Nonlocal Dispersals in Spatially Periodic Media

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Outline of Talk

- Introduction
- Principal Eigenvalues of Nonlocal Dispersal Operators
- Spatially Periodic Stationary Solutions of KPP Equation with Nonlocal Dispersal
- Spatial Spreading Speeds of KPP Equations with Nonlocal Dispersal
- Traveling Wave Solutions of KPP Equation with Nonlocal Dispersal
- Other Related Works

1. Introduction

Population growth model with nonlocal dispersal

$$\frac{\partial u}{\partial t} = \nu\Big[\int_{\mathbb{R}^N} \kappa(y-x)(u(t,y)-u(t,x))dy\Big] + uf(x,u) \qquad (1)$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^N$

$u(t,x)$ – population density

$\kappa(z) \geq 0$, $\kappa(0) > 0$, $\int_{\mathbb{R}^N} \kappa(z)dz = 1$

$\nu\big[\int_{\mathbb{R}^N} \kappa(y-x)(u(t,y)-u(t,x))dy\big]$ – nonlocal dispersal

ν – nonlocal dispersal rate

$f(x,u)$ – growth rate

$f(x+p_i\mathbf{e_i},u) = f(x,u)$ $(p_i > 0)$ – spatial periodicity

$f(x,u) < 0$ for $u \gg 1$, $f_u(x,u) < 0$ for $u \geq 0$

Random dispersal counterpart

$$\frac{\partial u(t,x)}{\partial t} = \nu \Delta u(t,x) + u(t,x) f(x,u) \qquad (1)'$$

If $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$, $\tilde{\kappa}(z) = \tilde{\kappa}(-z)$, $\mathrm{supp}\tilde{\kappa} = \{z \in \mathbb{R}^N \mid \|z\| < 1\}$

$$\nu \int_{\mathbb{R}^N} \kappa(y-x)[u(y)-u(x)]dy] = \nu \int_{\mathbb{R}^N} \tilde{\kappa}(z)\big[u(x+\delta z) - u(x)\big] dz$$

$$= \nu \int_{\mathbb{R}^N} \tilde{\kappa}(z) \left[\delta(\nabla u(x) \cdot z) + \frac{\delta^2}{2} \sum_{i,j=1}^{n} u_{x_i x_j}(x) z_i z_j + O(\delta^3) \right] dz$$

$$= \Big(\frac{\nu \delta^2}{2N} \int_{\mathbb{R}^N} \tilde{\kappa}(z) \|z\|^2 \, dz \Big) \Delta u(x) + O(\delta^3)$$

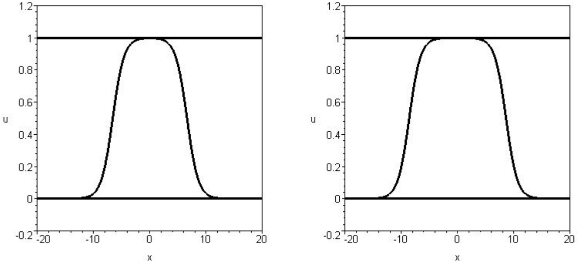
If $0 < \delta \ll 1$, expect (1) has similar dynamics as $(1)'$

However, the solutions of $(1)'$ have smoothness and compactness properties, while the solutions of (1) have no such properties

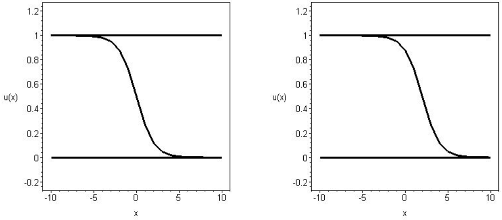
1. Introduction

Central problems

- Stability of $u \equiv 0$
- Existence of spatially periodic positive stationary solution $u^*(\cdot)$ (if $u \equiv 0$ is unstable)
- How fast does the population spread into the region where there is no population initially (if $u \equiv 0$ is unstable)?



