Stable traveling spots in a planar three-component FitzHugh-Nagumo system

Peter van Heijster
http://math.bu.edu/people/heijster/
BOSTON UNIVERSITY

Collaborator:
Björn Sandstede (Brown University)

Outline

BOSTON UNIVERSITY

- Introduction
- Stationary spots
- Bifurcation to traveling spots:
  - Asymptotic analysis
  - Direct solver
  - AUTO
- Work in progress

Model

BOSTON UNIVERSITY

Generalized FitzHugh-Nagumo Equation:

$$
\begin{aligned}
U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\
\tau V_t &= \Delta V + U - V \\
\theta W_t &= D^2 \Delta W + U - W
\end{aligned}
$$

where $0 < \varepsilon << 1$; $D>1$; $0<\tau,\theta$; α,β,γ are constants.

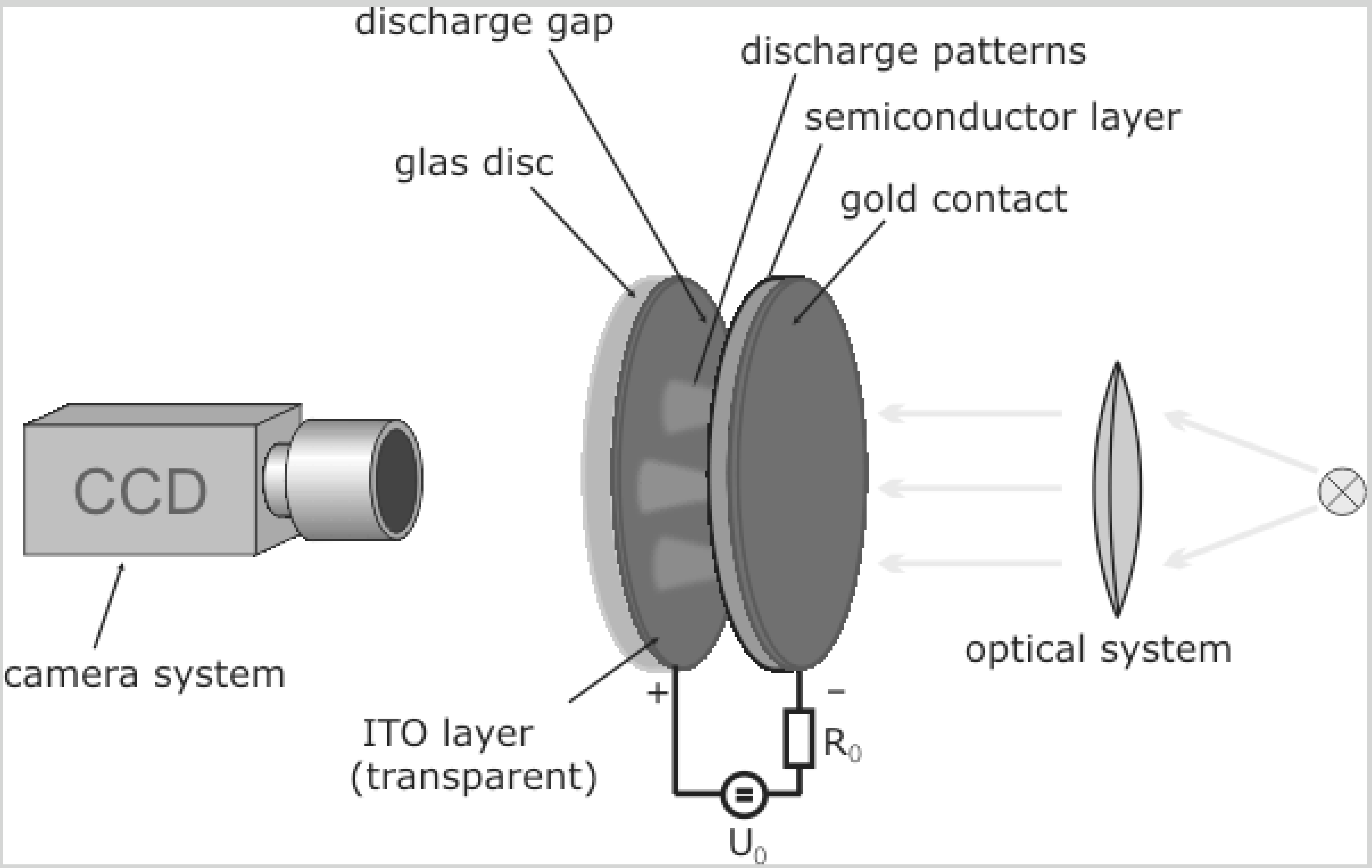
- U: fast component
  - ➡ bistable: $U = \pm 1$
  - ➡ nonlinear: U^3
  - ➡ coupling to the slow components is small

- V,W: slow components
  - ➡ linear
  - ➡ only coupled to the fast component

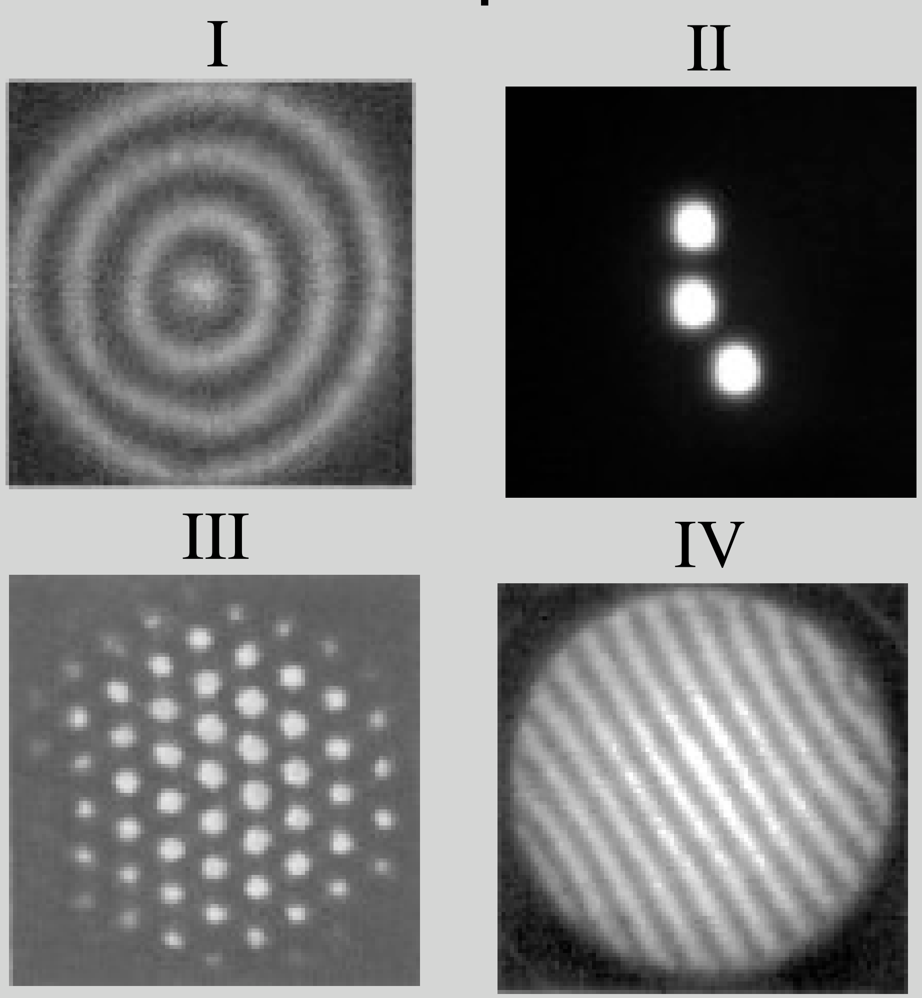
Gas-discharge experiments

BOSTON UNIVERSITY

Set up [Purwins et al.]:



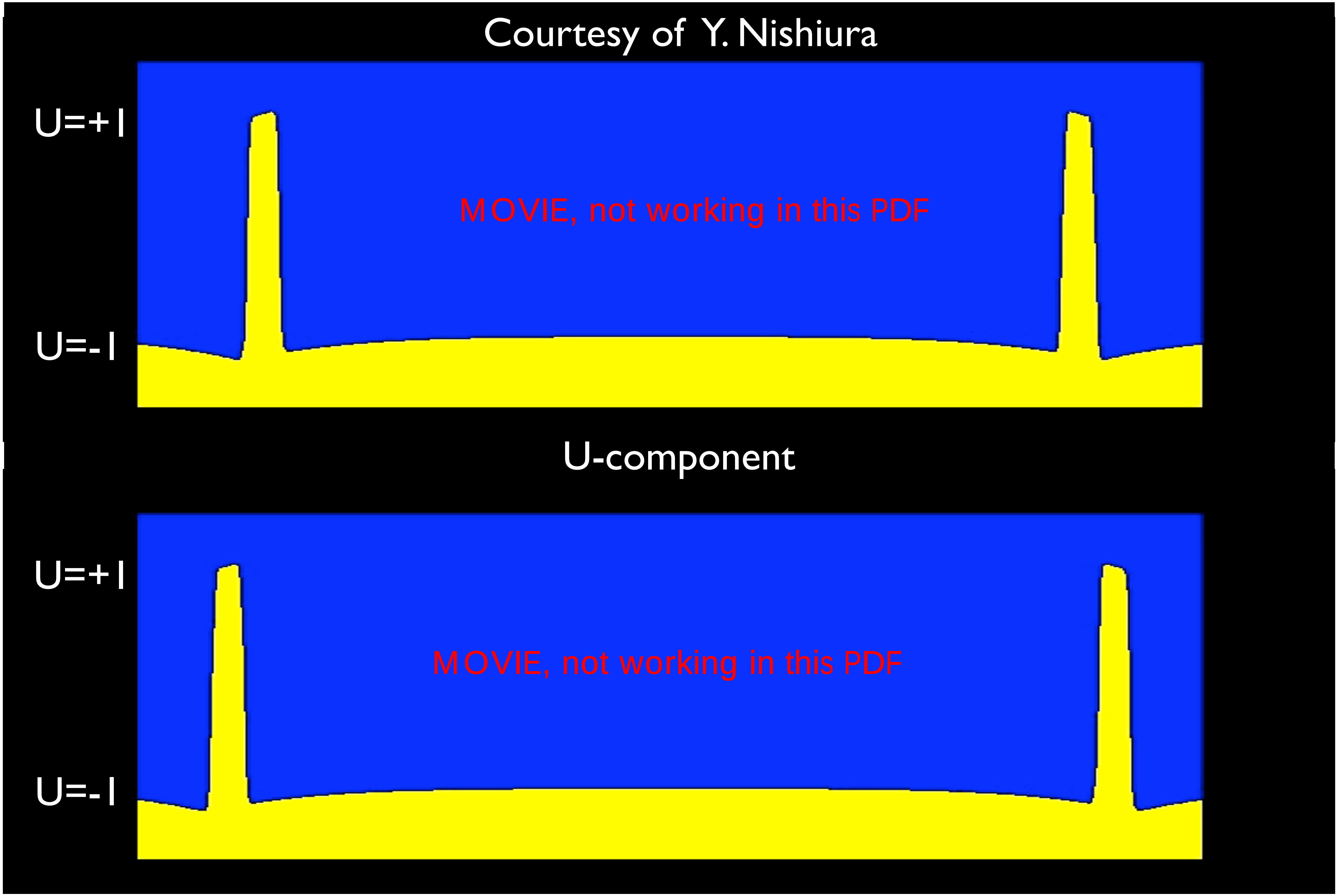
Observed patterns:



