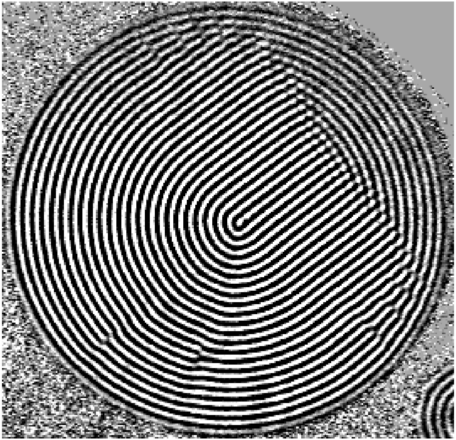
# Existence of defects in the Swift-Hohenberg equation

Mariana Haragus & Arnd Scheel

# Defects



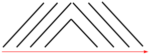
[Ercolani, Indik, Newell, Passot, 2000]

- **dislocations**
- **grain boundaries**
- *disclinations*

# Swift-Hohenberg equation

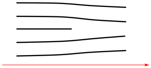
- **Swift-Hohenberg equation**

$$u_t = -(\Delta + 1)^2 u + \mu u - u^3$$

  - **grain boundaries** 

- **anisotropic Swift-Hohenberg equation**

$$u_t = -(\Delta + 1)^2 u + \mu u - u^3 + \beta u_{xx}$$

  - **dislocations** 

# Spatial dynamics

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{F}(\mathbf{U}; \mu, k, c, \beta)$$

**dislocation/grain boundary** $\longleftrightarrow$ **heteroclinic orbit**

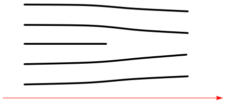
- **bifurcation problem :** *bifurcation points*
- **center manifold reduction :** *reduced system*
- **reduced system :**
  - leading order system : *existence of heteroclinic orbits*
  - full system : *persistence of heteroclinic orbits*

# Swift-Hohenberg equation

- **anisotropic Swift-Hohenberg equation**

$$u_t = -(\Delta + 1)^2 u + \mu u - u^3 + \beta u_{xx}$$

- **dislocations**

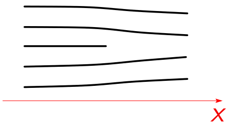


- *two-dimensional traveling waves*

$$\boxed{c u_x = -(\Delta + 1)^2 u + \mu u - u^3 + \beta u_{xx}}$$

# Spatial dynamics

- **dislocations**

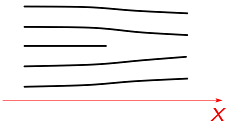



  - limits at $x = \pm\infty$ : **rolls**

    *(y-periodic solutions, x-independent)*
  - solutions connecting two rolls

# Spatial dynamics

- **dislocations**




  - limits at $x = \pm\infty$ : **rolls**

    *(y-periodic solutions, x-independent)*
  - solutions connecting two rolls

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{F}(\mathbf{U}; \mu, k, c, \beta)$$

  - **rolls** $\longleftrightarrow$ **equilibria**
  - **dislocations** $\longleftrightarrow$ **heteroclinic orbits**

# Dynamical system

- Swift-Hohenberg equation

$$cu_x = -(\partial_{xx} + k^2\partial_{yy} + 1)^2 u + \mu u - u^3 + \beta u_{xx}$$

- Ansatz : $\mathbf{U} = (u, u_1, v, v_1)$
- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}(\mu, k, c, \beta)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

$$\mathcal{A}(\mu, k, c, \beta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1 + k^2\partial_y^2) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta(1 + k^2\partial_y^2) + \mu & c & -(1 + k^2\partial_y^2) + \beta & 0 \end{pmatrix}, \quad \mathcal{F}(\mathbf{U}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u^3 \end{pmatrix}$$

# Dynamical system

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}(\mu, k, c, \beta)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

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- **phase space**

$$\mathcal{X} = H^3_{\rm per}(0,2\pi) \times H^2_{\rm per}(0,2\pi) \times H^1_{\rm per}(0,2\pi) \times L^2(0,2\pi)$$

- $\mathcal{A}(\mu, k, c, \beta)$ closed linear operator ; domain

$$\mathcal{Y} = H^4_{\rm per}(0,2\pi) \times H^3_{\rm per}(0,2\pi) \times H^2_{\rm per}(0,2\pi) \times H^1_{\rm per}(0,2\pi)$$

- $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}$ smooth

# Parameters

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}(\mu, k, c, \beta)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