- Existence of traveling wave solutions connecting 0 and a positive stationary solution $u^*(\cdot)$ (if $u \equiv 0$ is unstable and $u^*(\cdot)$ exists)



1. Introduction

The problems are well understood in the random dispersal case

$$\frac{\partial u}{\partial t} = \nu \Delta u + uf(x,u), \quad x \in \mathbb{R}^N \qquad (1)'$$

$f_u(x,u) < 0$ for $u > 0$, $f(x,u) < 0$ for $u \gg 1$

$u \equiv 0$ is linearly unstable

$\Longrightarrow$

- $\exists !$ spatially periodic positive stationary solution $u = u^*(x)$
- $\forall \xi \in S^{N-1}$, $\exists$ a spreading speed $c^*(\xi)$ in the direction of ξ
- $\forall \xi \in S^{N-1}$, $c \geq c^*(\xi)$, $\exists$ a traveling wave solution $u(t,x)$ propagating in the direction of ξ with speed c and connecting $u^*(\cdot)$ and 0

Fisher (1937), Kolmogorov, Petrowsky, Piscunov (1937), Aronson, Weinberger (1975, 1978), H. Weinberger (1982, 2002), M.A. Lewis, B. Li, H. Weinberger (2002), H. Berestycki, F. Hamel, L. Roques (2004, 2005), J.Nolen, J. Xin (2005), L. Xing, X.-Q. Zhao(2007, 2009), Grgoire Nadin (2009), ...

The problems are not well studied in the nonlocal dispersal case

A basic tool to study the problems:

Spectral theory, in particular, principal eigenvalue theory of nonlocal dispersal operators

$$\begin{cases} \nu[\int_{\mathbb{R}^N} \kappa(y-x)v(y)dy - v(x)] + a(x)v(x) = \lambda v(x) \\ v(x+p_i\mathbf{e_i}) = v(x) \end{cases} \tag{2}$$

$a(x+p_i\mathbf{e_i}) = a(x)$

If $a(x) \equiv$constant, by the Krein-Rutman theorem, (2) has a principal eigenvalue

However, in general, principal eigenvalue theory of nonlocal dispersal operators needs to be developed

Definition 2.1.

$$X_p = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u(x + p_i \mathbf{e_i}) = u(x)\}$$

Consider

$$\nu[\int_{\mathbb{R}^N} \kappa(y-x)v(y)dy - v(x)] + a(x)v(x) = \lambda v(x), \quad v \in X_p$$

or

$$\nu[\mathcal{K} - I]v + a(\cdot)v = \lambda v, \quad v \in X_p \qquad (EV)$$

$\mathcal{K}v = \int_{\mathbb{R}^N} \kappa(y-x)v(y)dy$, $a(\cdot) \in X_p$
$\sigma(\nu[\mathcal{K} - I] + a(\cdot)I)$ be the spectrum of $\nu[\mathcal{K} - I] + a(\cdot)I$
$\lambda(\nu, a) \in \mathbb{R}$ is called a **principal eigenvalue** of (EV) if
$\lambda(\nu, a)$ is an algebraically simple eigenvalue of $\nu[\mathcal{K} - I] + a(\cdot)I$ with a positive eigenfunction $\phi \in X_p$
and for any $\mu \in \sigma(\nu[\mathcal{K} - I] + a(\cdot)I)$, $\mathrm{Re}(\mu) < \lambda(\nu, a)$

2. Principal Eigenvalue of Nonlocal Dispersal Operators

Question:

- Does $\lambda(\nu, a)$ exists?

  $\lambda_0(\nu, a) = \max\{\mathrm{Re}\mu \,|\, \mu \in \sigma(\nu[\mathcal{K} - I] + a(\cdot)I)\}$.

  $\lambda_0(\nu, a) \in \sigma(\nu[\mathcal{K} - I] + a(\cdot)I)$

  $\lambda(\nu, a) = \lambda_0(\nu, a)$ if $\lambda(\nu, a)$ exists

- Is $\lambda_0(\nu, a)$ the principal eigenvalue (P.E.) of $\nu[\mathcal{K} - I] + a(\cdot)I$?