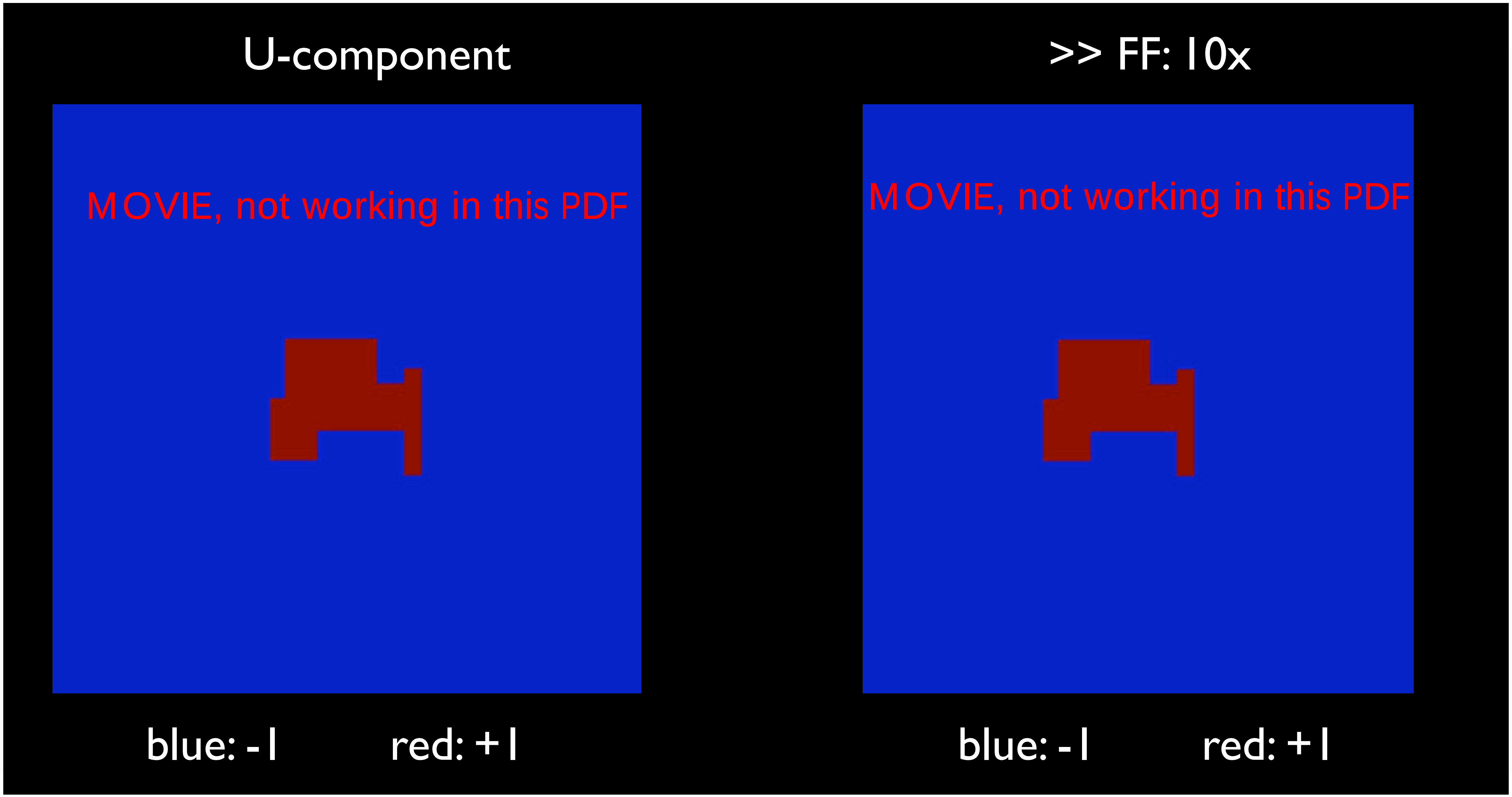
U: current density
V: voltage drop
W: surface charge

black: U=-1, white: U=+1

Inspiration



Stationary spot



$$\begin{aligned} U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\ \tau V_t &= \Delta V + U - V \\ \theta W_t &= D^2 \Delta W + U - W \end{aligned}$$

Theorem

BOSTON UNIVERSITY

Theorem [vH, Sandstede '11]:
Assume that $R_1>0$ solves:

$$\alpha v_0 + \beta w_0 + \gamma = -\frac{\sqrt{2}}{3R_1}$$

where v_0, w_0 are given by

$$v_0 = 1 - 2R_1 K_1(R_1) I_0(R_1)\,, \quad w_0 = 1 - 2\tfrac{R_1}{D} K_1\left(\tfrac{R_1}{D}\right) I_0\left(\tfrac{R_1}{D}\right)$$

Then there exists a stationary radially symmetric spot with radius R_1.

This spot is stable "if and only if" $\lambda(\ell)<0$ for all ℓ = 0,2,3,..., where

$$\begin{aligned} \lambda(\ell) = \; & 3\sqrt{2}\varepsilon^2 \alpha R_1 \left(K_1(R_1) I_1(R_1) - K_\ell(R_1) I_\ell(R_1)\right) + \\ & 3\sqrt{2}\varepsilon^2 \beta \tfrac{R_1}{D^2} \left(K_1\left(\tfrac{R_1}{D}\right) I_1\left(\tfrac{R_1}{D}\right) - K_\ell\left(\tfrac{R_1}{D}\right) I_\ell\left(\tfrac{R_1}{D}\right)\right) + \tfrac{\varepsilon^2}{R_1^2}(1-\ell^2) \end{aligned}$$

$$\begin{aligned} U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\ \tau V_t &= \Delta V + U - V \\ \theta W_t &= D^2 \Delta W + U - W \end{aligned}$$

Theorem

BOSTON UNIVERSITY

Theorem [vH, Sandstede '11]:
Assume that $R_1 > 0$ solves:

$$\alpha v_0 + \beta w_0 + \gamma = -\frac{\sqrt{2}}{3R_1}$$

where v_0, w_0 are given by

$$v_0 = 1 - 2R_1 K_1(R_1) I_0(R_1)\,, \quad w_0 = 1 - 2\frac{R_1}{D} K_1\left(\frac{R_1}{D}\right) I_0\left(\frac{R_1}{D}\right)$$

Then there exists a stationary radially symmetric spot with radius R_1.

This spot is stable "if and only if" $\lambda(\ell) < 0$ for all $\ell = 0,2,3,\ldots$, where

$$\begin{aligned} \lambda(\ell) = \; & 3\sqrt{2}\varepsilon^2 \alpha R_1 \left(K_1(R_1) I_1(R_1) - K_\ell(R_1) I_\ell(R_1)\right) + \\ & 3\sqrt{2}\varepsilon^2 \beta \frac{R_1}{D^2} \left(K_1\left(\frac{R_1}{D}\right) I_1\left(\frac{R_1}{D}\right) - K_\ell\left(\frac{R_1}{D}\right) I_\ell\left(\frac{R_1}{D}\right)\right) + \frac{\varepsilon^2}{R_1^2}(1 - \ell^2) \end{aligned}$$

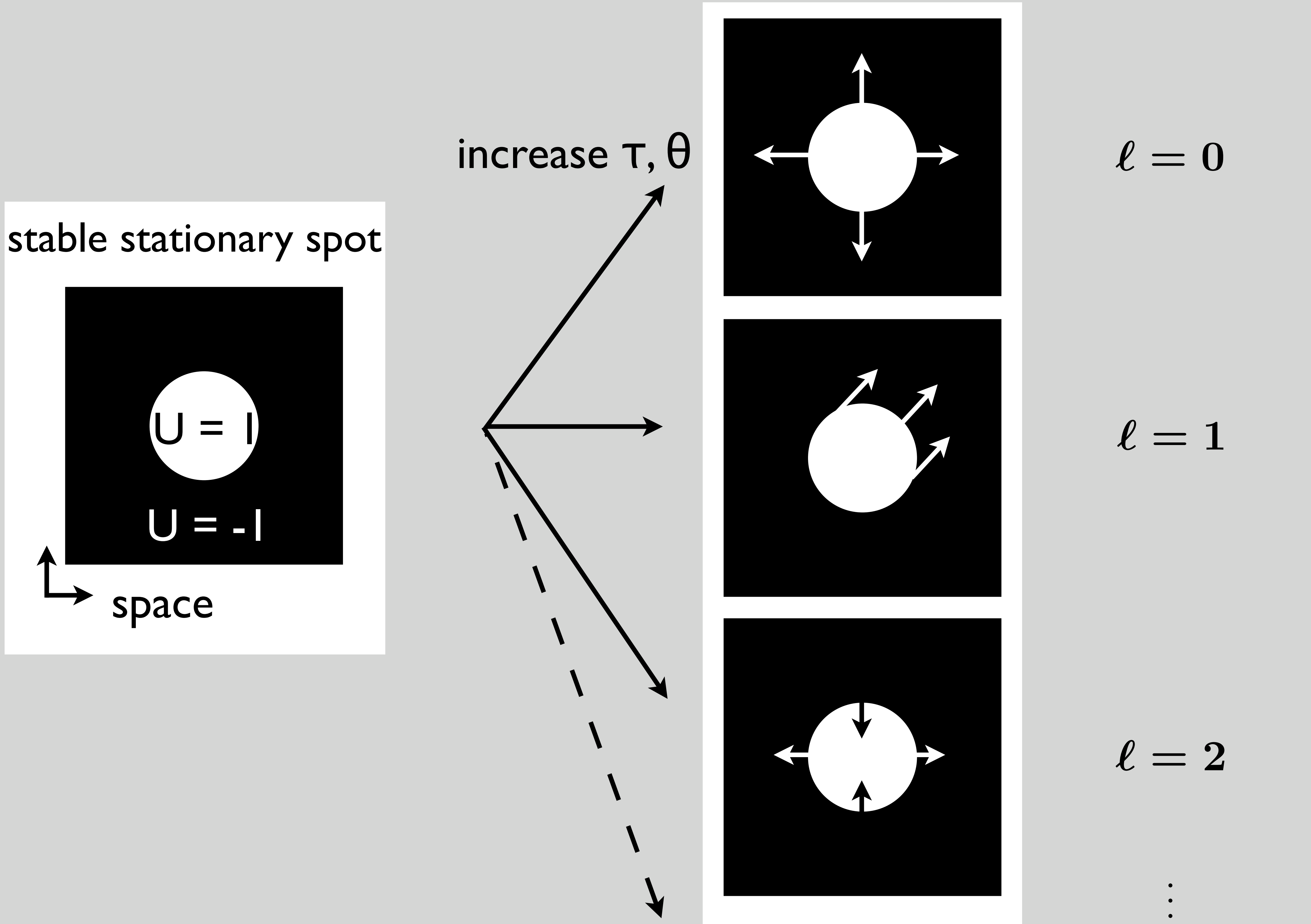
- Spot corresponding to the smallest zero of existence condition is unstable with respect to $\ell = 0$ (radial perturbations)
- $\alpha, \beta \leq 0$: Spot is unstable with respect to $\ell = 0$ (radial perturbations)

$$\begin{aligned} U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\ \tau V_t &= \Delta V + U - V \\ \theta W_t &= D^2 \Delta W + U - W \end{aligned}$$

Bifurcations

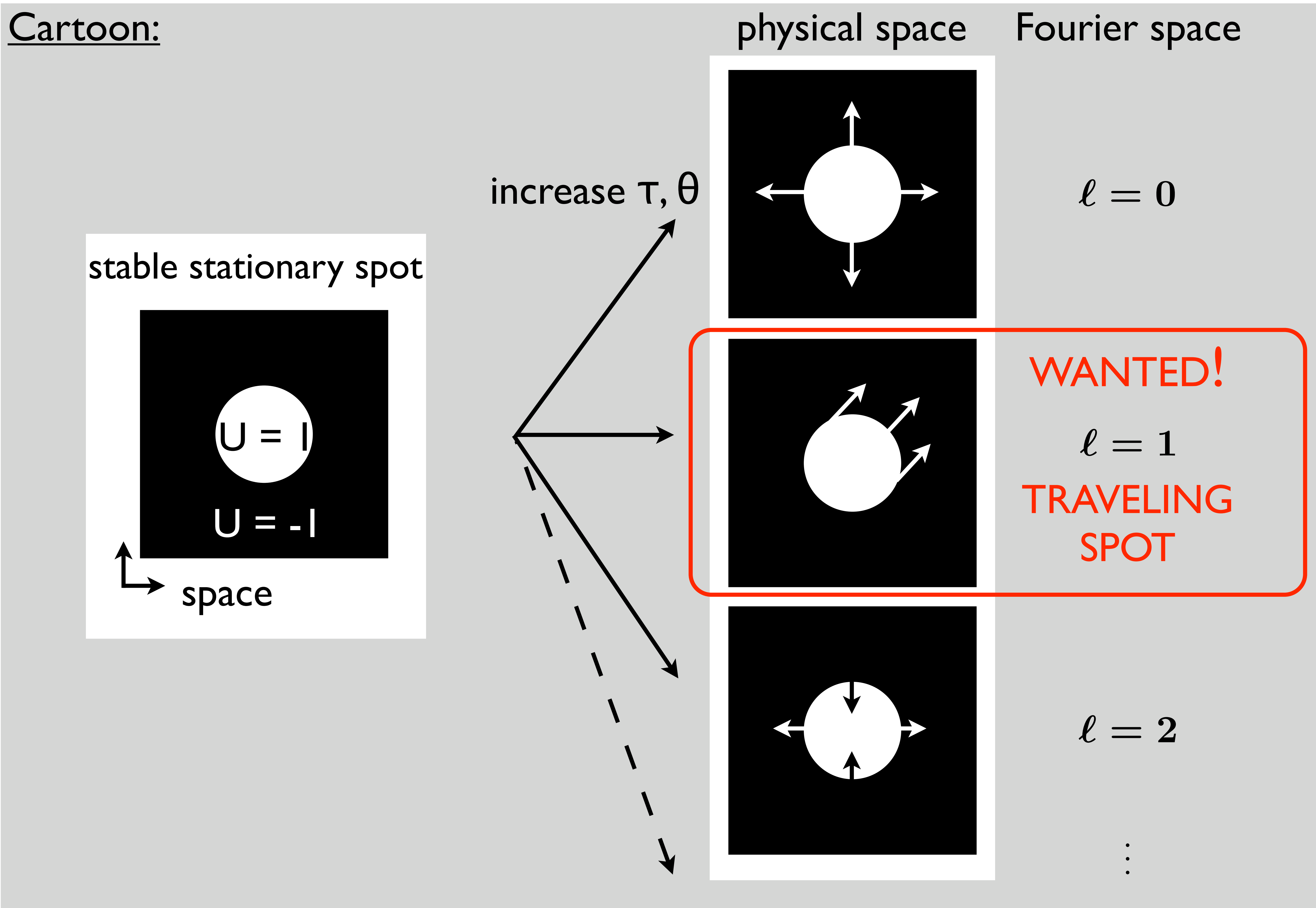
BOSTON UNIVERSITY

Cartoon:



$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$
$$\tau V_t = \Delta V + U - V$$
$$\theta W_t = D^2 \Delta W + U - W$$