  - *parameters :*
    - *equation :* $\mu$, $\beta$
    - *y-periodic solutions :* wavenumber $k$
    - *traveling waves :* speed $c$
  - choice of parameters : *co-existence of rolls with different wavenumbers*

- **dispersion relation**

# Dispersion relation

- **dispersion relation :** solutions of the form $\boxed{\mathbf{U} = e^{\nu x} e^{i\ell y} \mathbf{u}}$

$$\boxed{(\nu^2 + 1 - k^2\ell^2)^2 = \mu + \beta\nu^2 - c\nu}, \quad \ell \in \mathbb{Z}$$

- *co-existence of rolls with wavenumbers $\ell_- \neq \ell_+$ if*

$$\nu = 0, \quad \mu = (1 - k^2\ell_\pm^2)^2$$

- **choice of parameters :**
  - $|\ell_- - \ell_+| = 1$ : $\boxed{\ell_- = \ell_*,\ \ell_+ = \ell_* + 1}$, $\ell_* \in \mathbb{N}$
  - $\mu = (1 - k^2\ell_*^2)^2, \quad \mu = (1 - k^2(\ell_* + 1)^2)^2$

$$\boxed{k_*^2 = \frac{2}{2\ell_*^2 + 2\ell_* + 1}, \quad \sqrt{\mu_*} = \frac{2\ell_* + 1}{2\ell_*^2 + 2\ell_* + 1}}$$

  - $\boxed{\beta > 0,\ c = 0}$

# Bifurcation problem

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}(\mu, k, c, \beta)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

- **parameters**
  - $\beta > 0$, $k = k_* > 0$ fixed parameters
  - $c \sim 0$, $\mu \sim \mu_*$ *bifurcation parameters*

# Bifurcation problem

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}(\mu, k, c, \beta)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

- **parameters**
  - $\beta > 0$, $k = k_* > 0$ fixed parameters
  - $c \sim 0$, $\mu \sim \mu_*$ *bifurcation parameters*

- **dynamical system** : $\boxed{\mu = \mu_* + \bar{\mu}}$

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{A}_*\mathbf{U} + \mathcal{B}(\bar{\mu}, c)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

$$\mathcal{A}_* = \mathcal{A}(\mu_*, k_*, 0, \beta), \quad \mathcal{B}(\bar{\mu}, c) = \mathcal{A}(\mu_* + \bar{\mu}, k_*, c, \beta) - \mathcal{A}(\mu_*, k_*, 0, \beta).$$

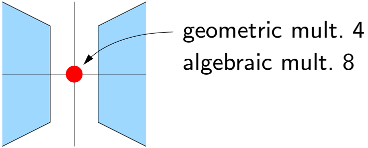
  - $\mu_*$ **small parameter**

# Linear operator

- **dynamical system**

$$\frac{d\mathbf{U}}{dx} = \mathcal{A}_*\mathbf{U} + \mathcal{B}(\bar{\mu}, c)\mathbf{U} + \mathcal{F}(\mathbf{U})$$

- *spectrum of* $\mathcal{A}_*$




*($\beta = 0$ : $4\ell_*$ additional eigenvalues!)*

# Center manifold

- **small bounded solutions**

$$\mathbf{U}(x) = U_c(x) + \mathbf{\Psi}_c(U_c(x); \bar{\mu}, c), \quad U_c(x) \in \mathcal{X}_c$$

  - $\mathcal{X}_c$ spectral subspace associated with the purely imaginary eigenvalues (dimension 8)

- **reduced system**

$$\frac{\mathrm{d}U_c}{\mathrm{d}x} = \mathcal{A}_* U_c + \mathbf{P}_c\Big(\mathcal{B}(\bar{\mu}, c)(U_c + \mathbf{\Psi}_c(U_c; \bar{\mu}, c)) + \mathcal{F}(U_c + \mathbf{\Psi}_c(U_c; \bar{\mu}, c))\Big)$$

  - $\mathbf{P}_c$ spectral projector on $\mathcal{X}_c$
  - *ODE ; dimension* 8

# Reduced system

- **reduced system**

$$\boxed{\frac{\mathbf{d}U_c}{\mathbf{d\,x}} = \mathcal{A}_* U_c + \mathbf{P}_c\Big(\mathcal{B}(\bar{\mu}, c)(U_c + \mathbf{\Psi}_c(U_c; \bar{\mu}, c)) + \mathcal{F}(U_c + \mathbf{\Psi}_c(U_c; \bar{\mu}, c))\Big)}$$