Theorem 2.1 (Necessary and sufficient conditions).

$a(\cdot) \in X_p$

$a_{\max} = \max_{x \in \mathbb{R}^N} a(x)$

$a_{\min} = \min_{x \in \mathbb{R}^N} a(x)$

- P. E. $\lambda(\nu, a)$ of $\nu[\mathcal{K} - I] + a(\cdot)I$ exists

  or $\lambda_0(\nu, a)$ is the P. E. of $\nu[\mathcal{K} - I] + a(\cdot)I$

  $\Longleftrightarrow$

  $\lambda_0(\nu, a) > -\nu + a_{\max}$

Theorem 2.2 (Sufficient conditions).

(1) If $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$, where $\mathrm{supp}(\tilde{\kappa}) = \{z \mid \|z\| < 1\}$, then $\exists \delta_0 > 0$ s. t. the P. E. $\lambda(\nu, a)$ of $\nu[\mathcal{K} - I] + a(\cdot)I$ exists for all $0 < \delta < \delta_0$.

(2) If $a_{\max} - a_{\min} < \nu$, then the P. E. $\lambda(\nu, a)$ of $\nu[\mathcal{K} - I] + a(\cdot)I$ exists.

(3) If $a(\cdot)$ is C^N and all the partial derivatives of $a(x)$ up to order $N-1$ at x_0 are zero, where $a(x_0) = a_{\max}$, then the P. E. $\lambda(\nu, a)$ of $\nu[\mathcal{K} - I] + a(\cdot)I$ exists for all $\delta > 0$.

2. Principal Eigenvalues of Nonlocal Dispersal Operators

Biological interpretation

The P. E. of $\nu[K_\delta - I] + a(\cdot)I$ exists in the following cases

- the nonlocal dispersal is "nearly" local ($\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$ and $0 < \delta \ll 1$)
- the periodic habitat is "nearly globally" homogeneous (i.e. $a_{\max} - a_{\min} < \nu$)
- the periodic habitat is "nearly" homogeneous in a region where it is most conducive to the population growth (i.e. the partial derivatives of $a(x)$ up to order $N-1$ are zero at some x_0 with $a(x_0) = a_{\max}$, **which is always satisfied when** $a(\cdot)$ **is** C^1 **and** $N = 1$ **or** 2)

Remarks.

- If δ is not small and the periodic habitat is not of the homogeneity mentioned above, the principal eigenvalue of $\nu[K_\delta - I] + a(\cdot)I$ may not exist

  which reveals some **essential difference between local and nonlocal dispersal operators**
- For any $a(\cdot) \in X_p$, $\exists a_n(\cdot) \in X_p$, which are C^N and "nearly" homogeneous in a region where it is most conducive to the population growth, such that $a_n(x) \to a(x)$ as $n \to \infty$ in X_p.
- If $a_n(\cdot), a(\cdot) \in X_p$ and $a_n(x) \to a(x)$ in X_p, then $\lambda_0(a_n) \to \lambda_0(a)$.
- A similar result as Theorem 2.2 (3) is obtained by J. Coville (2010)
- V. Hutson, S. Martinez, K. Mischaikow, and G. T. Vickers (2003) obtained the existence of P. E. for the case $N = 1$

Theorem 2.3 (Effects of spatial variations).

$$a(\cdot) \in X_p,\ a(x + p_i\mathbf{e_i}) = a(x)$$

$$\bar{a} = \frac{1}{p_1 p_2 \cdots p_N} \int_0^{p_1} \int_0^{p_2} \cdots \int_0^{p_N} a(x)dx$$

- $\lambda_0(\nu, a) \geq \lambda_0(\nu, \bar{a}) (= \bar{a})$
- $\lambda_0(\nu, a) = \lambda_0(\nu, \bar{a}) \iff a(x) \equiv \bar{a}$

Theorem 2.4 (Effects of dispersal rates).