Bifurcations



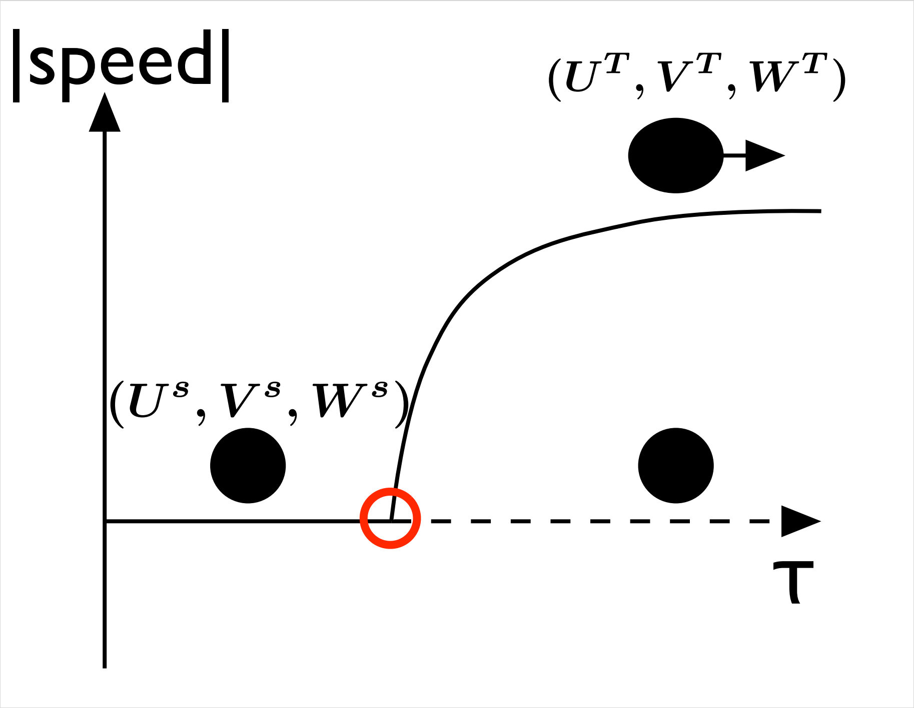
Drift: asymptotics

BOSTON UNIVERSITY

Goal: Determine for arbitrary small ε the points (τ,θ) at which the stationary spot bifurcates to a traveling spot

Method: Weakly nonlinear analysis

Cartoon:



- Speed is small (second small parameter):

$$c = \delta\,, \quad 0 < \varepsilon \ll \delta \ll 1$$

- Traveling spot retains to leading order the shape of the stationary spot:

$$(U^T, V^T, W^T) = (U^s, V^s, W^s) + \delta(u, v, w)$$

- Determine eq[ns] for (u,v,w) and use singular perturbation techniques to derive the drift line

Result: The drift line is given by

$$\frac{\sqrt{2}}{3R_1^2} = \alpha\hat{\tau}\left(I_1(R_1)K_2(R_1) - I_0(R_1)K_1(R_1)\right) + \frac{\beta\hat{\theta}}{D^3}\left(I_1\left(\frac{R_1}{D}\right)K_2\left(\frac{R_1}{D}\right) - I_0\left(\frac{R_1}{D}\right)K_1\left(\frac{R_1}{D}\right)\right)$$

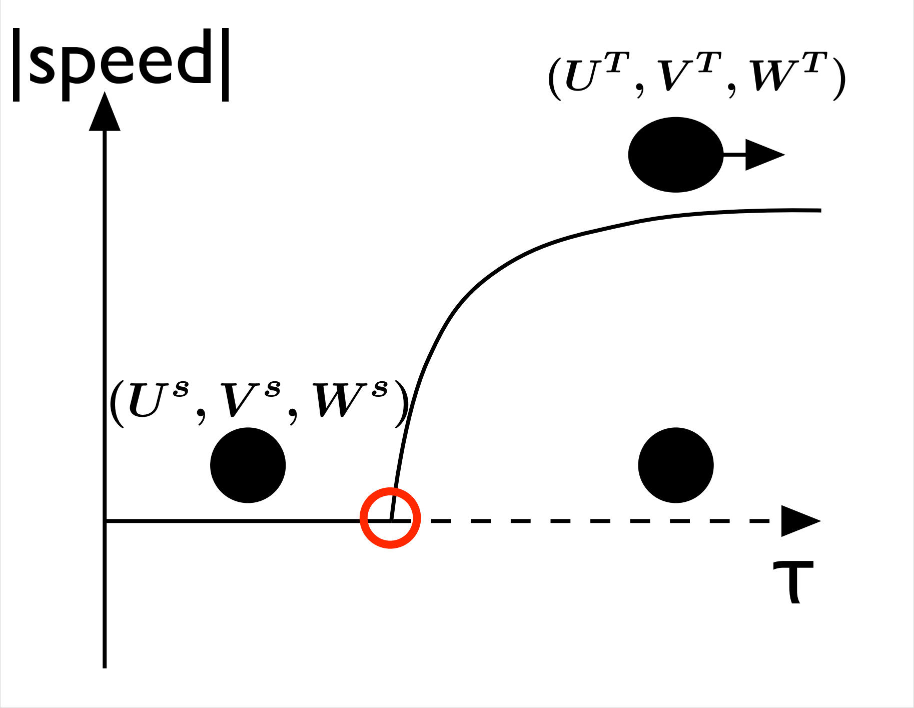
Drift: asymptotics

BOSTON UNIVERSITY

Goal: Determine for arbitrary small ε the points (τ,θ) at which the stationary spot bifurcates to a traveling spot

Method: Weakly nonlinear analysis

Cartoon:



- Speed is small (second small parameter):

$$c = \delta\,, \quad 0 < \varepsilon \ll \delta \ll 1$$

- Traveling spot retains to leading order the shape of the stationary spot:

$$(U^T, V^T, W^T) = (U^s, V^s, W^s) + \delta(u, v, w)$$

- Determine eq^{ns} for (u,v,w) and use singular perturbation techniques to derive the drift line

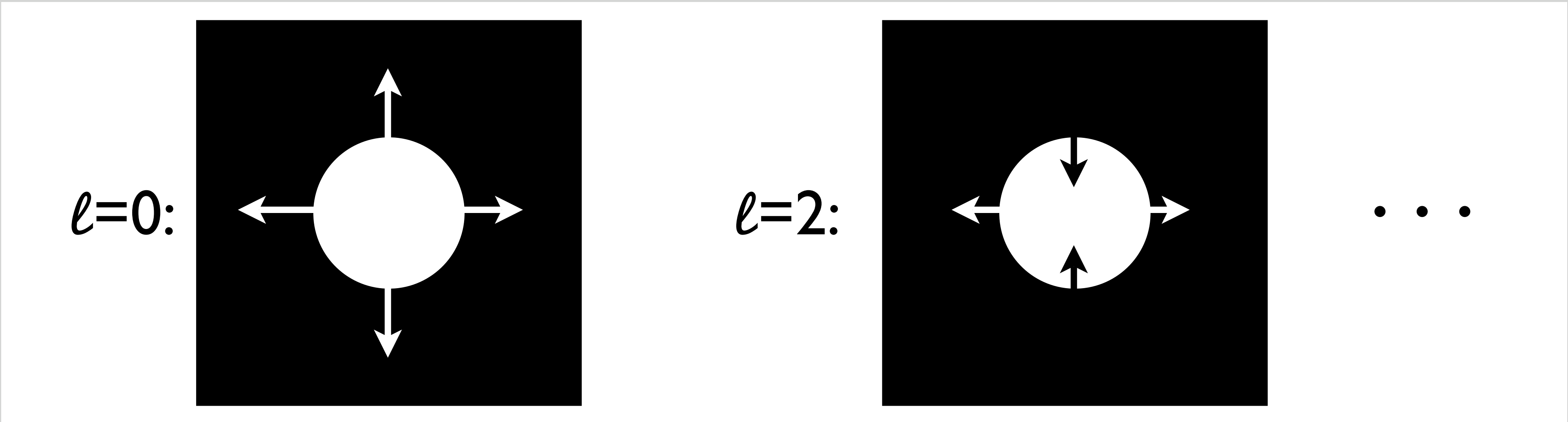
Result: The drift line is given by

$$(\hat{\tau}, \hat{\theta}) = \varepsilon^2(\tau, \theta)$$

$$\frac{\sqrt{2}}{3R_1^2} = \alpha\hat{\tau}\left(I_1(R_1)K_2(R_1) - I_0(R_1)K_1(R_1)\right) + \frac{\beta\hat{\theta}}{D^3}\left(I_1\left(\frac{R_1}{D}\right)K_2\left(\frac{R_1}{D}\right) - I_0\left(\frac{R_1}{D}\right)K_1\left(\frac{R_1}{D}\right)\right)$$

Other bifurcations

BOSTON UNIVERSITY



The other bifurcations (ℓ=0,2,3,...), if present, will be Hopf bifurcations.
These Hopf lines are implicitly given by:

$$
\begin{aligned}
0 &= \frac{1}{R_1^2}(1-\ell^2) + 3\sqrt{2}\alpha R_1 \left(I_1(R_1)K_1(R_1) - \Re\left[I_\ell\left(\sqrt{1+i\hat{\tau}|\hat{\lambda}(\ell)|}R_1\right) K_\ell\left(\sqrt{1+i\hat{\tau}|\hat{\lambda}(\ell)|}R_1\right)\right]\right) \\
&\quad + 3\sqrt{2}\frac{\beta}{D^2}R_1 \left(I_1\left(\frac{R_1}{D}\right) K_1\left(\frac{R_1}{D}\right) - \Re\left[I_\ell\left(\sqrt{1+i\hat{\theta}|\hat{\lambda}(\ell)|}\frac{R_1}{D}\right) K_\ell\left(\sqrt{1+i\hat{\theta}|\hat{\lambda}(\ell)|}\frac{R_1}{D}\right)\right]\right) \\
|\hat{\lambda}(\ell)| &= -3\sqrt{2}\alpha R_1 \left(\Im\left[I_\ell\left(\sqrt{1+i\hat{\tau}|\hat{\lambda}(\ell)|}R_1\right) K_\ell\left(\sqrt{1+i\hat{\tau}|\hat{\lambda}(\ell)|}R_1\right)\right]\right) \\
&\quad - 3\sqrt{2}\frac{\beta}{D^2}R_1 \left(\Im\left[I_\ell\left(\sqrt{1+i\hat{\theta}|\hat{\lambda}(\ell)|}\frac{R_1}{D}\right) K_\ell\left(\sqrt{1+i\hat{\theta}|\hat{\lambda}(\ell)|}\frac{R_1}{D}\right)\right]\right).
\end{aligned}
$$

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$
$$\tau V_t = \Delta V + U - V$$
$$\theta W_t = D^2 \Delta W + U - W$$

Specific parameters

BOSTON UNIVERSITY

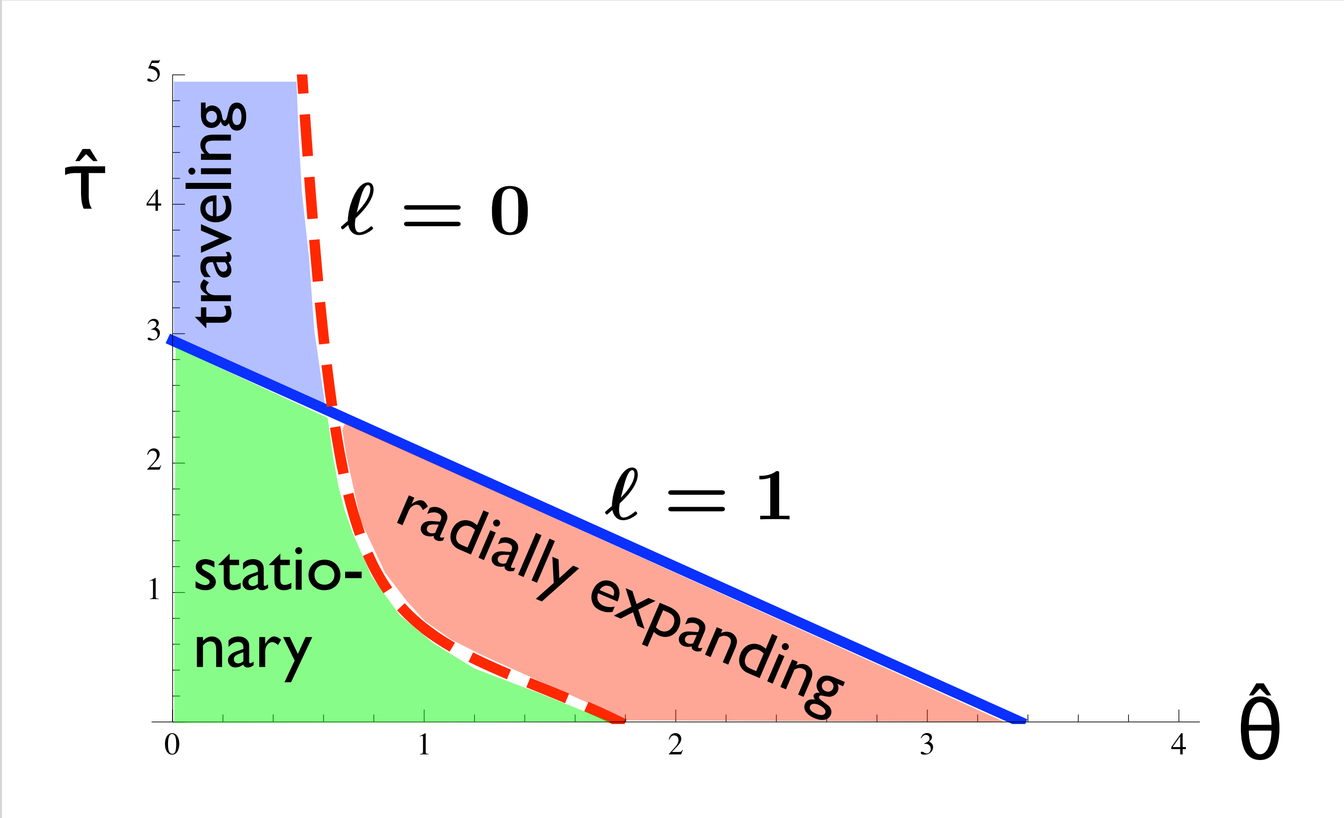
Choose the following set of parameters (for the remainder of presentation):

$$\alpha = 0.5\ ,\ \beta = 2,\ \gamma = 1,\ D = 2$$

Then, there exists a stationary stable spot solution with (leading order) width

$$R_1 = 1.86$$

The bifurcation diagram of this stationary spot looks like:



$$
\begin{aligned}
U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\
\tau V_t &= \Delta V + U - V \\
\theta W_t &= D^2 \Delta W + U - W
\end{aligned}
$$