- **first difficulties**
  - **dimension 8** *(too large)*

    – *even solutions in y* –
  - $\mu_*$ **small parameter**

    – *appropriate scaling* –

# Ansatz

- **small bounded solutions**

$$\mathbf{U}(x) = U_c(x) + \mathbf{\Psi}_c(U_c(x); \bar{\mu}, c), \quad U_c(x) \in \mathcal{X}_c$$

- **base of $\mathcal{X}_c$**

$$E_{\pm\ell_*} = \begin{pmatrix} e^{\pm i\ell_* y} \\ 0 \\ \sqrt{\mu_*} e^{\pm i\ell_* y} \\ 0 \end{pmatrix}, \; E_{\pm(\ell_*+1)} = \begin{pmatrix} e^{\pm i(\ell_*+1)y} \\ 0 \\ -\sqrt{\mu_*} e^{\pm i(\ell_*+1)y} \\ 0 \end{pmatrix}, \; F_{\pm\ell_*} = \begin{pmatrix} 0 \\ e^{\pm i\ell_* y} \\ 0 \\ \sqrt{\mu_*} e^{\pm i\ell_* y} \end{pmatrix}, \; F_{\pm(\ell_*+1)} = \begin{pmatrix} 0 \\ e^{\pm i(\ell_*+1)y} \\ 0 \\ -\sqrt{\mu_*} e^{\pm i(\ell_*+1)y} \end{pmatrix}$$

  - *even solutions*

$$\begin{aligned} U_c(x, y) \;=\; & a_0(x)\left(E_{+\ell_*}(y) + E_{-\ell_*}(y)\right) + b_0(x)\left(E_{+(\ell_*+1)}(y) + E_{-(\ell_*+1)}(y)\right) \\ & + a_1(x)\left(F_{+\ell_*}(y) + F_{-\ell_*}(y)\right) + b_1(x)\left(F_{+(\ell_*+1)}(y) + F_{-(\ell_*+1)}(y)\right) \end{aligned}$$

  - *scaling*

$$\bar{\mu} = \mu_*^3 \tilde{\bar{\mu}}, \quad c = \mu_*^{3/2} \tilde{c}, \quad a_j = \mu_*^{3/2} \tilde{a}_j, \quad b_j = \mu_*^{3/2} \tilde{b}_j, \quad j = 0, 1$$

# Reduced system

- **reduced system**

$$
\begin{aligned}
a_0' &= a_1 \\
a_1' &= -\frac{1}{\beta}\left(\mu_*^3\bar{\mu}a_0 + \mu_*^{3/2}ca_1 - 3\mu_*^3 a_0(a_0^2 + 2b_0^2)\right) + \dots \\
b_0' &= b_1 \\
b_1' &= -\frac{1}{\beta}\left(\mu_*^3\bar{\mu}b_0 + \mu_*^{3/2}cb_1 - 3\mu_*^3 b_0(2a_0^2 + b_0^2)\right) + \dots
\end{aligned}
$$

- *second scaling*

$$
X = \frac{1}{\sqrt{\beta}}\mu_*^{3/2}\sqrt{\bar{\mu}}\,x, \quad c = \sqrt{\bar{\mu}}\,\bar{c}, \quad a_0 = \frac{1}{\sqrt{3}}\sqrt{\bar{\mu}}\,A_0, \quad b_0 = \frac{1}{\sqrt{3}}\sqrt{\bar{\mu}}\,B_0
$$

# Reduced system

- **new system**

$$
\begin{aligned}
A_0'' &= -A_0 - \bar{c}A_0' + A_0(A_0^2 + 2B_0^2) + \mathcal{O}(\mu_*^{1/2}) \\
B_0'' &= -B_0 - \bar{c}B_0' + B_0(2A_0^2 + B_0^2) + \mathcal{O}(\mu_*^{1/2})
\end{aligned}
$$

- **leading order system :** $\boxed{\bar{c} = \mu_* = 0}$

$$
\begin{aligned}
A_0'' &= -A_0 + A_0(A_0^2 + 2B_0^2) \\
B_0'' &= -B_0 + B_0(2A_0^2 + B_0^2)
\end{aligned}
$$

  - *rich dynamics (9 equilibria, several heteroclinic orbits, ...)*
  - *first integral*

$$
H(A_0, B_0, A_0', B_0') = (A_0')^2 + (B_0')^2 - \frac{1}{2}\left(A_0^2 + B_0^2 - 1\right)^2 - A_0^2 B_0^2
$$