$\kappa(z) = \kappa(-z)$

- $\nu_1 < \nu_2 \Longrightarrow \lambda_0(\nu_1, a) > \lambda_0(\nu_2, a)$
- $\lambda_0(\nu_0, a)$ is P. E. $\Longrightarrow \lambda_0(\nu, a)$ is P. E. for $\nu \geq \nu_0$
- $\lambda_0(\nu, a)$ is P. E. for $\nu \gg 1$

2. Principal Eigenvalues of Nonlocal Dispersal Operators

Problems.

- $\nu_1 < \nu_2 \Longrightarrow \lambda_0(\nu_1, a) > \lambda_0(\nu_2, a)$ for general $\kappa(\cdot)$?
- $a^*(x)$ is the (Schwarz) Steiner periodic rearrangement $\Longrightarrow$ $\lambda_0(\nu, a^*) \geq \lambda_0(\nu, a)$?
- If $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$, put $\lambda_0(\delta, \nu, a) = \lambda_0(\nu, a)$

  $\delta_1 < \delta_2 \Longrightarrow \lambda_0(\delta_1, \nu, a) \geq \lambda_0(\delta_2, \nu, a)$?

  $\lambda_0(\delta_0, \nu, a)$ is the P.E. $\Longrightarrow \lambda_0(\delta, \nu, a)$ is the P.E. for $\delta < \delta_0$?

2. Principal Eigenvalues of Nonlocal Dispersal Operators

Problems.

• Principal eigenvalue theory for nonlocal dispersal operators with "Dirichlet type" boundary condition:

$$\nu\Big[\int_D \kappa(y-x)v(y)dy - v(x)\Big] + a(x)v(x) = \lambda v(x),\ \ x \in \bar{D}$$

C. Cortazar, M. Elgueta, and J. D. Rossi (2009) obtained some relation between

$$v \to \int_D \kappa(y-x)v(y)dy - v(x)$$

and

$$v \to \Delta v \quad \text{with} \quad v = 0 \quad \text{on} \quad \partial D$$

2. Principal Eigenvalues of Nonlocal Dispersal Operators

Problems.

• Principal eigenvalue theory for nonlocal dispersal operators with "Neumann type" boundary condition:

$$\nu \int_D \kappa(y-x)[v(y)-v(x)]dy + a(x)v(x) = \lambda v(x), \ x \in \bar{D}$$

C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski (2007) obtained some relation between

$$u \to \int_D \kappa(y-x)(v(y)-v(x))dy$$

and

$$v \to \Delta v \quad \text{with} \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial D$$

3. Spatially Periodic Stationary Solutions of KPP Equations

Monostablility assumptions

$$\frac{\partial u}{\partial t} = \nu\left[\int_{\mathbb{R}^N} \left[\kappa(y-x)(u(t,y)-u(t,x))dy\right] + uf(x,u) \qquad (1)$$

$f(x+p_i\mathbf{e_i},u) = f(x,u),\ p_i > 0\ (i = 1,2,\cdots,N$

(H1) $u \equiv 0$ is linearly unstable, i.e., $\lambda_0(\nu, f(\cdot,0)) > 0$

(H2) $f_u(x,u) < 0$ for $x \in \mathbb{R}^N$, $u \geq 0$, $f(x,u) < 0$ for $x \in \mathbb{R}^N$, $u \gg 1$ (e.g. $f(x,u) = r(x) - u$)

3. Spatially Periodic Stationary Solutions of KPP Equations

Basic properties

$$\frac{\partial u}{\partial t} = \nu[\int_{\mathbb{R}^N} \left[\kappa(y-x)(u(t,y)-u(t,x))dy\right] + uf(x,u) \qquad (1)$$

$X = C^b_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R})$

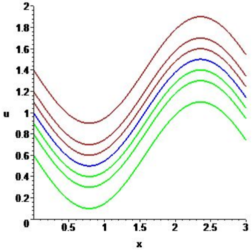
$\forall u_0 \in X$, (1) has a unique (local) solution $u(t,x;u_0)(\in X)$ with $u(0,x;u_0) = u_0(x)$

If $u_0 \geq 0$, then $u(t,x;u_0)$ exists for all $t \geq 0$ and $u(t,x;u_0) \geq 0$ for $t \geq 0$.

