $\Omega = \mathcal{S}'(\mathbb{R})$, $H = L_2(\mathbb{R})$,
$\mathrm{Q} = -\boldsymbol{\Delta} + x^2 + 1$, $q_k = 2k$.

More generally: Have $\mathbb{F}$; the equation determines $\mathfrak{q} = \{q_k,\ k \geq 1\}$.

Gaussian Chaos Expansion

Noise: $\dot{W} = \boldsymbol{\xi} = \{\xi_k,\ k \geq 1\}$, iid $\mathcal{N}(0,1)$. **Chaos space:** $L_2(\Omega; V)$

Index set: $\mathcal{J} = \left\{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots) : \alpha_k \in \{0,1,2,\ldots\},\ \sum_k \alpha_k < \infty\right\}$

Notations: $\mathbf{(0)} = (0,0,0\ldots)$,

$|\boldsymbol{\alpha}| = \sum_k \alpha_k,\ \alpha! = \prod_k \alpha_k!,\ \boldsymbol{\beta} < \boldsymbol{\alpha} \Leftrightarrow \beta_k \leq \alpha_k, \boldsymbol{\beta} \neq \boldsymbol{\alpha}.$

$\mathrm{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},\quad \mathfrak{q}^{\boldsymbol{\alpha}} = \prod_k q_k^{\alpha_k}.$

Basis elements: $\xi_{\boldsymbol{\alpha}} = \frac{1}{\sqrt{\boldsymbol{\alpha}!}} \prod_k \mathrm{H}_{\alpha_k}(\xi_k)$

Chaos expansion: $v = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} v_{\boldsymbol{\alpha}} \xi_{\boldsymbol{\alpha}},$

Weighted chaos spaces: $\sum_{\boldsymbol{\alpha} \in \mathcal{J}} r_{\boldsymbol{\alpha}} \|v_{\boldsymbol{\alpha}}\|_V^2 < \infty$

Generalized expectation: $\mathbb{E}v = v_{\mathbf{(0)}}$

Hida-Kondratiev spaces

V — another Hilbert space. For $\rho \in [0,1]$ and $\ell \geq 0$,

• the space $(\mathcal{S})_{\rho,\ell}(V)$ is the collection of $\Phi \in \mathbb{L}_2(\boldsymbol{\xi}; V)$ such that $\|\Phi\|^2_{\rho,\ell;V} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} (\boldsymbol{\alpha}!)^{\rho} \mathfrak{q}^{\ell\boldsymbol{\alpha}} \|\Phi_{\boldsymbol{\alpha}}\|^2_V < \infty$;

• the space $(\mathcal{S})_{-\rho,-\ell}(V)$ is the closure of $\mathbb{L}_2(\boldsymbol{\xi}; V)$ with respect to the norm $\|\Phi\|^2_{-\rho,-\ell;V} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} (\boldsymbol{\alpha}!)^{-\rho} \mathfrak{q}^{-\ell\boldsymbol{\alpha}} \|\Phi_{\boldsymbol{\alpha}}\|^2_V$;

• the space $(\mathcal{S})_{\rho}(V)$ is the **projective** limit (intersection endowed with a special topology) of the spaces $(\mathcal{S})_{\rho,\ell}(V)$, as ℓ varies over all integers;

• the space $(\mathcal{S})_{-\rho}(V)$ is the **inductive** limit (union endowed with a special topology) of the spaces $(\mathcal{S})_{-\rho,-\ell}(V)$, as ℓ varies over all integers.

References: Hida et al. ($\rho = 0$); Kuo ($0 < \rho < 1$); Holden et al. ($\rho = 1$)

- $\langle \Psi, \eta \rangle = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \Psi_{\boldsymbol{\alpha}} \eta_{\boldsymbol{\alpha}} \in V, \quad \Psi \in (\mathcal{S})_{-\rho}(V), \ \eta \in (\mathcal{S})_{\rho}(\mathbb{C})$.
- $\mathcal{E}(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}}{\sqrt{\boldsymbol{\alpha}!}} \xi_{\boldsymbol{\alpha}}, \ \boldsymbol{z} = (z_1, z_2, \ldots) \in \ell_2(\mathbb{C})$.
- $|\boldsymbol{\alpha}|! \leq C \mathfrak{q}^{\gamma \boldsymbol{\alpha}} \boldsymbol{\alpha}!$
- $0 \leq \rho \leq 1 \ \Leftrightarrow \ \mathcal{E}(\boldsymbol{z}) \in (\mathcal{S})_{\rho}(\mathbb{C})$
- S-transform: $\widetilde{\Phi}(\boldsymbol{z}) = \langle \Phi, \mathcal{E}(\boldsymbol{z}) \rangle = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \frac{\Phi_{\boldsymbol{\alpha}}}{\sqrt{\boldsymbol{\alpha}!}} \boldsymbol{z}^{\boldsymbol{\alpha}}$.
- **(Simplified) characterization theorems** (making everything intrinsic)

(a) If $0 \leq \rho < 1$ and $\Phi \in (\mathcal{S})_{-\rho}(V)$, then $\widetilde{\Phi}(z\boldsymbol{p} + \boldsymbol{q})$ is entire ($\boldsymbol{p}, \boldsymbol{q}$ real).

