Exploring Function and Distribution Structure in Interactive Computing Through Examples

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NSF

Preliminary remarks

- Focus:
  - Lossless distributed function computation in source networks
  - Nodes connected by bidirectional rate-limited error-free bit-pipes
  - Discrete Memoryless Stationary Sources
- New degree of freedom: multi-round interaction
- Disclaimers:
  - No structured coding ensembles
  - No Gaussian Quadratic problem
  - Some theory but no proofs
  - Lots of simple but striking examples

Outline

- Introduction
- General two-terminal problem
- Co-located network with independent sources
- General multi-terminal problem: some observations

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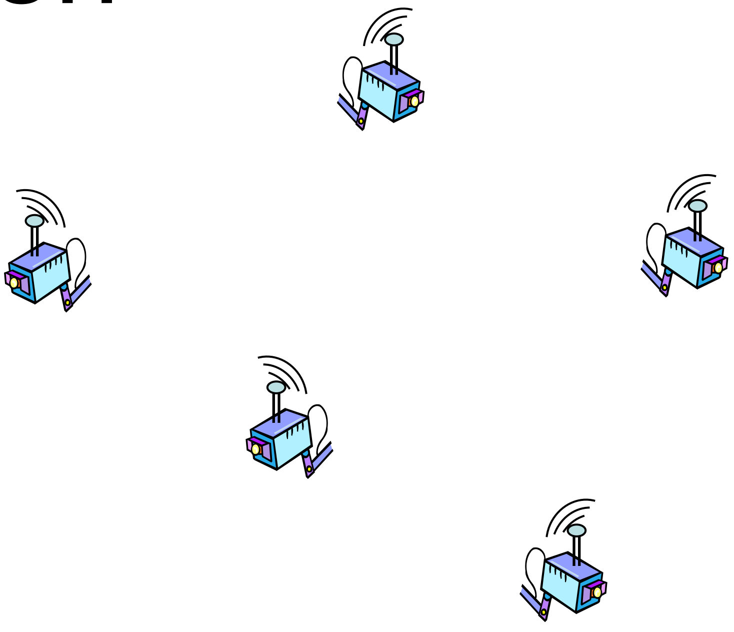
Motivation

- Wireless sensor networks:
  - Provide only information of interest, not the entire data



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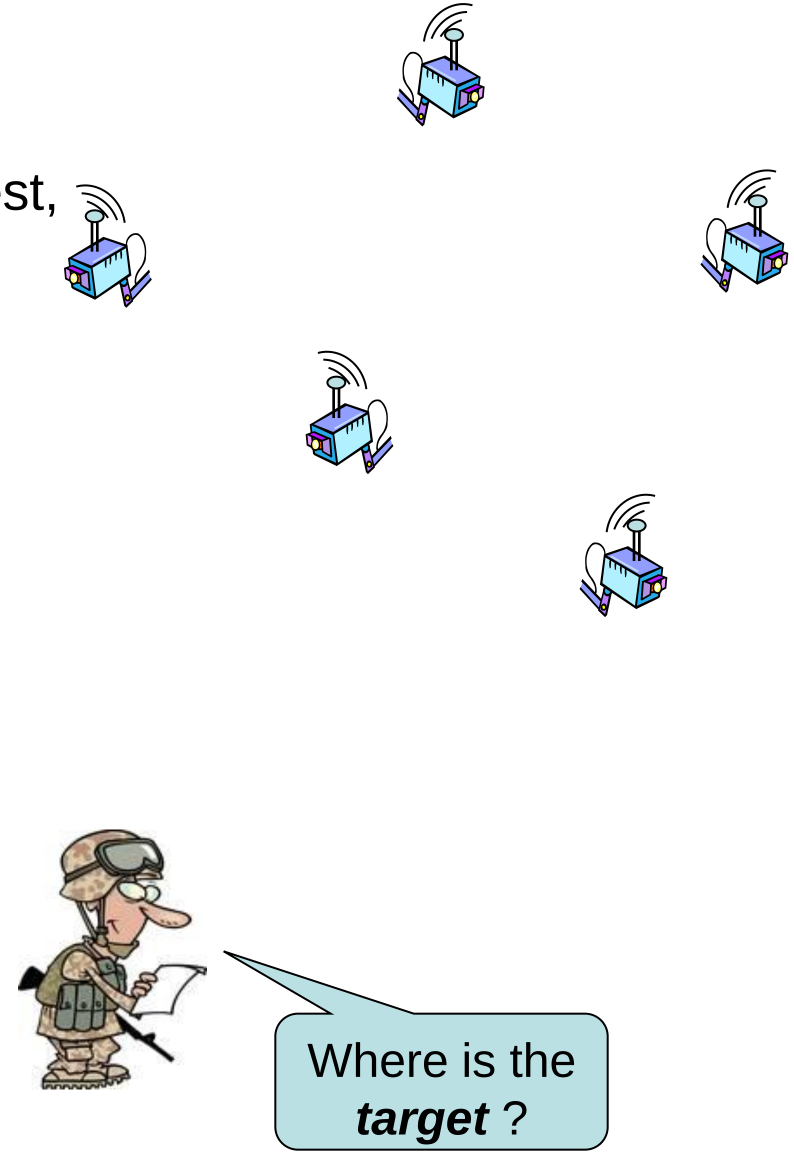