If $u_0 \in X_p$ and $u_0 \geq 0$, then $u(t,\cdot;u_0) \in X_p$ for $t \geq 0$

3. Spatially Periodic Stationary Solutions of KPP Equations

Theorem 3.1. Assume (H1) and (H2). There is a unique spatially periodic positive stationary solution $u = u^*(x)$ of (1) which is asymptotically stable with respect to any $u_0 \in X_p$ with $u_0(x) \geq 0$, $u_0(x) \not\equiv 0$.



Works on the existence and uniqueness of $u^*(\cdot)$: V. Hutson, S. Martinez, K. Mischaikow, G. T. Vickers (2004); P.W. Bates and Guanyu Zhao (2007); C.-Y. Kao, Y. Lou, and W. Shen (2010); J. Coville (2010)

Idea of proof.

(H1) $\Longrightarrow \exists\ a_0(\cdot) \in X_p$ with $a_0(x) \le f(x,0)$, $\lambda_0(\nu, a_0) > 0$ and $\lambda_0(\nu, a_0)$ is the principal eigenvalue of $\nu[\mathcal{K} - I] + a_0(\cdot)I$.

Let $\phi_0(\cdot) \in X_p$ be a positive principal eigenfunction of $\nu[\mathcal{K} - I] + a_0(\cdot)I$ and $0 < \epsilon \ll 1$.
Then $u = \epsilon\phi_0$ is a subsolution of (1) $\Longrightarrow u(t,\cdot;\epsilon\phi_0)$ increases as t increases

(H2) $\Longrightarrow u \equiv M$ is a supersolution of (1) for $M \gg 1 \Longrightarrow$
$u(t,\cdot;M)$ decreases increases as t increases
$\Longrightarrow u^*(x) := \lim_{t\to\infty} u(t,x;M) \ge u_*(x) := \lim_{t\to\infty} u(t,x;\epsilon\phi_0) \ge \epsilon\phi_0(x) > 0$

Prove $u^*(x) = u_*(x)$ and $u^* \in X_p$.

3. Spatially Periodic Stationary Solutions of KPP Equations

Remarks. Let $\bar{f}(u) := \frac{1}{p_1 p_2 \cdots p_N} \int_0^{p_1} \int_0^{p_2} \cdots \int_0^{p_N} f(x,u)dx$

- $\bar{f}(0) > 0 \Longrightarrow$ (H1), i.e., $u \equiv 0$ is unstable
- $u \equiv 0$ is linearly unstable solution of

$$\frac{\partial u}{\partial t} = \nu[\int_D \kappa(y-x)u(y)dy - u(x)] + u\bar{f}(u)$$

  $\Longrightarrow$

  $u \equiv 0$ is linearly unstable solution of (1), but not the viceversa
  Hence **spatial variation favors the population persistence**
- Theorem 3.1 requires $\lambda_0(\nu, f(\cdot,0)) > 0$, but it is not necessary
  $\lambda_0(\nu, f(\cdot,0))$ is the P. E. of $\nu[\mathcal{K} - I] + f(\cdot,0)I$

4. Spatial Spreading Speeds of KPP Equations

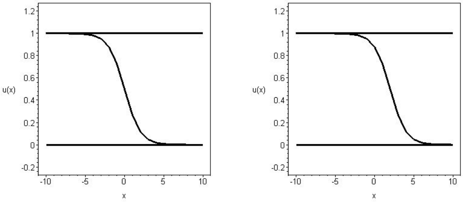
Definition 4.1. Assume (H1) and (H2). Given $\xi \in S^{N-1}$, let

$$X^+(\xi) = \{u \in X \mid u \geq 0, \ \liminf_{x\cdot\xi \to -\infty} u(x) > 0, \ u(x) = 0 \text{ for } x\cdot\xi \gg 1\}$$