(b) If $\Phi \in (\mathcal{S})_{-1}(V)$, then $\widetilde{\Phi}(\boldsymbol{z})$ is analytic "at the origin".

Wick product $\Phi \diamond \Psi$**:** $\widetilde{\Phi \diamond \Psi}(z) = \widetilde{\Phi}(z)\, \widetilde{\Psi}(z)$; $\mathbb{E}\Phi \diamond \Psi = \big(\mathbb{E}\Phi\big)\big(\mathbb{E}\Psi\big)$

$$(\Phi \diamond \Psi)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \sqrt{\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}} \Psi_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Phi_{\boldsymbol{\beta}}$$

- $\xi \diamond \eta = \xi\eta$ for non-random ξ and/or η
- If W is a standard BM and $\eta(t_i)$ is $\mathcal{F}^W_{t_i}$-measurable, then

$\eta(t_i)(W(t_{i+1} - W(t_i)) = \eta(t_i) \diamond (W(t_{i+1} - W(t_i))$

so $\displaystyle\int_0^T \eta(t) \diamond dW(t) = \int_0^T \eta(t)dW(t)$ for adapted η.

- For BM, (Usual product, Itô calculus) $\Leftrightarrow$ (Wick product, usual calculus).

- $\diamond$ as the divergence operator (Itô-Skorokhod-Wick integral)

More: Holden, Øksendal, Ubøe, Zhang (1996).

A comparison

$$\mathrm{H}_m(x)\mathrm{H}_n(x) = x^{m+n} + (\text{lower order terms})$$
$$= \mathrm{H}_{m+n}(x) + (\text{lower order terms}),$$

that is,

$$\xi\eta = \xi \diamond \eta + (\text{“lower order terms”})$$

Our basic example.

(a) $u = 1 + u \diamond \xi,\ u = \sum_k u_k \mathrm{H}_k(\xi)$:

$$\xi \diamond \mathrm{H}_k(\xi) = \mathrm{H}_{k+1}(\xi)$$

$$\boxed{u_0 = 1,\ u_k = u_{k-1}}$$

(b) $v = 1 + v\xi,\ v = \sum_k v_k \mathrm{H}_k(\xi)$:

$$\xi \mathrm{H}_k(\xi) = \mathrm{H}_{k+1}(\xi) - k\mathrm{H}_{k-1}(\xi)$$

$$\boxed{v_0 = 1 + v_1,\ v_k = v_{k-1} + (k+1)v_{k+1}}$$

(Our) Renormalization

- Is a special multiplication of random objects
- Does not affect deterministic equation
- Is related to Wick multiplication and stochastic integration
- Can increase the class of admissible equations
- Can force the solution to become a generalized random element
- Leads to a probabilistically strong solution
- Leads to unusual uniqueness results

- Extensions to non-Gaussian noise: pending
- Notations: $\diamond$, : : as in $:\xi^2: = \xi^{\diamond 2} = \mathrm{H}_2(\xi)$, $\boldsymbol{\delta}$, $\mathcal{R}$.

Deterministic equation:
$\dot{U} = \mathrm{A}u + F + \mathcal{P}\big(\mathrm{B}_1 U, \ldots, \mathrm{B}_n U\big),\; U(0) = U_0$, $\mathcal{P}$ is a polynomial.

Renormalization:
$\dot{u} = \mathrm{A}u + f + \mathcal{R}\mathcal{P}\big(\mathrm{B}_1 u, \ldots, \mathrm{B}_n u\big) + \mathcal{R}\big(\mathbf{L}u\,\dot{\boldsymbol{W}}\big),\; u(0) = u_0$, with
linear operators $\mathrm{A}, \mathrm{B}_i, \mathbf{L}$,
random initial condition u_0, $\mathbb{E}\dot{\boldsymbol{W}} = 0$.

If $\mathbb{E}f = F$ and $\mathbb{E}u_0 = U_0$, then $\mathbb{E}u(t) = U(t)$.
Zero uniqueness of U implies uniqueness of zero-mean u

Moreover, under suitable conditions, $\mathbb{E}\|u\|^2 \asymp \sum_{\boldsymbol{\alpha}} q^{\boldsymbol{\alpha}} |\boldsymbol{\alpha}|!$

Here $\boldsymbol{\alpha} = (\alpha_k,\; k \geq 1), \sum_k \alpha_k < \infty$, $q^{\boldsymbol{\alpha}} = \prod_{k\geq 1} q_k^{\alpha_k}$.