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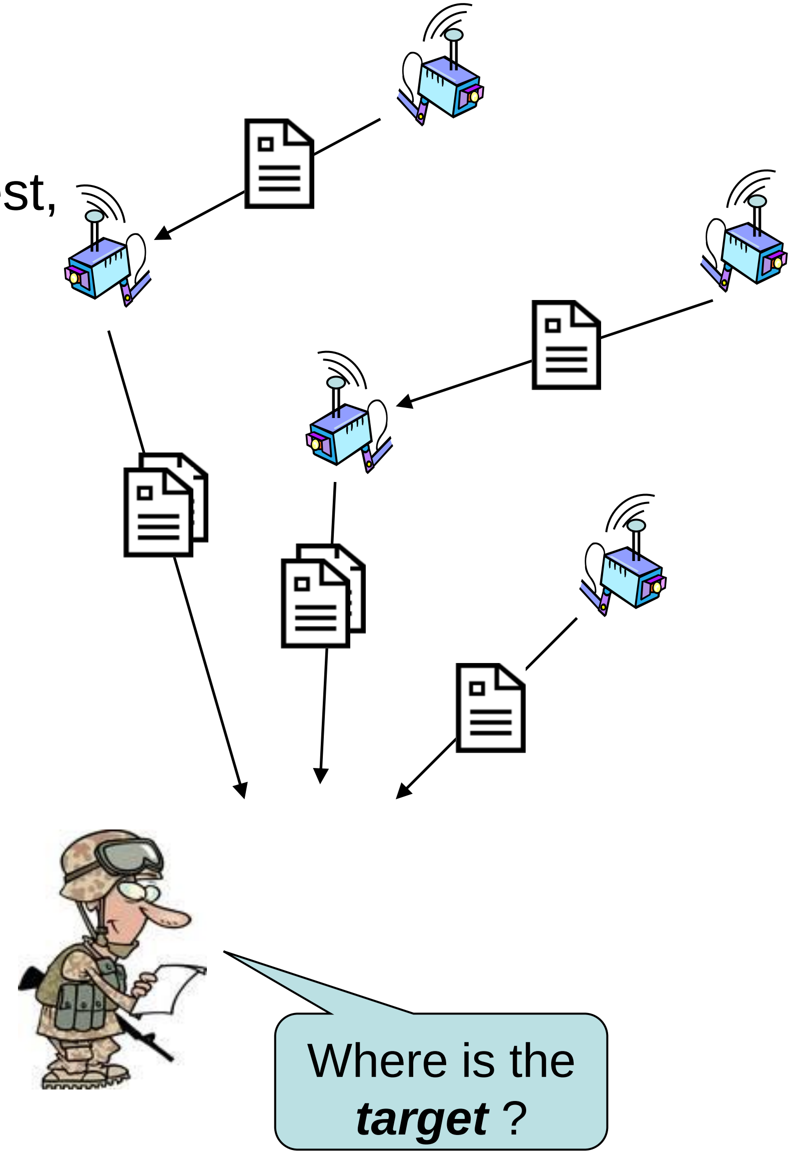