$c^*(\xi) \in \mathbb{R}$ is called the **spreading speed** of (1) in the direction of ξ if for any $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi \geq c' t, t\to\infty} u(t,x;u_0) = 0 \quad \forall c' > c^*(\xi)$$

$$\liminf_{x\cdot\xi \leq c'' t, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0 \quad \forall c'' < c^*(\xi)$$



Theorem 4.1. Assume (H1) and (H2).

(1) For given $\xi \in S^{N-1}$, $c^*(\xi)$ exists

(2)

$$c^*(\xi) = \inf_{\mu>0} \frac{\lambda_0(\nu, a, \xi, \mu)}{\mu},$$

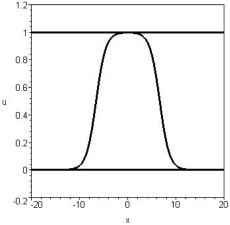
$\lambda_0(\nu, a, \xi, \mu) = \max\{\mathrm{Re}\mu \,|\, \mu \in \sigma(\nu[\mathcal{K}_{\xi,\mu} - I] + a(\cdot)I),$

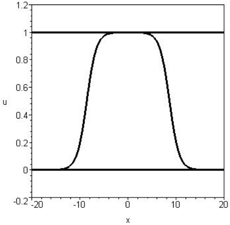
$(\mathcal{K}_{\xi,\mu} u)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} \kappa(y-x)u(y)dy$ for $u \in X_p$,

$a(x) = f(x, 0)$

(3) If $\kappa(z) = \kappa(-z)$, $c^*(\xi) = c^*(-\xi)$

(4) For any $u_0 \in X$ with $u_0 \geq 0$, $u_0(x) > 0$ for $\|x\| = O(1)$, $u_0(x) = 0$ for $\|x\| \gg 1$,





$\lim_{\|x\| \geq c' t, t \to \infty} u(t, x; u_0) = 0$ if $c^{'} > \sup_{\xi \in S^{N-1}} c^*(\xi)$

$\lim_{\|x\| \leq c'' t, t \to \infty} [u(t, x; u_0) - u^*(x)] = 0$ if $0 < c^{''} < \inf_{\xi \in S^{N-1}} c^*(\xi)$

(5)

$$c^*(\xi) \geq \hat{c}^*(\xi) \quad \forall \xi \in S^{N-1}$$

and $c^*(\xi) = \hat{c}^*(\xi)$ for some $\xi \in S^{N-1}$ iff $f(x,0) \equiv \hat{f}(0)$ (provided $\hat{f}(0) > 0$), $\hat{c}^*(\xi)$ be the spreading speed of

$$u_t = \nu[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + u\hat{f}(u)$$

in the direction of $\xi \in S^{N-1}$,
$\hat{f}(u) = \frac{1}{p_1 p_2 \cdots p_N} \int_0^{p_1} \int_0^{p_2} \cdots \int_0^{p_N} f(x,u)dx$

Spatial variation speeds up the spatial spreading!

4. Spatial Spreading Speeds of KPP Equations

(6) Write $c^*(\xi)$ as $c^*(\nu, \xi)$ to indicate the dependence of the spreading speed on the dispersal rate ν

$f(x, u) \equiv f(u) \Longrightarrow c^*(\nu, \xi)$ increases as ν increases

(7) If $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$, write $c^*(\xi)$ as $c^*(\delta, \xi)$ to indicate the dependence of the spreading speed on the dispersal distance δ

$f(x, u) \equiv f(u) \Longrightarrow c^*(\delta, \xi)$ increases as δ increases

Remarks and problems.