Specific parameters

BOSTON UNIVERSITY

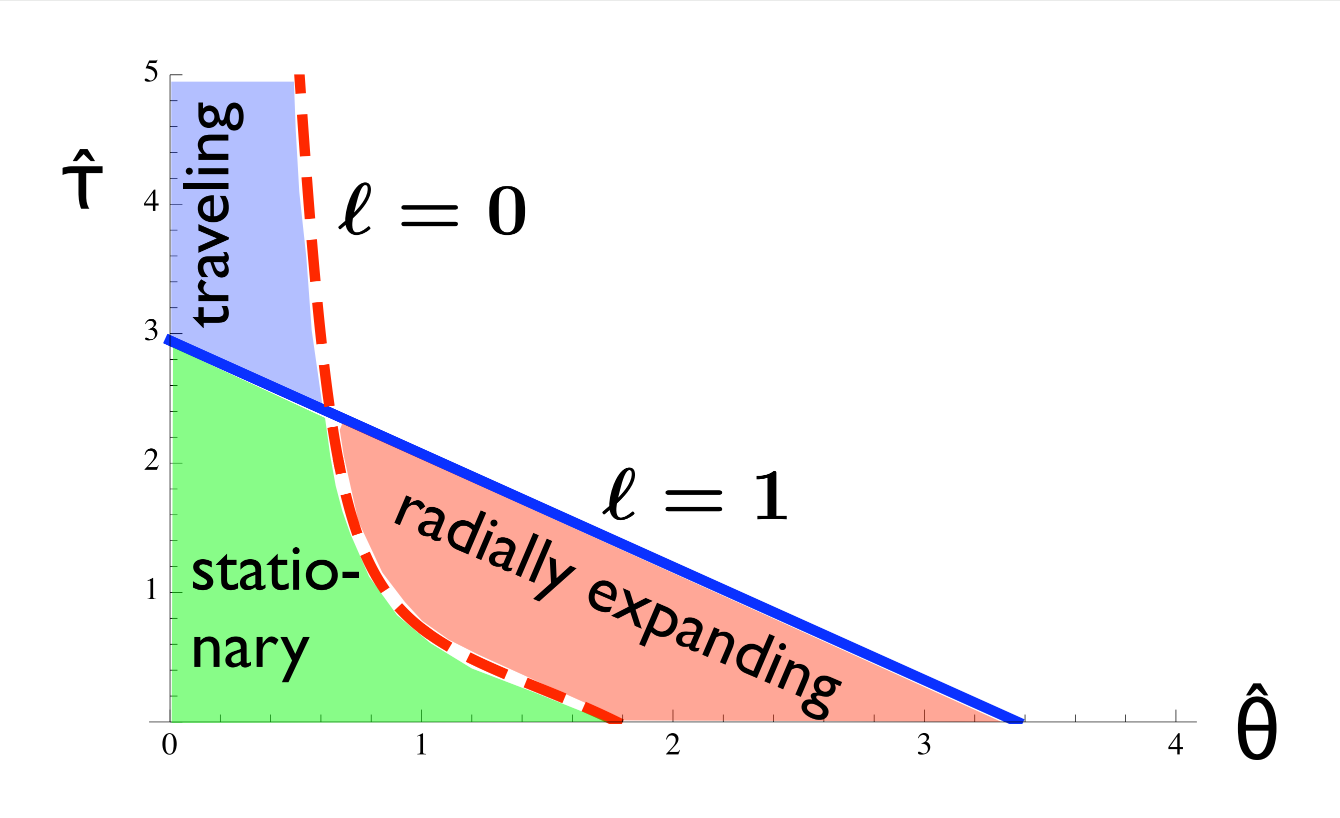
Choose the following set of parameters (for the remainder of presentation):

$$\alpha = 0.5 , \beta = 2, \gamma = 1, D = 2$$

Then, there exists a stationary stable spot solution with (leading order) width

$$R_1 = 1.86$$

The bifurcation diagram of this stationary spot looks like:



Note: We found bifurcations of stable stationary spots to traveling spots only when both α and β are positive. In all the other cases the ℓ=0-line is always below the ℓ=1-line.

$$U_t = \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)$$
$$\tau V_t = \Delta V + U - V$$
$$\theta W_t = D^2 \Delta W + U - W$$

Specific parameters

BOSTON UNIVERSITY

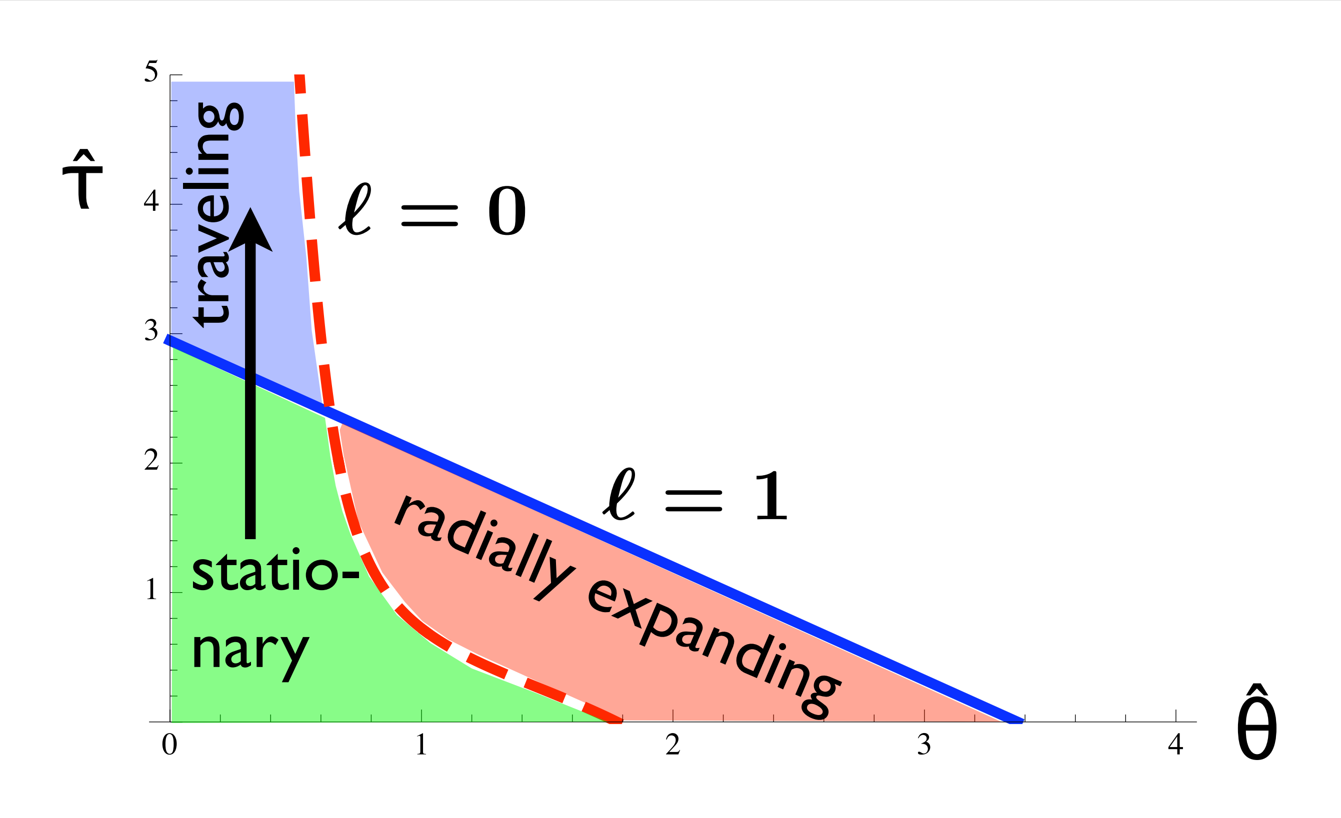
Choose the following set of parameters (for the remainder of presentation):

$$\alpha = 0.5\ ,\ \beta = 2,\ \gamma = 1,\ D = 2$$

Then, there exists a stationary stable spot solution with (leading order) width

$$R_1 = 1.86$$

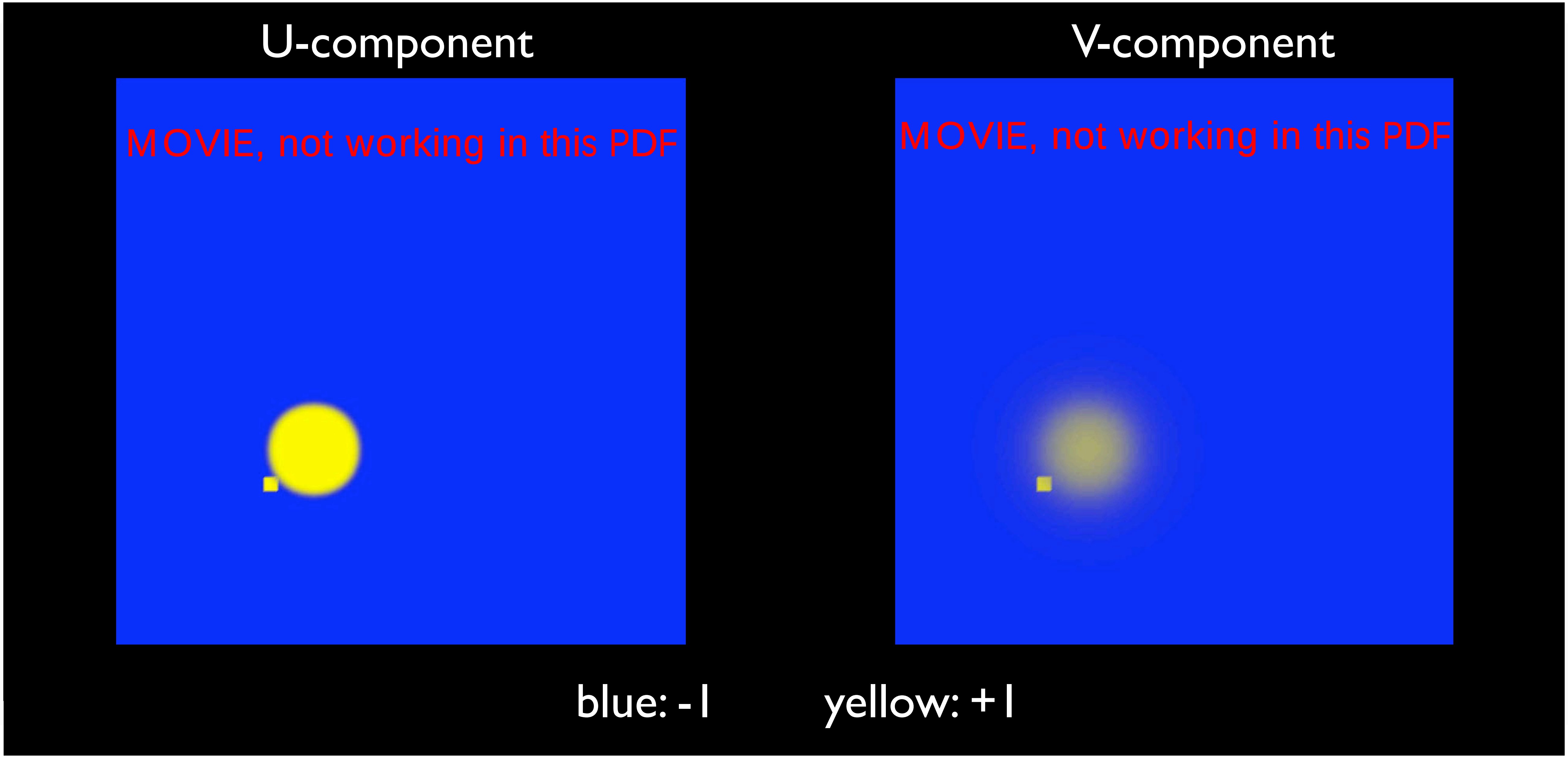
The bifurcation diagram of this stationary spot looks like:



Note: We found bifurcations of stable stationary spots to traveling spots only when both α and β are positive. In all the other cases the ℓ=0-line is always below the ℓ=1-line.