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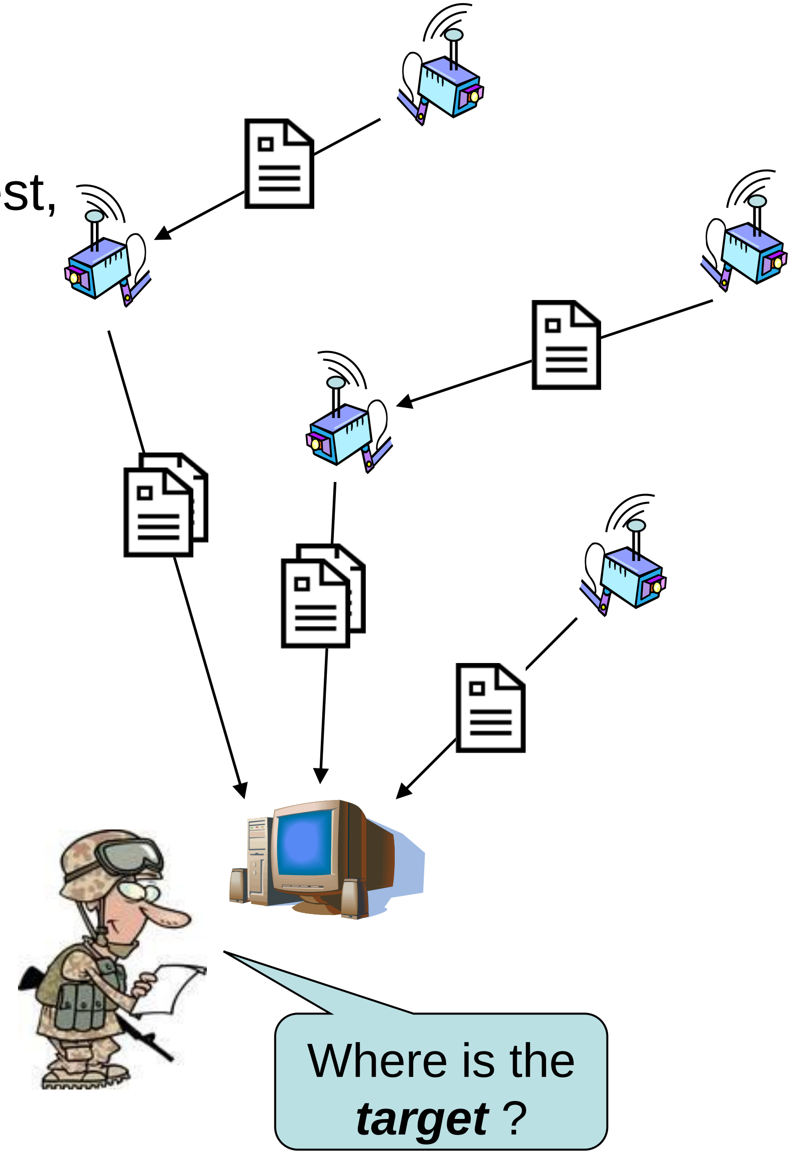

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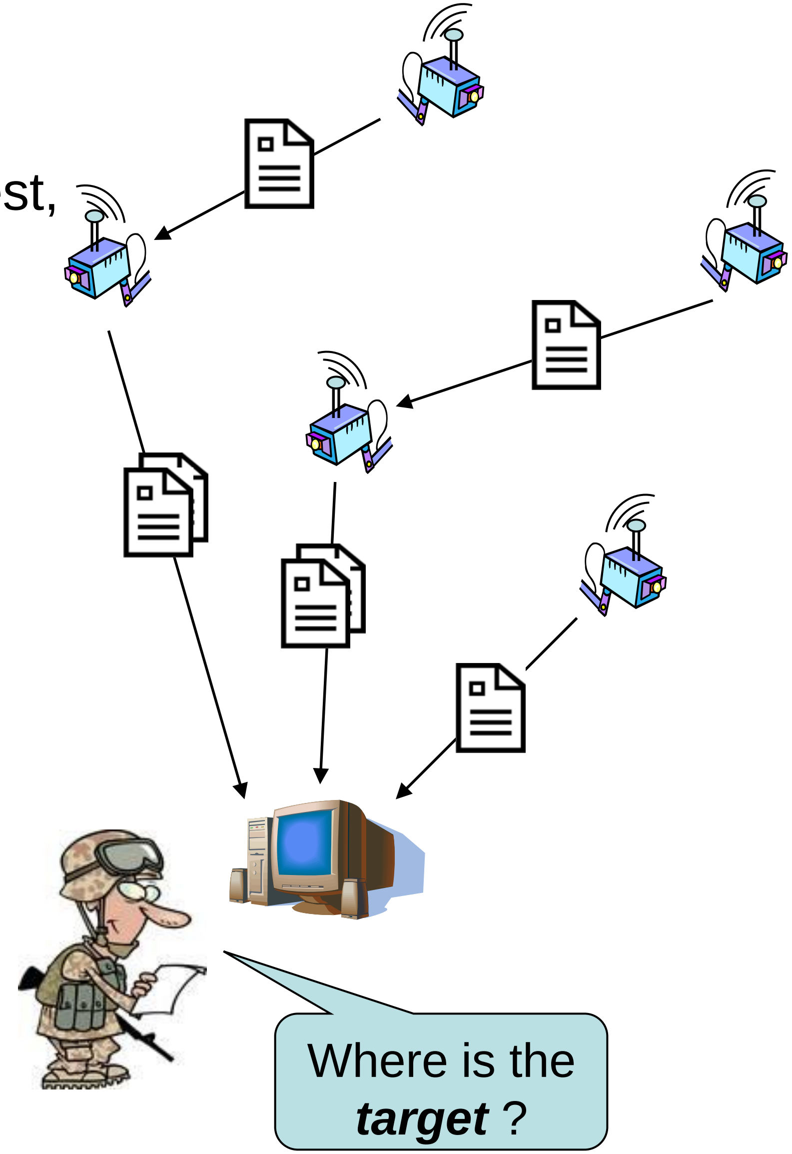
  - Inefficient communication