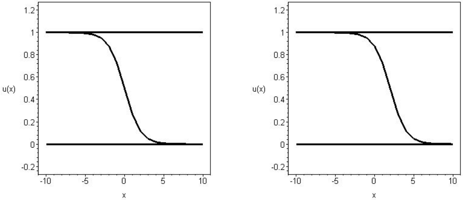
- Whether $c^*(\nu, \xi)$ increases as ν increases in general?
- Whether $c^*(\delta, \xi)$ increases as δ increases in general?
- Theorem 4.1 requires $\lambda_0(\nu, f(\cdot, 0)) > 0$, but it is not necessary $\lambda_0(\nu, f(\cdot, 0))$ is the P. E. of $\nu[\mathcal{K} - I] + f(\cdot, 0)I$

Definition 5.1. Assume (H1) and (H2).

A solution $u(t,x)$ is called a **traveling wave solution** of (1) in the direction of $\xi \in S^{N-1}$ with speed c if

$$u(t,x) = \phi(x - ct\xi, ct\xi)$$

for some $\phi(x,z)$ satisfying that $\phi(x,z) \geq 0$, $\phi(\cdot,z) \in X$, $\phi(x,\cdot) \in X_p$,
$\lim_{x\cdot\xi\to-\infty}[\phi(x,z) - u^*(x+z)] = 0$, $\lim_{x\cdot\xi\to\infty} \phi(x,z) = 0$



Equivalent definition

Assume $u(t,x)=\phi(x-ct\xi, ct\xi)$ is a T. W. solution.
Let $\psi(x,z)=\phi(x,z-x)$. Then
$u(t,x)=\psi(x-ct\xi, x)$
$\psi(x,\cdot)\in X_p,\ \psi(x,z)\geq 0$
$\lim_{x\cdot\xi\to-\infty}[\psi(x,z)-u^*(z)]=0,\ \lim_{x\cdot\xi\to\infty}\psi(x,z)=0$

We can also define a T. W. solution to be a solution of the form $u(t,x)=\psi(x-ct\xi, x)$
$\psi(x,\cdot)\in X_p,\ \psi(x,z)\geq 0$
$\lim_{x\cdot\xi\to-\infty}[\psi(x,z)-u^*(z)]=0,\ \lim_{x\cdot\xi\to\infty}\psi(x,z)=0$

Theorem 5.1. Assume (H1) and (H2). Additionally, assume $\mathrm{supp}(\kappa)$ is compact and $\lambda_0(\nu, f(\cdot, 0), \mu, \xi)$ is P.E.

(1) (Nonexistence) For given $\xi \in S^{N-1}$, there is no T. W. in the direction of ξ with speed $c < c^*(\xi)$.

(2) (Existence) For any $\xi \in S^{N-1}$ and $c > c^*(\xi)$, $\exists \phi(\cdot, \cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^+)$ such that $u(t, x) = \phi(x - ct\xi, ct\xi)$ is a T. W. of (1).

(3) (Uniqueness) For given $\xi \in S^{N-1}$ and $c > c^*(\xi)$, if $u = \tilde{\phi}(x - ct\xi, ct\xi)$ is also a traveling wave solution of (1) and $\lim_{x\cdot\xi\to\infty} \frac{\tilde{\phi}(x,z)}{\phi(x,z)} = 1$, then $\tilde{\phi}(x, z) \equiv \phi(x, z)$.

(4) (Stability) For given $\xi \in S^{N-1}$, $c > c^*(\xi)$, and $u_0 \in C^b_{\rm unif}(\mathbb{R}^N, \mathbb{R}^+)$ with $\liminf_{x\cdot\xi\to-\infty} u_0(x) > 0$ and $\lim_{x\cdot\xi\to\infty} \frac{u_0(x)}{\phi(x,0)} = 1$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \Big| \frac{u(t,x,u_0)}{\phi(x-ct\xi, ct\xi)} - 1 \Big| = 0.$$

Remarks and problems

- $\mathrm{supp}(\kappa)$ is compact is a technical assumption
- In a recent work of J. Coville, J. Dávila, S. Martínez, under the same conditions of Theorem 5.1, the authors showed the existence of T. W. for any $c \geq c^*(\xi)$
- The uniqueness and stability of T. W. in the direction of ξ with speed $c = c^*(\xi)$ have not been studied
- It is open whether T. W. exists if $\lambda_0(\nu, f(\cdot, 0), \xi, \mu)$ is not P. E.