Direct PDE solver

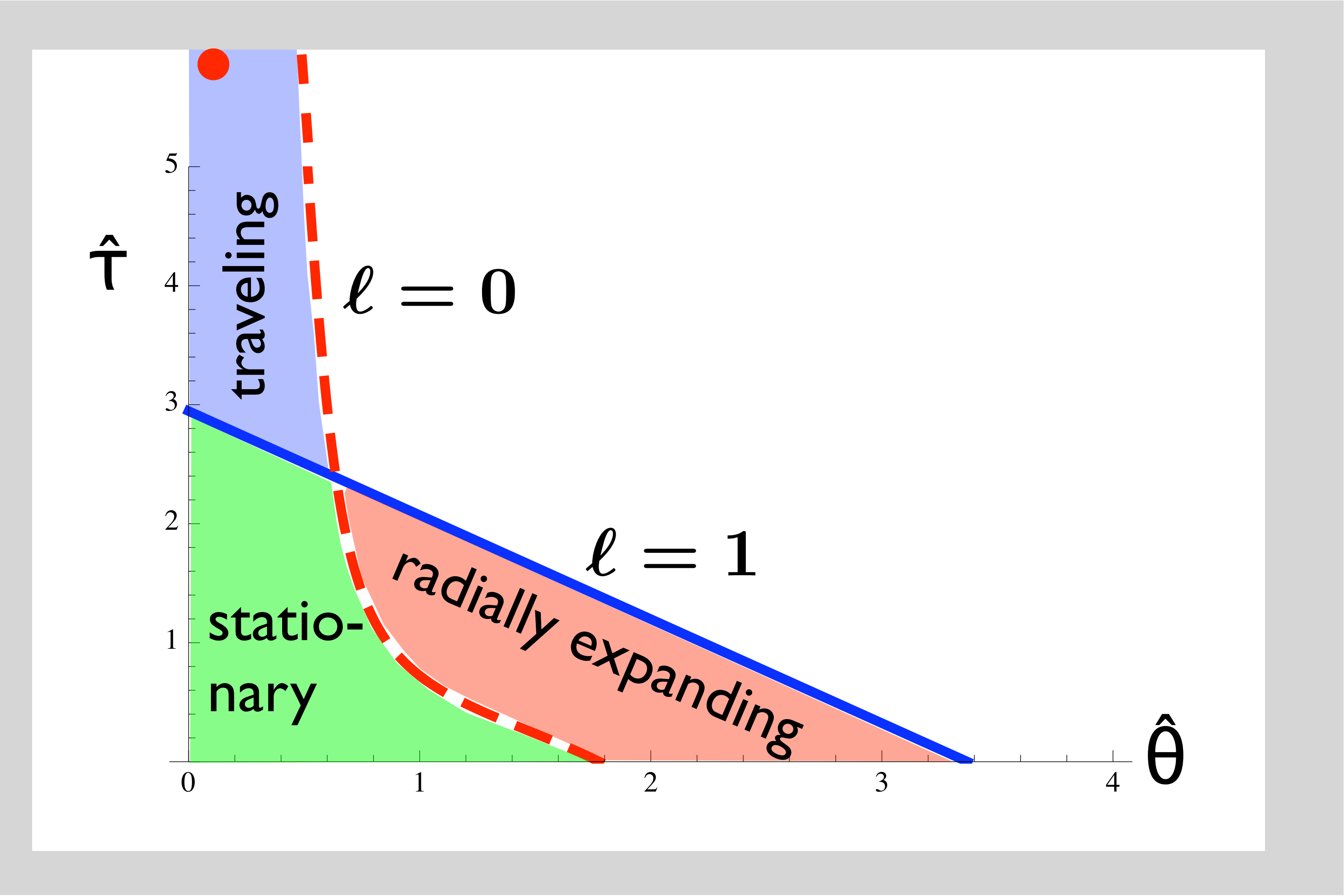
- Code written by K.-I. Ueda:
  ➡ 5-point discretization of the Laplacian on a 20 by 20 square with 200 equidistance mesh points
  ➡ Semi-implicit time scheme: conjugate gradients with incomplete Cholesky
- Parameter values: $\alpha = 0.5, \beta = 2, \gamma = 1, D = 2, \varepsilon = 0.1, \hat{\tau} = 6, \hat{\theta} = 0.01$



Why AUTO?

BOSTON UNIVERSITY

- Direct simulations with the PDE solver are slow and costly since the speed of a traveling spot is slow, especially for small ε.
- For example, simulation shown was done far away from the drift bifurcation line, with relatively large ε:



Want: better numerical evidence for the drift bifurcation line and more flexibility.
Tool: AUTO

AUTO

BOSTON UNIVERSITY

Rescale and co-moving frame:

$$(\hat{\tau}, \hat{\theta}) = \varepsilon^2(\tau, \theta), \quad (x_1, x_2, t) \to (x_1 - \varepsilon^2 ct, x_2, t)$$

Stationary solution in moving frame:

$$\begin{aligned}
-\varepsilon^2 c U_{x_1} &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\
-c\hat{\tau} V_{x_1} &= \Delta V + U - V \\
-c\hat{\theta} W_{x_1} &= D^2 \Delta W + U - W
\end{aligned}$$

Polar coordinates: $x_1 = r\cos\phi, \quad x_2 = r\sin\phi$

$$\begin{aligned}
-\varepsilon^2 c\left(\cos\phi\, U_r - \frac{\sin\phi}{r} U_\phi\right) &= \varepsilon^2\left(U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\phi\phi}\right) + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\
-c\hat{\tau}\left(\cos\phi\, V_r - \frac{\sin\phi}{r} V_\phi\right) &= \left(V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\phi\phi}\right) + U - V \\
-c\hat{\theta}\left(\cos\phi\, W_r - \frac{\sin\phi}{r} W_\phi\right) &= D^2\left(W_{rr} + \frac{1}{r}W_r + \frac{1}{r^2}W_{\phi\phi}\right) + U - W
\end{aligned}$$

AUTO (cont.)

BOSTON UNIVERSITY

Write as a first order system (in the radial variable)

$$
\begin{aligned}
u_r &= \frac{p}{\varepsilon} \\
p_r &= -\frac{p}{r} - \frac{\varepsilon}{r^2} u_{\phi\phi} - \frac{u}{\varepsilon} - \frac{u^3}{\varepsilon} + (\alpha v + \beta w + \gamma) - cp\cos\phi + \varepsilon c \frac{\sin\phi}{r} u_\phi \\
v_r &= q \\
q_r &= -\frac{q}{r} - \frac{1}{r^2} v_{\phi\phi} - u + v - \hat{\tau} c q \cos\phi + \hat{\tau} c \frac{\sin\phi}{r} v_\phi \\
w_r &= z \\
z_r &= -\frac{z}{r} - \frac{1}{r^2} w_{\phi\phi} - \frac{u}{D^2} + \frac{w}{D^2} - \frac{\hat{\theta} c z}{D^2} \cos\phi + \frac{\hat{\theta} c}{D^2} \frac{\sin\phi}{r} w_\phi
\end{aligned}
$$

Fourier in ϕ:

$$\bar{U}(r,\phi) = \sum_{\ell=-\infty}^{\ell=\infty} \bar{U}^\ell(r) e^{i\ell\phi}$$

recall:

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}), \quad \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}), \quad \frac{\partial \bar{U}}{\partial \phi} = i \sum_{\ell=-\infty}^{\infty} \ell u^\ell e^{i\ell\phi} \quad \frac{\partial^2 \bar{U}}{\partial \phi^2} = -\sum_{\ell=-\infty}^{\infty} \ell^2 u^\ell e^{i\ell\phi}$$

AUTO (cont.)

BOSTON UNIVERSITY

So, we get:

$$
\begin{aligned}
u_r^\ell &= \frac{p^\ell}{\varepsilon}\\
p_r^\ell &= -\frac{p^\ell}{r} + \frac{\varepsilon \ell^2}{r^2}u^\ell - \frac{u^\ell}{\varepsilon} - \frac{nonl}{\varepsilon} + (\alpha v^\ell + \beta w^\ell + \gamma)\\
&\quad -\frac{c}{2}(p^{\ell-1} + p^{\ell+1}) + \frac{\varepsilon c}{2r}\left((\ell-1)u^{\ell-1} - (\ell+1)u^{\ell+1}\right)\\
v_r^\ell &= q^\ell\\
q_r^\ell &= -\frac{q^\ell}{r} + \frac{\ell^2}{r^2}v^\ell - u^\ell + v^\ell - \frac{\hat{\tau} c}{2}(q^{\ell-1} + q^{\ell+1})\\
&\quad +\frac{\hat{\tau} c}{2r}\left((\ell-1)v^{\ell-1} - (\ell+1)v^{\ell+1}\right)\\
w_r^\ell &= z^\ell\\
z_r^\ell &= -\frac{z^\ell}{r} + \frac{\ell^2}{r^2}w^\ell - \frac{u^\ell}{D^2} + \frac{w^\ell}{D^2} - \frac{\hat{\theta} c}{2D^2}(z^{\ell-1} + z^{\ell+1})\\
&\quad +\frac{\hat{\theta} c}{2D^2 r}\left((\ell-1)w^{\ell-1} - (\ell+1)w^{\ell+1}\right)
\end{aligned}
$$

- Solutions need to be even: restrict ourself to $\ell \geq 0$
- γ only appears in the $\ell = 0$-term!
- nonl-term contains infinitely many coupled terms

AUTO: We have to truncate to a finite number of Fourier modes

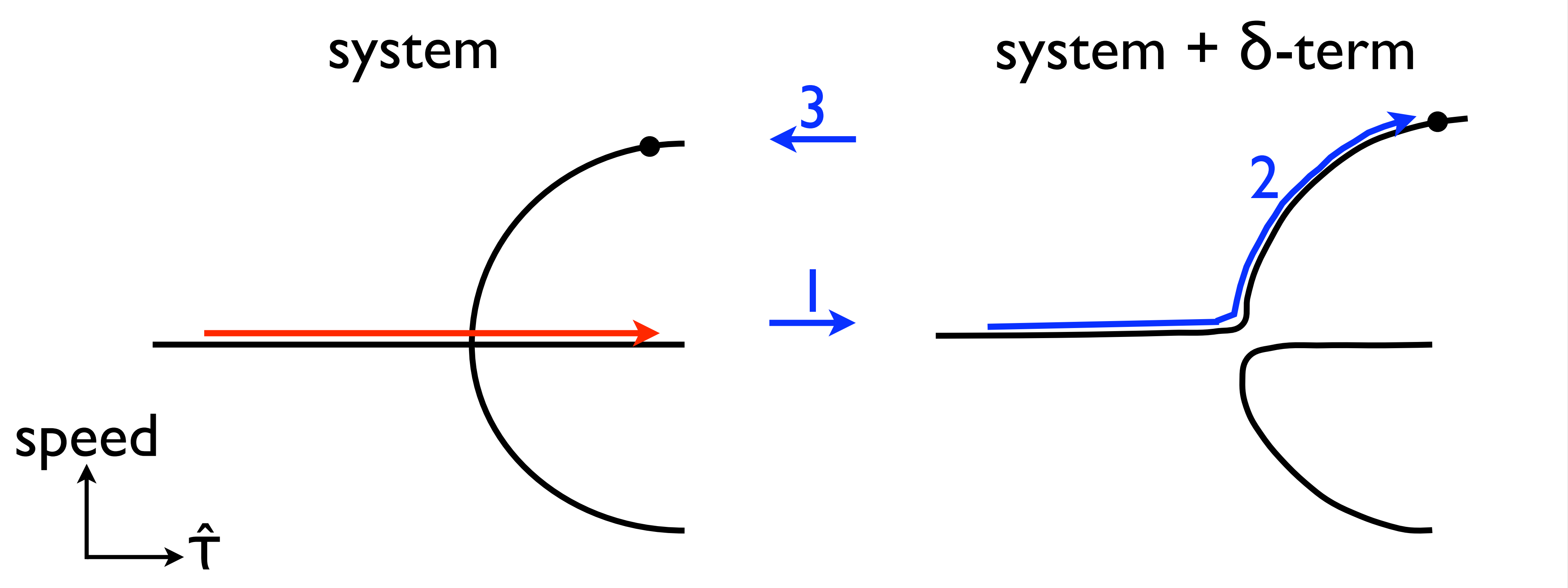
AUTO: Difficulties

BOSTON UNIVERSITY

Implement model in AUTO for a finite number of Fourier modes and on a finite domain [0, L] with appropriate boundary conditions.

2 Major difficulties:

- AUTO does not switch onto the traveling branch for increasing $\hat{\tau}$
  - ➡ Add a small symmetry breaking term: $$p_r^1 = p_r^1 - \delta \frac{r^2}{L^2}$$
  - ➡ Continue in $\hat{\tau}$ beyond bifurcation point (speed becomes non-zero)
  - ➡ Continue δ down to 0 (check that speed stays nonzero)



$$\begin{aligned} U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma) \\ \tau V_t &= \Delta V + U - V \\ \theta W_t &= D^2 \Delta W + U - W \end{aligned}$$

Difficulties (cont.)

BOSTON UNIVERSITY

Second major difficulty:

- AUTO detects many branch points, so it is not possible to detect the correct drift point and continue the drift line in the $(\hat{\tau},\hat{\theta})$-plane.