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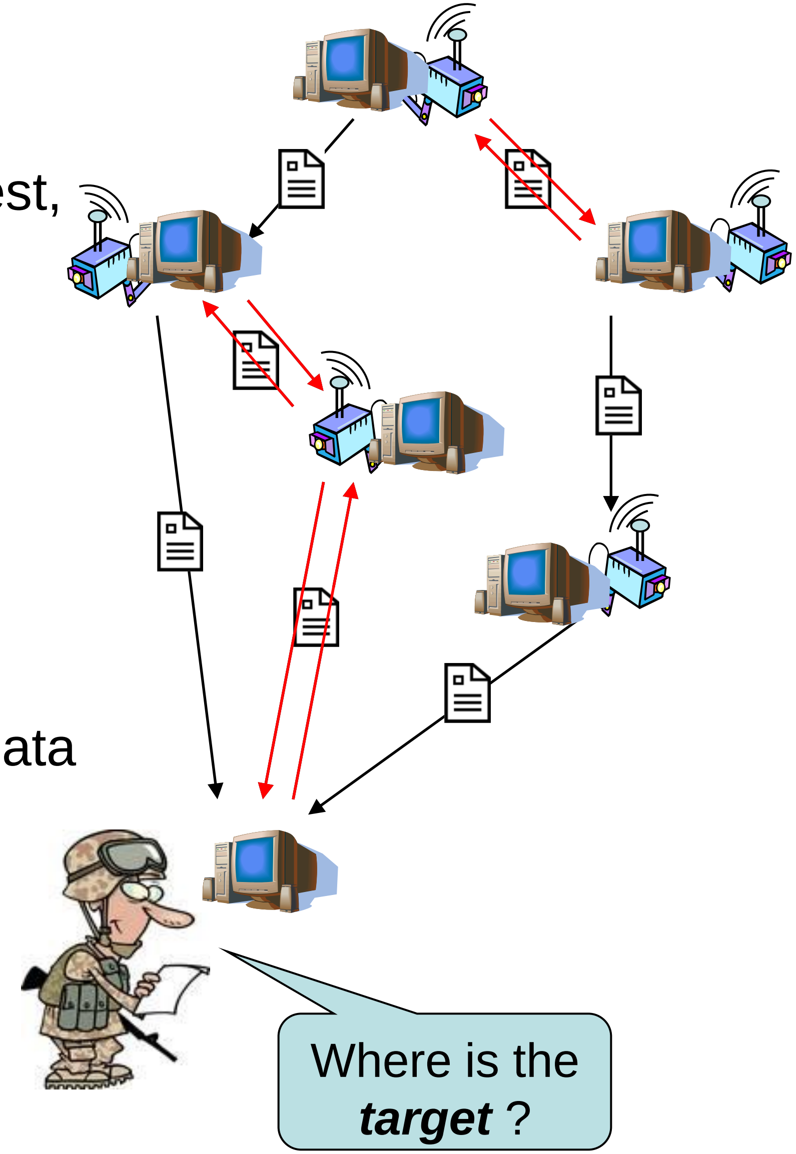
  - Distributed computing: process data as it moves