# Rolls and dislocations

- **rolls**
  - *equilibria* $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$
- **dislocations**
  - *heteroclinic orbit* $(A_0^\star, B_0^\star)$ *connecting* $(1, 0)$ *and* $(0, 1)$
  - *existence of this heteroclinic orbit :*

    [van den Berg & van der Vorst, 1997]

# Persistence of the heteroclinic orbit

- **solve** $\mathcal{T}(A_0, B_0, \bar{c}, \bar{\mu}, \mu_*) = 0$

$$\mathcal{T}(A_0, B_0, \bar{c}, \bar{\mu}, \mu_*) = \begin{pmatrix} A_0'' + A_0 - A_0(A_0^2 + 2B_0^2) + \bar{c}A_0' - \mathcal{R}_A^*(A_0, A_0', B_0, B_0', \bar{\mu}, \bar{c}; \mu_*) \\ B_0'' + B_0 - B_0(2A_0^2 + B_0^2) + \bar{c}B_0' - \mathcal{R}_B^*(A_0, A_0', B_0, B_0', \bar{\mu}, \bar{c}; \mu_*) \end{pmatrix}.$$

  - *particular solution :* $(A_0^\star, B_0^\star, 0, 0, 0)$
  - $D_{(A_0,B_0)}\mathcal{T}(A_0^\star, B_0^\star, 0, 0, 0) = \mathcal{L}_\star$, $\quad \partial_{\bar{c}}\mathcal{T}(A_0^\star, B_0^\star, 0, 0, 0) = (A_0^{\star\prime}, B_0^{\star\prime})$

- **linear operator**

$$\mathcal{L}_\star = \begin{pmatrix} \partial_{xx} + 1 - 3A_0^{\star 2} - 2B_0^{\star 2} & -4A_0^\star B_0^\star \\ -4A_0^\star B_0^\star & \partial_{xx} + 1 - 2A_0^{\star 2} - 3B_0^{\star 2} \end{pmatrix}$$

  - *Fredholm operator with index* 0
  - 0 **simple eigenvalue ; eigenvector** $(A_0^{\star\prime}, B_0^{\star\prime})$

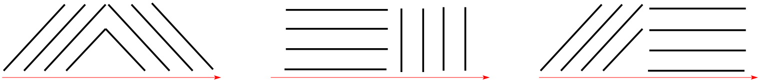
- **implicit function theorem**
  - **solution** $(A_0, B_0) = (A_0, B_0)(\bar{\mu}, \mu_*), \ \bar{c} = \bar{c}(\bar{\mu}, \mu_*)$

# Swift-Hohenberg equation

- **Swift-Hohenberg equation**

$$u_t = -(\Delta + 1)^2 u + \mu u - u^3$$

- **grain boundaries**

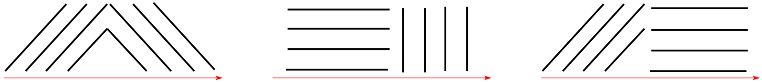


- *two-dimensional steady waves*

$$\boxed{0 = -(\Delta + 1)^2 u + \mu u - u^3}$$

# Spatial dynamics

- **grain boundaries**



  - solutions connecting two rolls with different orientations

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{F}(\mathbf{U}; \mu, k_*)$$

  - **rolls** $\longleftrightarrow$ **equilibria**
  - **grain boundaries** $\longleftrightarrow$ **heteroclinic orbit**

# Bifurcation problem

- **dynamical system**

$$\frac{\mathbf{dU}}{\mathbf{dx}} = \mathcal{F}(\mathbf{U}; \mu, k_*)$$

- **parameters** *(co-existence of rolls with different orientations)*
  - $\frac{1}{2} < k_* < 1$ fixed parameter
  - $\mu \sim 0$ *bifurcation parameter*
- **center manifold reduction**
  - *spectrum of* $\mathcal{A}_* = D_{\mathbf{U}}\mathcal{F}(0; 0, k_*)$
  - **reduced system**

# Reduced system

- *spectrum of* $\sigma_c(\mathcal{A}_*) = \{\pm i, \pm ik_x\}$;    4 eigenvalues