➡ Detect drift bifurcation as points where the linearization L_1 has a generalized eigenfunction ψ:

$$L_1 \psi = M \bar{U}_r^s$$

➡ M is a diagonal matrix with $1, 1/\hat{\tau}, D^2/\hat{\theta}$ on its diagonal and $\bar{U}_r^s$ is the radial derivative of stationary spot and thus lies in the null space of L_1

➡ Add small term $\delta \bar{U}_r^s$ to the eq^n (makes the system onto)

$$L_1 \psi + \delta \bar{U}_r^s = M \bar{U}_r^s$$

➡ Add integral condition to ensure that the kernel is 0 (solvability condition):

$$\langle \psi, \bar{U}_r^s \rangle = 0$$

➡ Unique solution (ψ,δ)

➡ We are at a drift bifurcation iff δ = 0, so we continue δ down to 0

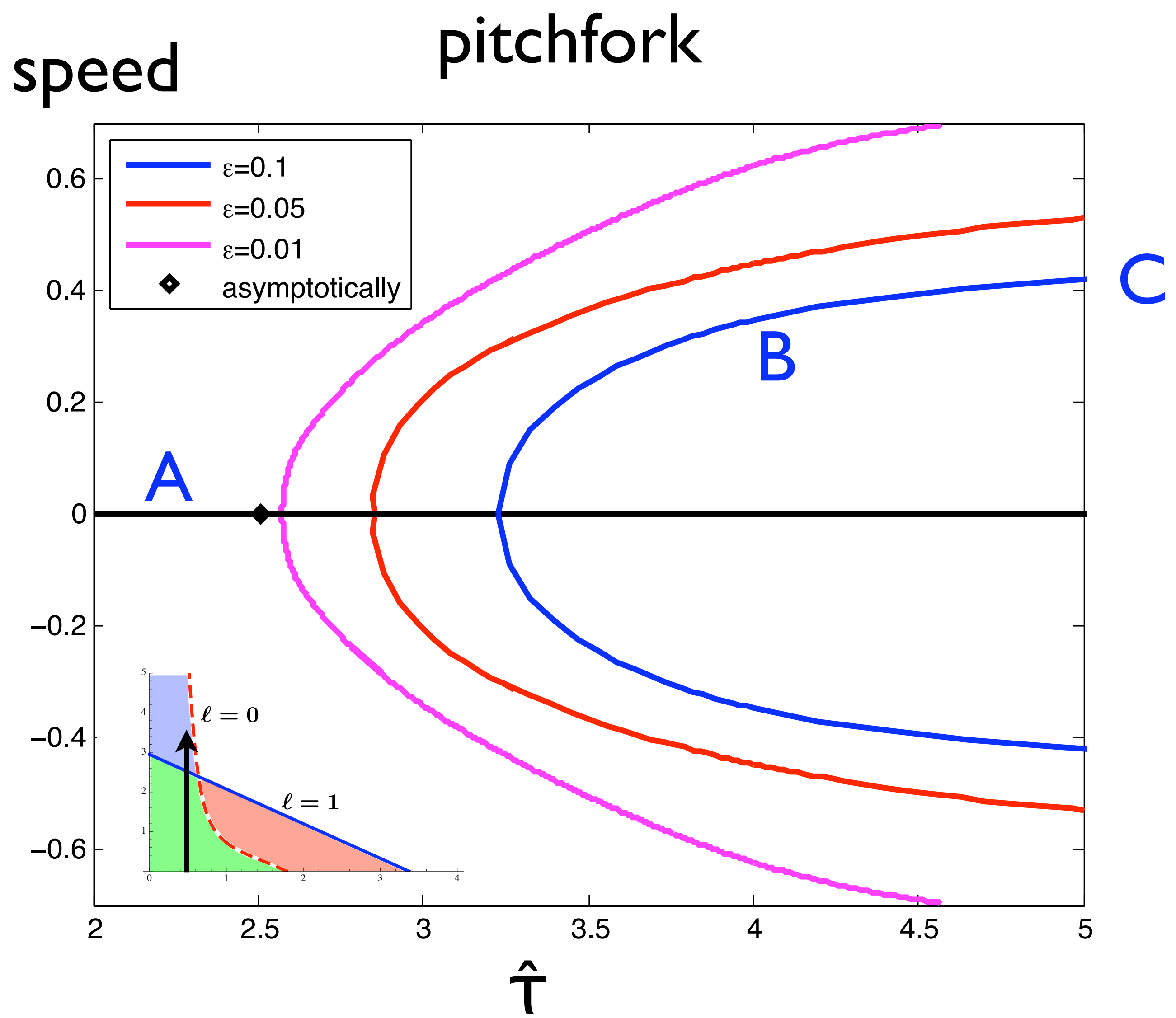
➡ Remove the δ-term, and continue in $\hat{\tau}$ or $\hat{\theta}$

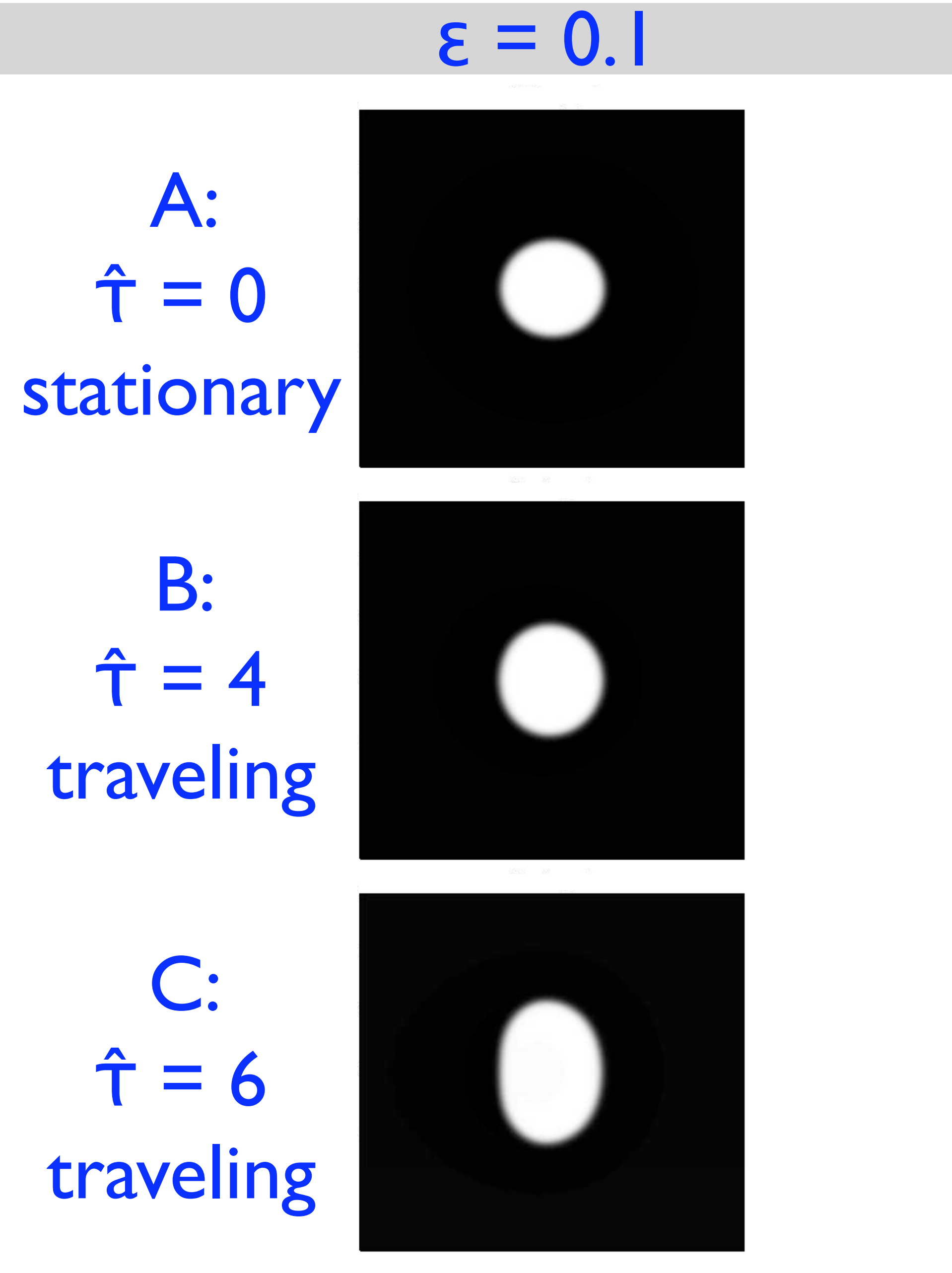
AUTO: Results

Results:

- standard parameter values: $\alpha = 0.5, \beta = 2, \gamma = 1, D = 2, \hat{\theta} = 0.5$