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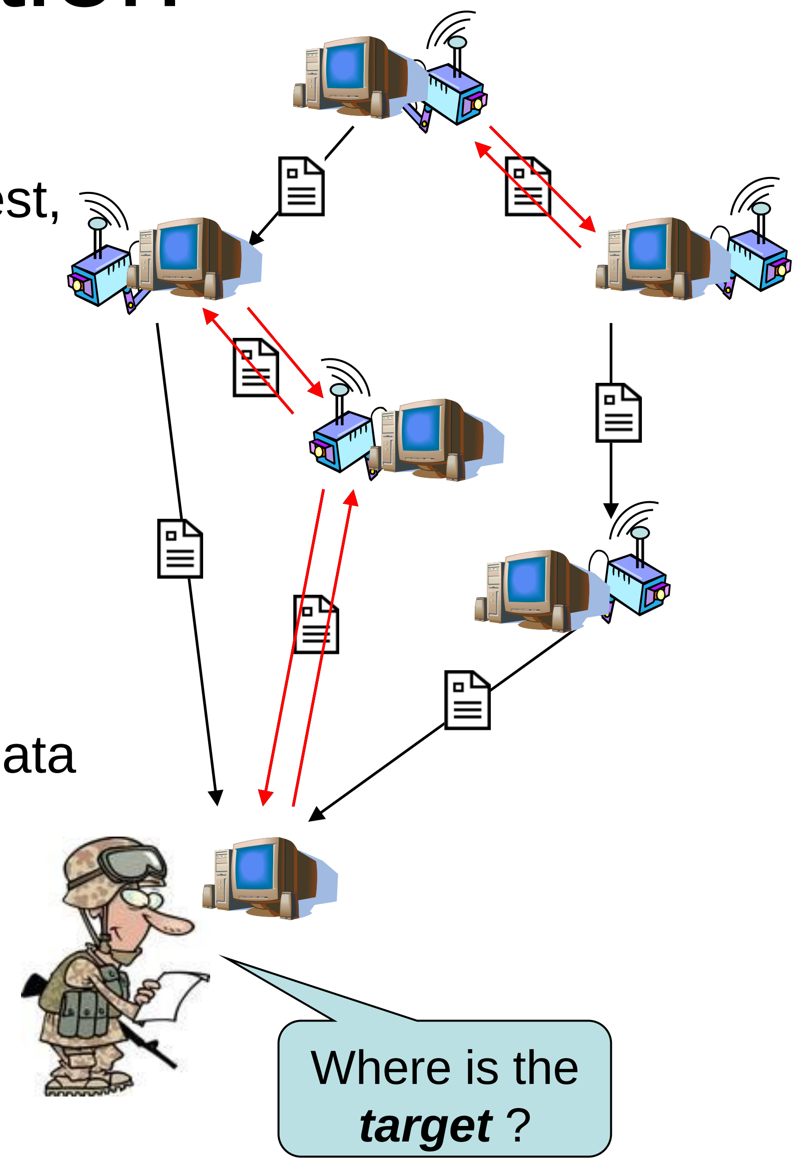

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- Traditional data networks:
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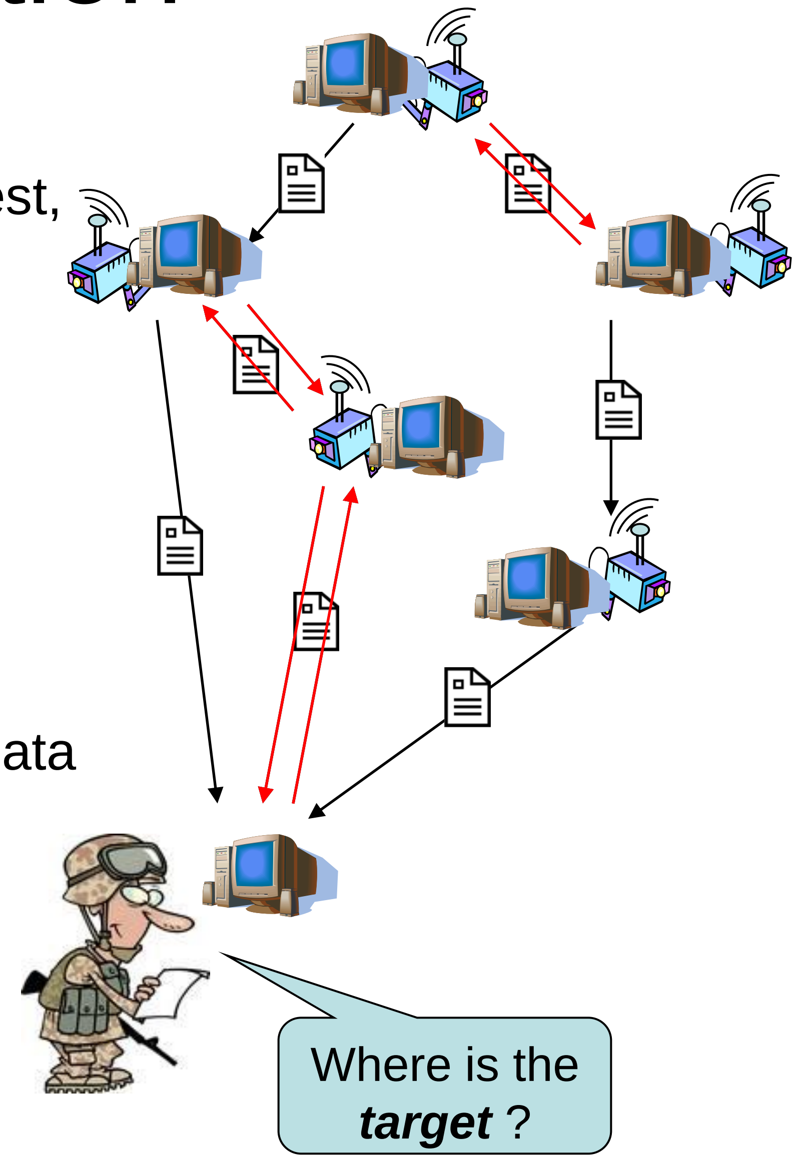
- In-network computing:
  - Distributed computing: process data as it moves
  - Efficient communication
  - Two-way communication (interaction)



Motivation

- Is interaction useful?