  $\pm i$ geometric mult. 1, algebraic mult. 2

  $\pm ik_x$ geometric mult. 2, algebraic mult. 4

- **reduced system :** *ODE in* $\mathbb{R}^{12}$

$$\begin{aligned}
A_0' &= \mathrm{i}A_0 + B_0 - \tfrac{\mathrm{i}}{4}\left(\mu a_0 - a_0(a_0^2 + 6a_+\overline{a_+})\right)\\
B_0' &= \mathrm{i}B_0 - \tfrac{1}{4}\left(\mu a_0 - a_0(a_0^2 + 6a_+\overline{a_+})\right)\\
A_+' &= \mathrm{i}k_x A_+ + B_+ - \tfrac{\mathrm{i}}{4k_x^3}\left(\mu a_+ - 3a_+(a_0^2 + a_+\overline{a_+})\right)\\
B_+' &= \mathrm{i}k_x B_+ - \tfrac{1}{4k_x^2}\left(\mu a_+ - 3a_+(a_0^2 + a_+\overline{a_+})\right)\\
A_-' &= \mathrm{i}k_x A_- + B_- - \tfrac{\mathrm{i}}{4k_x^3}\left(\mu \overline{a_+} - 3\overline{a_+}(a_0^2 + a_+\overline{a_+})\right)\\
B_-' &= \mathrm{i}k_x B_- - \tfrac{1}{4k_x^2}\left(\mu \overline{a_+} - 3\overline{a_+}(a_0^2 + a_+\overline{a_+})\right)
\end{aligned}$$

$a_0 = A_0 + \overline{A_0},\ b_0 = B_0 + \overline{B_0},\ a_+ = A_+ + \overline{A_-},\ a_- = A_+ - \overline{A_-},\ b_+ = B_+ + \overline{B_-}$

# Normal form

- normal form $(A_\kappa, B_\kappa) \longrightarrow (C_\kappa, D_\kappa)$
- solutions of the form

$$C_0(x) = \mathrm{e}^{\mathrm{i}x}\widetilde{C_0}, \quad D_0(x) = \mathrm{e}^{\mathrm{i}x}\widetilde{D_0}, \quad C_\pm(x) = \mathrm{e}^{\mathrm{i}k_x x}\widetilde{C_\pm}, \quad D_\pm(x) = \mathrm{e}^{\mathrm{i}k_x x}\widetilde{D_\pm}.$$

- scaling

$$\widehat{x} = |\mu|^{1/2}x, \quad c = |\mu|\widehat{c}, \quad C_\kappa = |\mu|^{1/2}\widehat{C_\kappa}, \quad D_\kappa = |\mu|\widehat{D_\kappa}, \quad \kappa \in \{0, \pm\},$$

$$\begin{aligned}
C_0'' &= -\frac{1}{4}C_0 + \frac{3}{4}C_0(|C_0|^2 + 2|C_+|^2 + 2|C_-|^2) \\
C_+'' &= -\frac{1}{4k_x^2}C_+ + \frac{3}{4k_x^2}C_+(2|C_0|^2 + |C_+|^2 + 2|C_-|^2) \\
C_-'' &= -\frac{1}{4k_x^2}C_- + \frac{3}{4k_x^2}C_-(2|C_0|^2 + 2|C_+|^2 + |C_-|^2)
\end{aligned} \quad + O(|\mu|^{1/2})$$

# Heteroclinic orbit

- **leading order system :** $\mu = 0$

$$\begin{aligned} C_0'' &= -\tfrac{1}{4}C_0 + \tfrac{3}{4}C_0(|C_0|^2 + 2|C_+|^2 + 2|C_-|^2) \\ C_+'' &= -\frac{1}{4k_x^2}C_+ + \frac{3}{4k_x^2}C_+(2|C_0|^2 + |C_+|^2 + 2|C_-|^2) \\ C_-'' &= -\frac{1}{4k_x^2}C_- + \frac{3}{4k_x^2}C_-(2|C_0|^2 + 2|C_+|^2 + |C_-|^2) \end{aligned}$$

  - *heteroclinic orbit* $(0, C_+^\star, C_-^\star)$ $(C_0^\star, C_+^\star, 0)$ ?

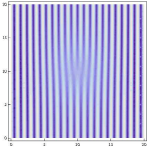
[van den Berg & van der Vorst, 1997]

- **reduced system :** *persistence of these heteroclinic orbits*

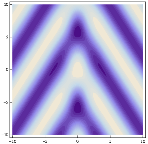
$\Longrightarrow$ **existence of grain boundaries**

# Defects

- **dislocations**




- **grain boundaries**




- *disclinations ?*