6. Other Related Works

KPP equations in locally spatially inhomogeneous media

$$\frac{\partial u}{\partial t} = \nu[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + uf(x,u), \quad x \in \mathbb{R}^N \quad (3)$$

$f_u(x,u) < 0$ for $u \geq 0$, $f(x,u) < 0$ for $u \gg 1$
$f(x,u) = f_0(u)$ for $\|x\| \gg 1$
$f_0(0) > 0$ (hence $\exists! u_0^* > 0$ s. t. $f_0(u_0^*) = 0$)
$\Longrightarrow$

• (3) has a unique positive stationary solution $u^*(\cdot)$ with

$$\lim_{\|x\| \to \infty} u^*(x) = u_0^*$$

• $\forall \xi \in S^{N-1}$, (3) has a spreading speed $c^*(\xi)$ in the direction of ξ and $c^*(\xi) = c_0^*(\xi)$, $c_0^*(\xi)$ is the spreading speed of

$$\frac{\partial u}{\partial t} = \nu[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + uf_0(u), \quad x \in \mathbb{R}^N$$

6. Other Related Works

Competition system with nonlocal dispersal

$$\begin{cases} \frac{\partial u}{\partial t} = \nu_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] \\ \qquad\qquad\qquad + u(a_1(x) - b_1(x)u - c_1(x)v), \quad x \in \mathbb{R}^N \\ \\ \frac{\partial v}{\partial t} = \nu_2[\int_D \kappa(y-x)v(t,y)dy - v(t,x)] \\ \qquad\qquad\qquad + v(a_2(x) - b_2(x)u - c_2(x)v), \quad x \in \mathbb{R}^N \end{cases} \tag{4}$$

$a_j(x + p_i\mathbf{e_i}) = a_i(x)$, $b_j(x + p_i\mathbf{e_i}) = b_j(x)$, $c_j(x + p_i\mathbf{e_i}) = c_j(x)$

Consider (4) in $X_p \times X_p$

- Which species can invade when rare?
- When both species can coexist?

(Georg Hetzer, Tung Nguyen, Wenxian Shen)

6. Other Related Works

Random dispersal vs nonlocal dispersal

$$\begin{cases} \frac{\partial u}{\partial t} = \nu \Delta u + u(a(x) - u - v), \quad x \in \mathbb{R}^N, \\ \\ \frac{\partial v}{\partial t} = \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v + v(a(x) - u - v), \quad x \in \mathbb{R}^N \end{cases} \tag{5}$$

$a(x + p + i\mathbf{e_i}) = a(x)$

Consider (5) in $X_p \times X_p$

• Which species can invade when rare?

(Chiu-Yen Kao, Yuan Lou, and Wenxian Shen)

6. Other Related Works

Evolution of mixed dispersal

$$\begin{cases} \frac{\partial u}{\partial t} = \nu_1 \Big[\tau_1 \Delta u + (1-\tau_1)\mathcal{K}u \Big] + u\left[a(x) - u - v\right], \ x \in \mathbb{R}^N, \\ \frac{\partial v}{\partial t} = \nu_2 \Big[\tau_2 \Delta v + (1-\tau_2)\mathcal{K}v \Big] + v\left[a(x) - u - v\right], \ x \in \mathbb{R}^N, \end{cases} \tag{6}$$

$a(x + p_i \mathbf{e_i}) = a(x)$

Consider (6) in $X_p \times X_p$

- Which species can invade when rare?
- When both species can coexist?

(Chiu-Yen Kao, Yuan Lou, and Wenxian Shen)

THANK YOU!