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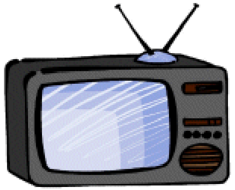

Without interaction (one-way):
Inefficient communication

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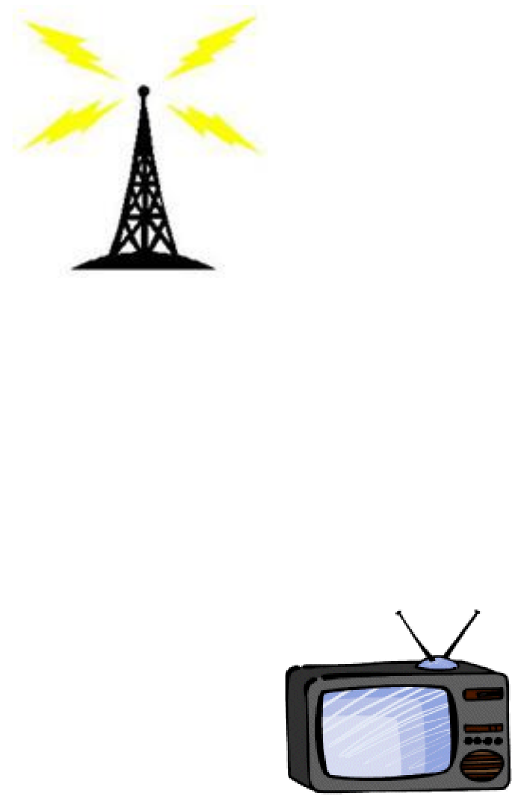

With interaction (two-way):
Efficient communication

Motivation

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Without interaction (one-way):
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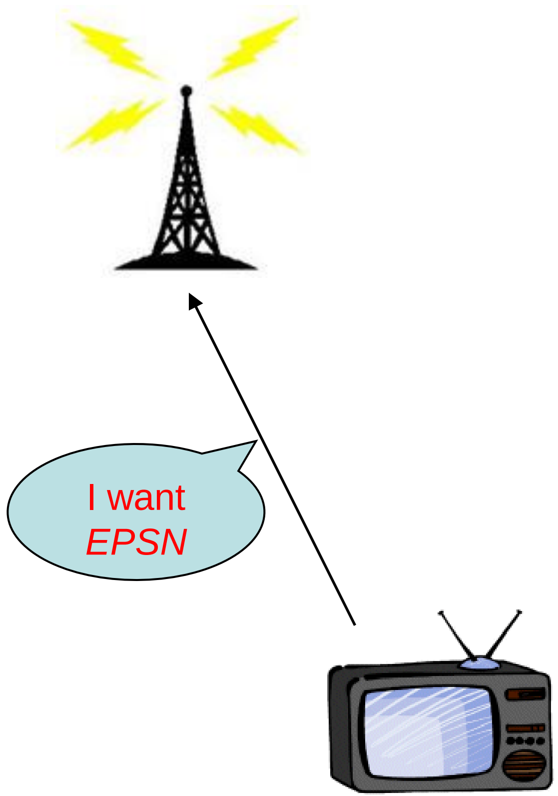

With interaction (two-way):
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Without interaction (one-way):
Inefficient communication

With interaction (two-way):
Efficient communication

Motivation

- Under what conditions is interaction useful?

Motivation

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- How useful is interaction?