Some examples

$u_t = \boldsymbol{\Delta} u + u\dot{W}(t,x)$, $d > 1$;

$u_t = \boldsymbol{\Delta} u + (u \diamond \nabla)u + \nabla u \dot{W}(t,x)$, $d > 1$;

$\nabla \cdot \big((1 + \dot{W}(x)) \diamond \nabla u\big) = f(x)$;

$\nabla \cdot \big((1 + e^{\diamond \dot{W}(x)}) \diamond \nabla u\big) = f(x)$;

$u_t = \big(1 + \dot{W}(t,x)\big) u_{xx}$;

Equations driven by fractional noise.

A more detailed example: Burgers equation

Equation: $u_t + u \diamond u_x = u_{xx}, x \in \mathbb{R},\ u(0,x) = \xi\,\phi(x)$.

Solution: $u(t,x) = \sum_{n\geq 0} u_n(t,x)\mathrm{H}_n(\xi)$.

Propagator:

$$\big(u_n\big)_t + \sum_{k=0}^{n} u_k\big(u_{n-k}\big)_x = \big(u_n\big)_{xx},\quad u_n(0,x) = \phi(x) I_{(n=1)}.$$

$\boldsymbol{n = 0}$: $\big(u_0\big)_t + u_0\big(u_0\big)_x = \big(u_0\big)_{xx},\ u_0(0,x) = 0$, so $u_0 \equiv 0$;

$\boldsymbol{n = 1}$: $\big(u_1\big)_t = \big(u_1\big)_{xx} - (u_0 u_1)_x,\ u_1(0,x) = \phi(x)$;

$\boldsymbol{n > 1}$: $\big(u_n\big)_t = (u_n)_{xx} - (u_0 u_n)_x - \sum_{k=1}^{n-1} u_k\big(u_{n-k}\big)_x,\ u_n(0,x) = 0.$

$$\|u_k(t,\cdot)\big(u_{n-k}(t,\cdot)\big)_x\|_0 \leq \left(\sup_x |u_k(t,x)|\right)\|\big(u_{n-k}(t,\cdot)\big)_x\|_0$$
$$\leq K_0\|u_k(t,\cdot)\|_1\,\|u_{n-k}(t,\cdot)\|_1.$$

Conclusion: Can run an induction.

Estimates Of The Norms

Define: $L_n^2 = \int_0^T \|u_n(t,\cdot)\|_2^2 dt + \sup_{t\in[0,T]} \|u_n(t,\cdot)\|_1^2$
Then:

$$L_n \le K(T,\phi) \sum_{k=1}^{n-1} L_k L_{n-k}.$$

Note: ((Very) Well Known) *Catalan numbers* $C_n,\ n \ge 0$, satisfy

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k},\ C_0 = 1; \quad C_n \sim \frac{4^n}{\sqrt{\pi}\, n^{3/2}}$$

Conclusion: $L_n \le A^n C_{n-1} \le q^n.$

One more comparison

Equation $v_t + vv_x = v_{xx}$ with initial condition $v(0,x) = \xi\phi(x)$ has a closed-form solution by Hopf-Cole transform; $\mathbb{E}|v(t,x)|^2 < \infty$.
On the other hand, if $v(t,x) = \sum_{k\geq 1} v_k(t,x)\mathrm{H}_k(\xi)$, then

$$(v_k)_t = (v_k)_{xx} - \sum_{m\geq 0}\sum_{\ell=0}^{k}\sqrt{\frac{(\ell+m)!(k+m-\ell)!}{\ell!(k-\ell)!m!}}v_{\ell+m}(v_{k+m-\ell})_x$$

In particular,
$(v_0)_t = (v_0)_{xx} - v_0(v_0)_x - \sum_{m\geq 1}\sqrt{m!}\,v_m(v_m)_x.$

Now, recall: if $u_t + u \diamond u_x = u_{xx}$ and $u(0,x) = \xi\phi(x)$, then $(k \geq 2)$

$$(u_k)_t = (u_k)_{xx} - \sum_{\ell=0}^{k} u_\ell(u_{k-\ell})_x = (u_k)_{xx} - (u_0u_k)_x - \sum_{\ell=1}^{k-1} u_\ell(u_{k-\ell})_x.$$

Can show that $u(t,x) \notin (\mathcal{S})_{-\rho}$ for every $\rho < 1$.

The equation $u_t + uu_x = u_{xx}$

Harry Bateman (1882–1946): British-American (1915, *Monthly Weather Review*)

Johannes Martinus Burgers (1895–1981): Dutch (1920–1940)

Conclusions

- Could be a new class of useful models
- Could be just a mathematical curiosity