- 15 fourier modes, domain size = 12



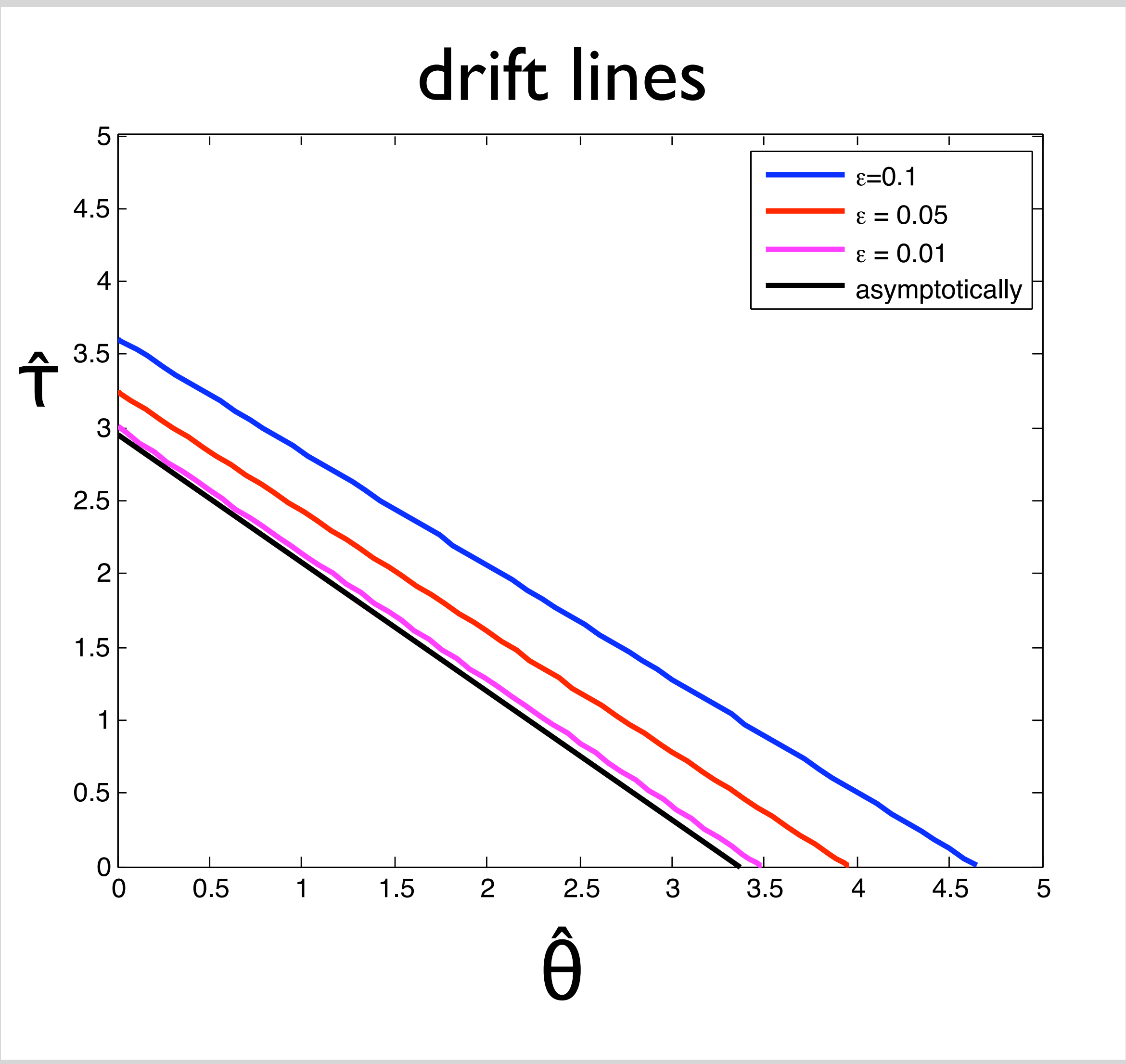


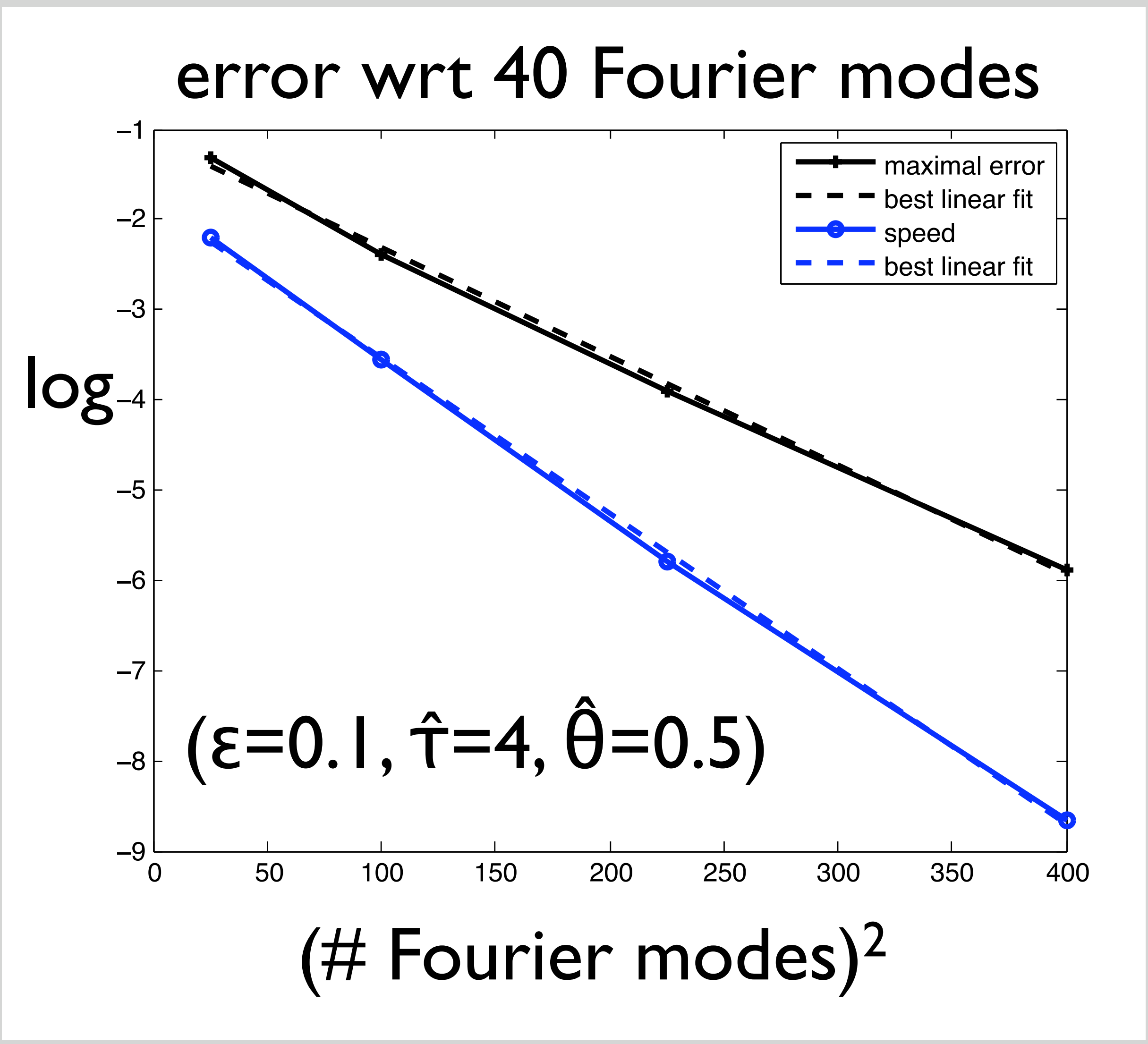
Results (cont.)

Results:

- standard parameter values: $\alpha = 0.5, \beta = 2, \gamma = 1, D = 2$

- 15 fourier modes, domain size = 12

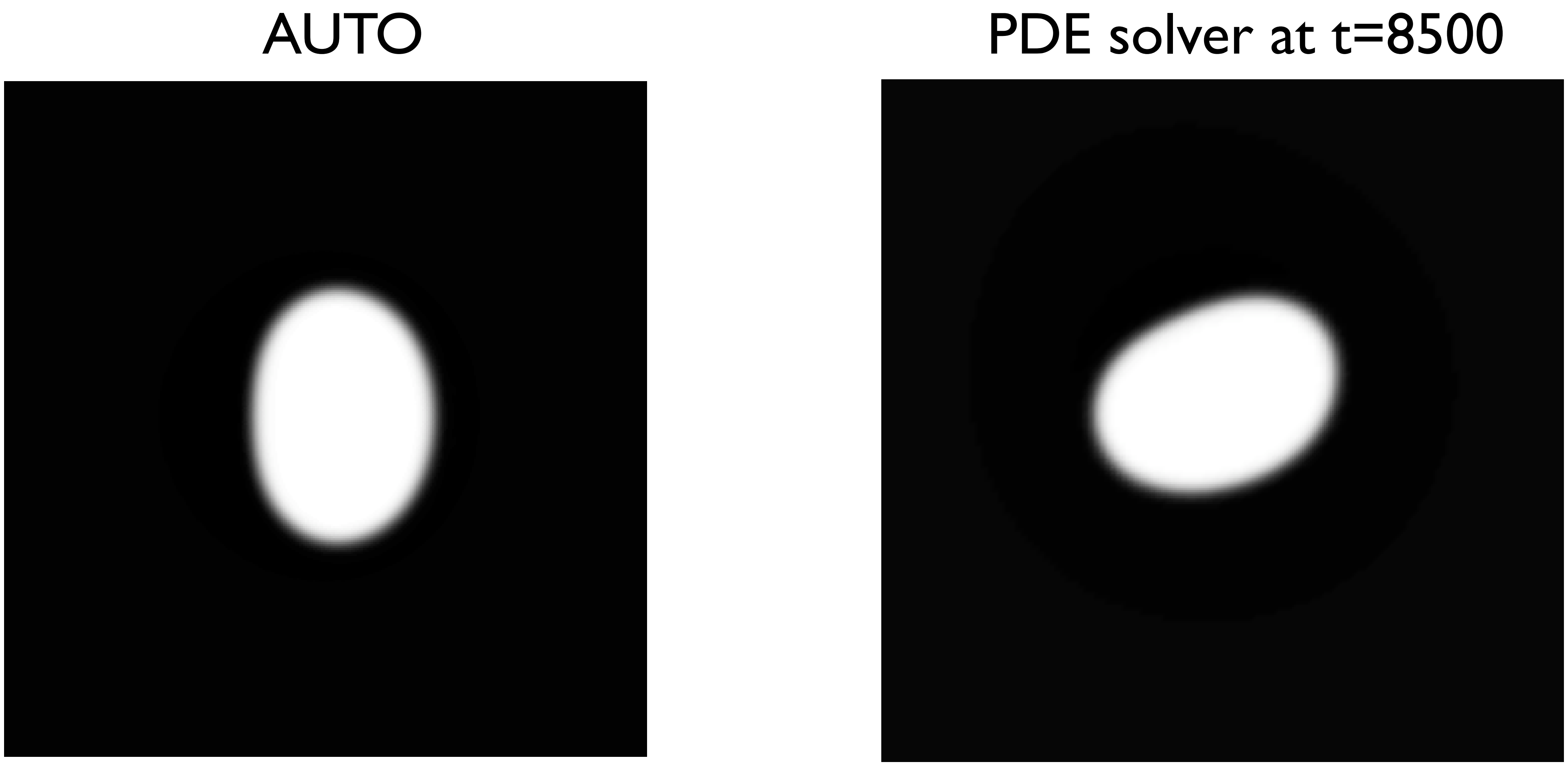




Compare: profile

- Parameter values:

$$\alpha = 0.5,\ \beta = 2,\ \gamma = 1,\ D = 2,\ \varepsilon = 0.1,\ \hat{\tau} = 6,\ \hat{\theta} = 0.01$$

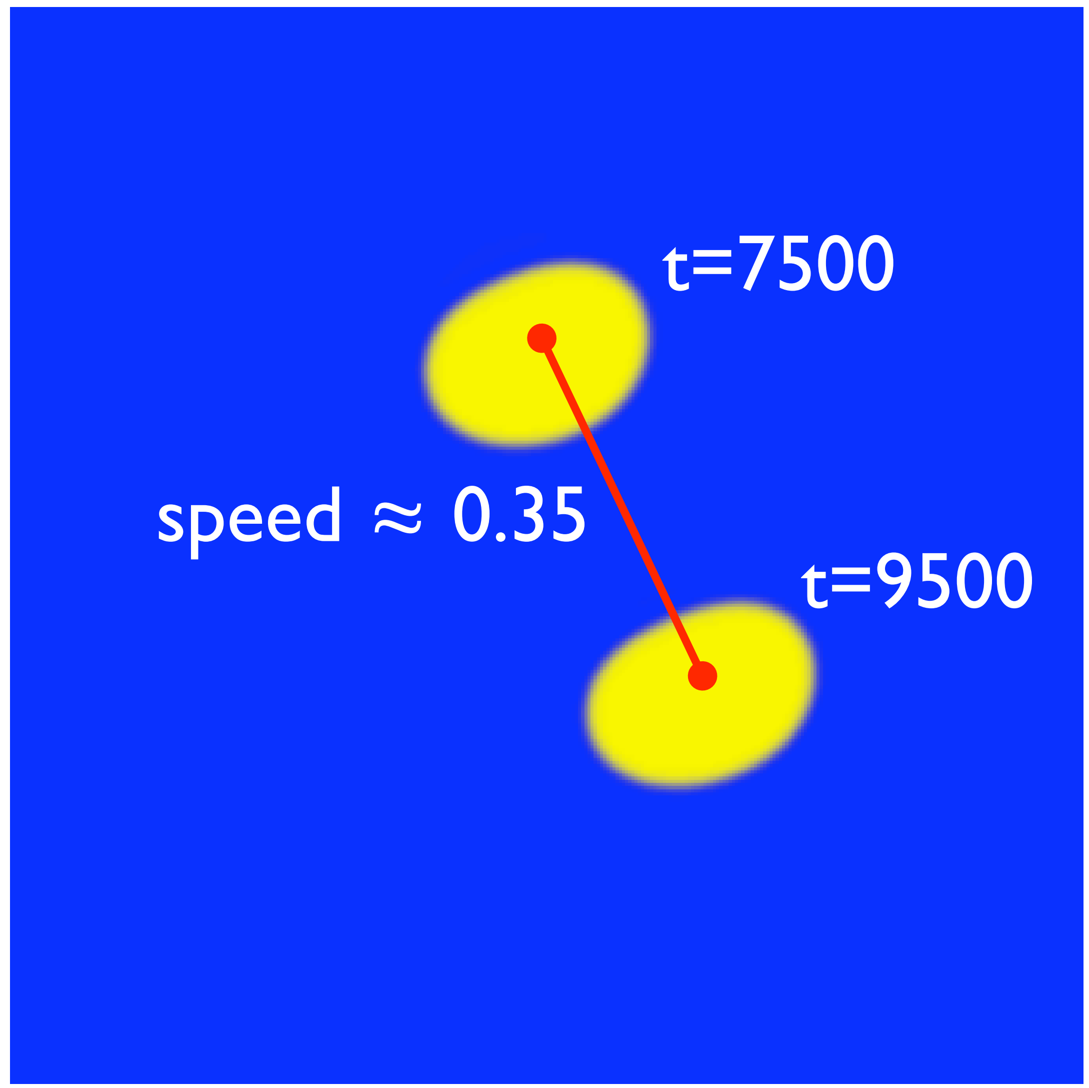


Compare: speed

BOSTON UNIVERSITY

- PDE solver:

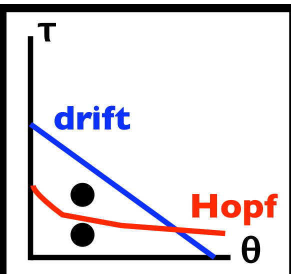
Snapshots of U-component at t=7500 and t=9500



- AUTO: predicted speed = 0.38

Hopf

- Different set of parameter values!