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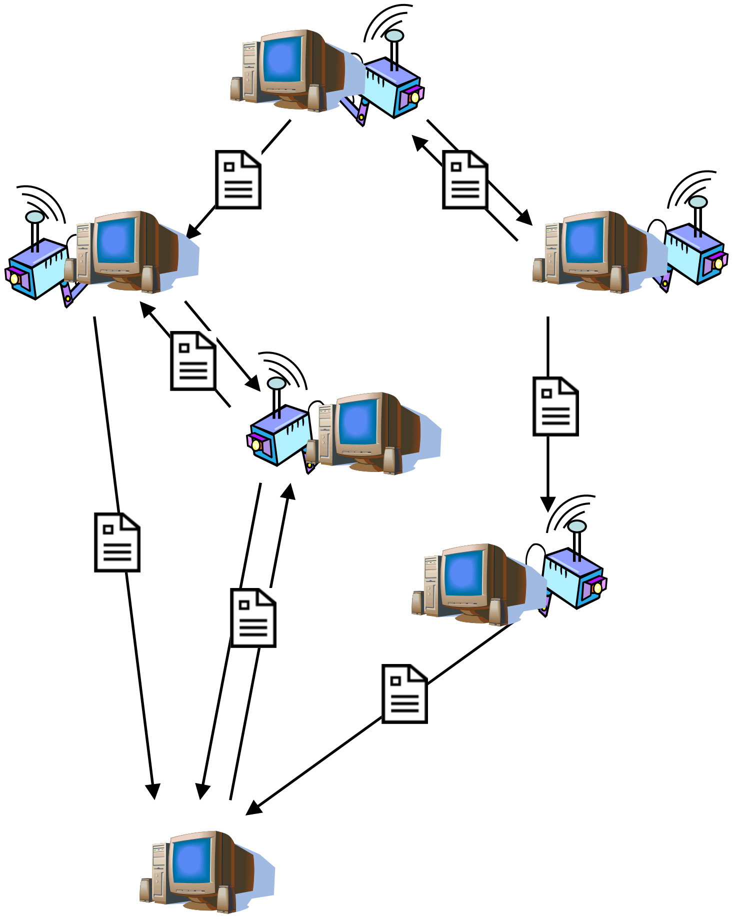
- Under what conditions is interaction useful?
- How useful is interaction?
- What is the best way to interact?
  - At *what time*, *who* should send *what* message to *whom*?



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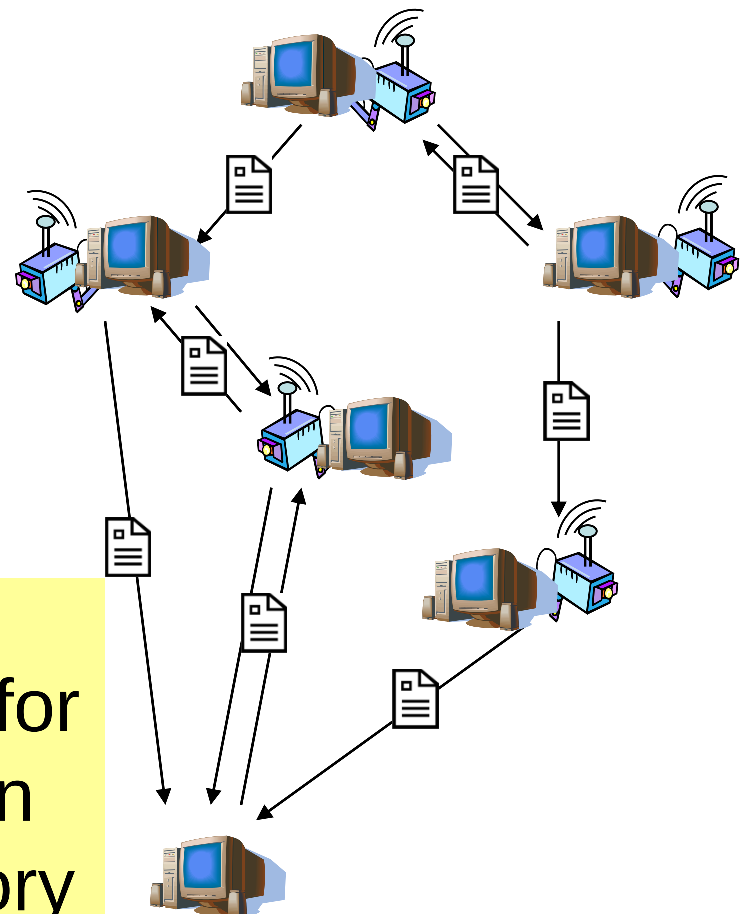

<u>Main Goal:</u>

Explore the benefit of *interaction* for distributed *function computation* in the framework of information theory

Related work

- Communication complexity (Yao, …, [Kushilevitz & Nisan])

There	Here
Focus on Pr(comp.error) = 0	• Pr(comp.error) $\rightarrow$ 0 as #samples $\rightarrow \infty$ • E [distortion] $\leq D$
Bits	Rate (bits per source sample)

- Two-way source coding [Kaspi'86]

There	Here
Source reproduction	Function computation
No example to show interaction useful	Many examples show interaction useful

- Coding for computing [Orlitsky & Roche'00]

There	Here
Two terminals, two messages	Multiple terminals, t messages

Related work (continued 1)