U-component

blue: -1 yellow: +1

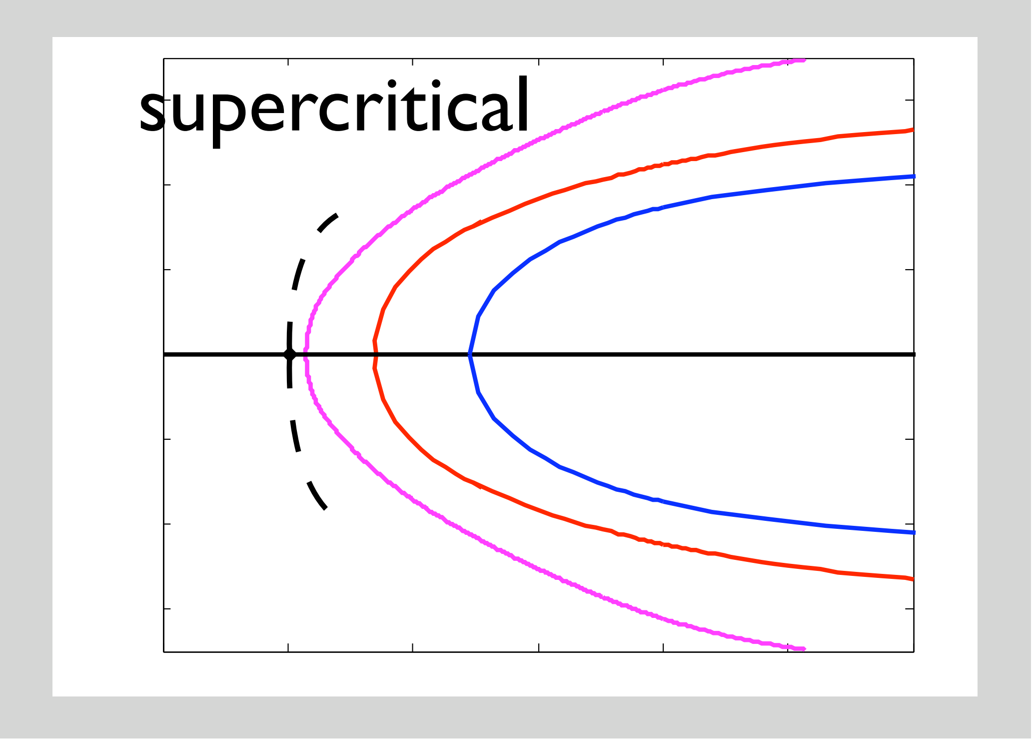
$\hat{\tau}$=0.31: below the Hopf line

$\hat{\tau}$=0.32: above the Hopf line

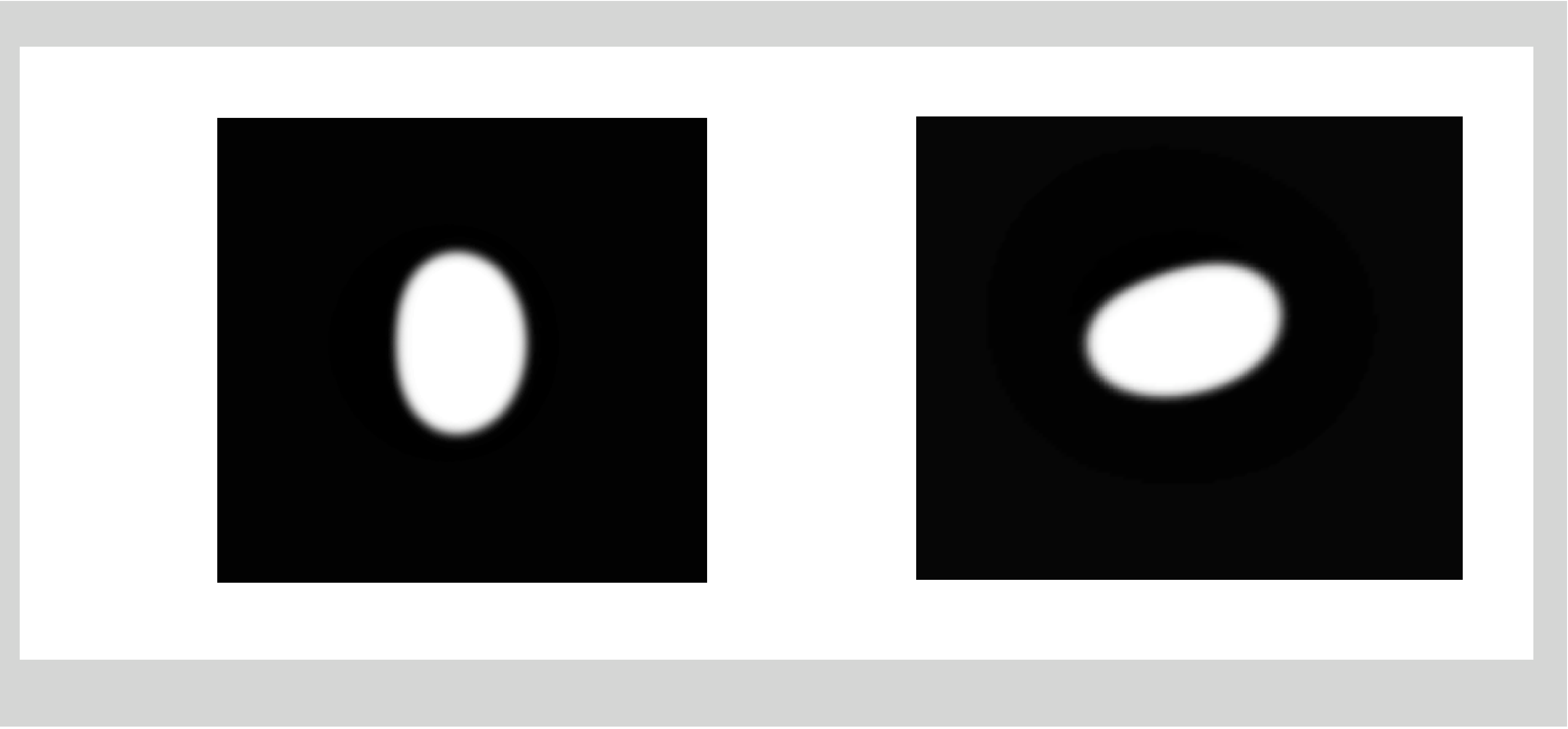
Work in progress I

BOSTON UNIVERSITY

- Super vs subcritical? [Ei, Mimura, Nagayama 2006]



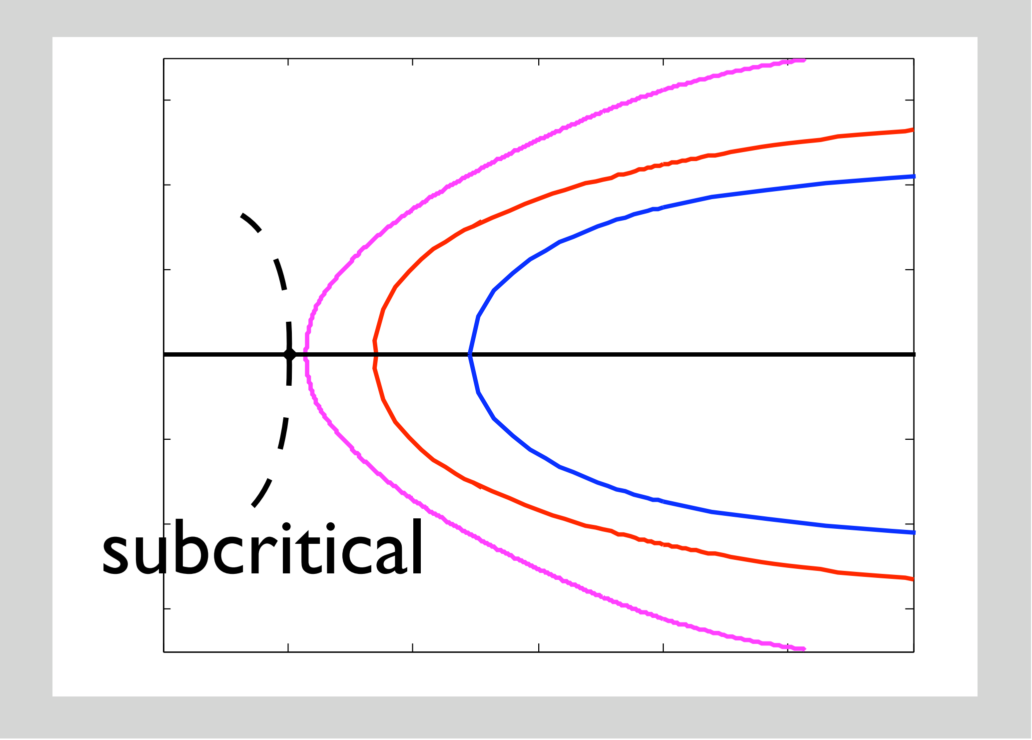
- Compare AUTO with PDE solver



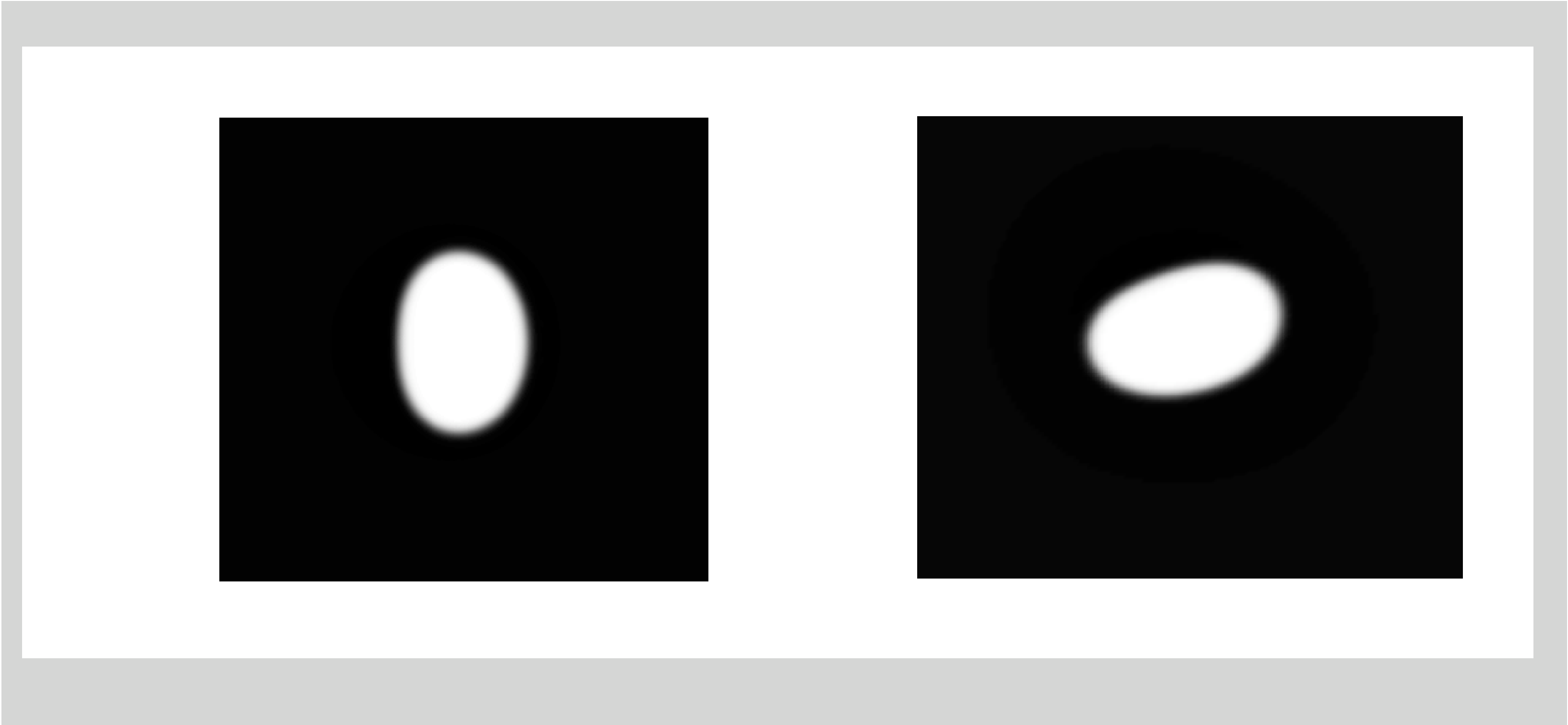
Work in progress I

BOSTON UNIVERSITY

- Super vs subcritical? [Ei, Mimura, Nagayama 2006]



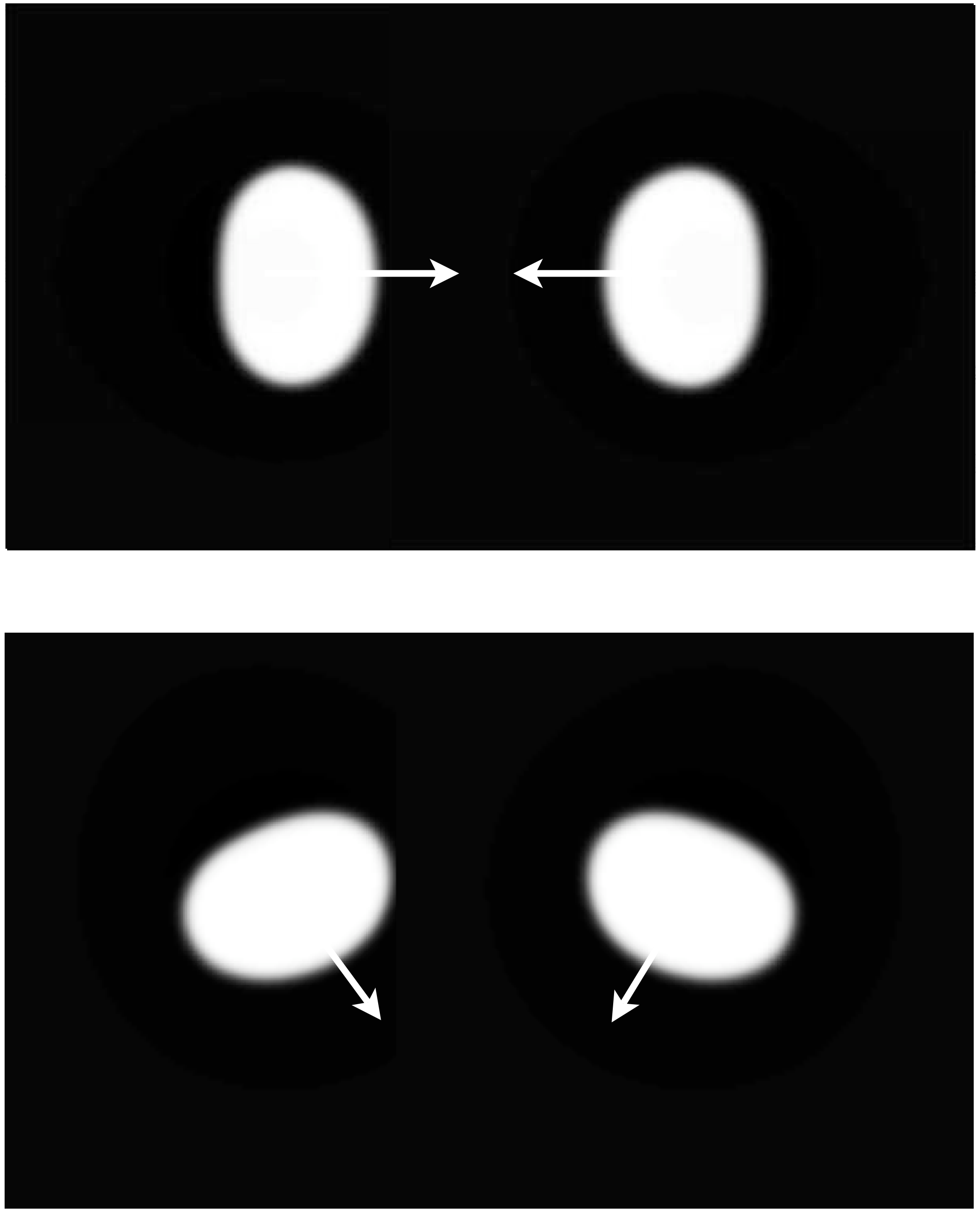
- Compare AUTO with PDE solver



Work in progress II

BOSTON UNIVERSITY

- Interaction of traveling spots (cartoon)



Questions??