- Scaling laws of max. rate of computation [Giridhar & Kumar'05]
  - For divisible, type-sensitive, type-threshold function classes
  - In random planar and collocated networks
  - Communication complexity flavor … (Here: distributed block source coding flavor)
  - Pr(compute. error) = 0 … (Here: Pr(comput. error) $\rightarrow$ 0 as #samples $\rightarrow \infty$, and also expected distortion criteria)

- Networks of finite max degree [Subramanian, Gupta, & Shakkottai'07]
  - Further subdivision of type-sensitive function class
  - If allow Pr(sample error) < ε then for some type-sensitive functions like AVERAGE, computation rate increases to type-threshold class

- Acyclic networks [Appuswami, Franceschetti, Karamchandani, & Zeger'07]
  - No interaction over multiple rounds of communication
  - Min-cut bound, tight for divisible functions in multi-edge tree networks
  - Bound not tight in general

Related work (continued 2)

- CEO-style rate-distortion problem [Prabhakaran, Ramchandran, & Tse’04]
  - Multiple rounds of communication
  - Conditioned on desired (hidden) source, observations of agents are independent
  - Lower bound on minimum sum-rate (for given distortion)
  - Bound tight for jointly Gaussian sources and MSE

- Network coding [many refs. too long to list]
  - Mainly non interactive, focus on data dissemination than function computation

- Gossip/Consensus algorithms [many refs]
  - Single sample at each node (zero block-coding rate)
  - Real-valued message exchanges (infinite # bits); With quantization: [Kashyap, Basar, & Srikant]
  - Focus on rate of convergence

- Computation over noisy channels
  - [Nazer & Gastpar]: for specialized classes of “matched” source-channel pairs
  - [Gallager’88], [Ying,Srikant,&Dullerud’07], [Ayaso,Shah,&Dahleh’08]: single sample at each node (zero block-coding rate)

Related work (continued 3)

- Network communication problems with conferencing decoders
- Secrect key agreement problems with public discussion [U. Maurer et al., I.Csiszar, P.Narayanan et al.]
- Feedback problems
- Secure multi-party computation problems [large CS theory literature]
- …

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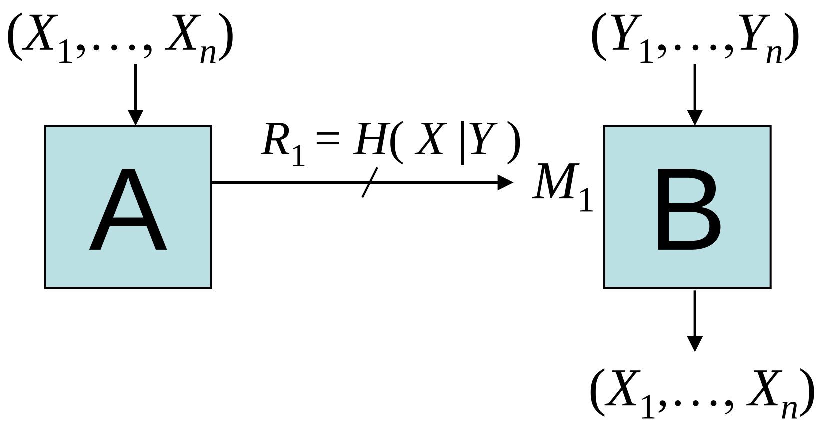
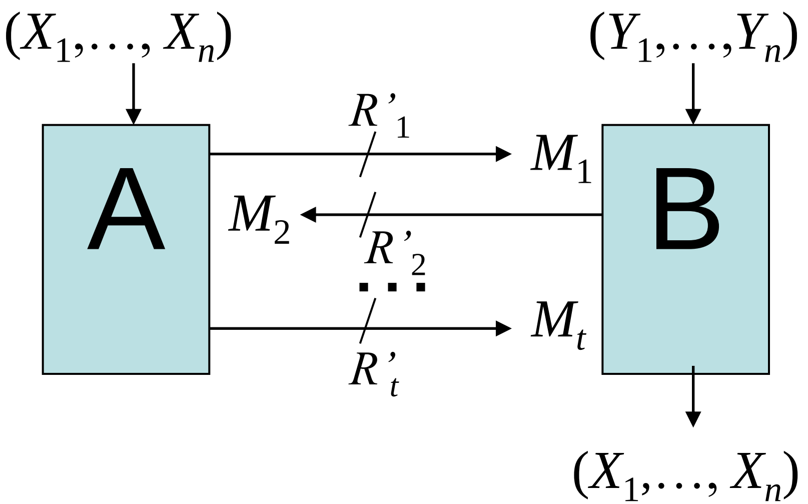
Example 1: Source reproduction

- *Goal:* only B reproduces $(X_1,\ldots,X_n):\ \ f_B(x,y)=x,\ \ f_A(x,y)=0$



Single msg (t =1)

$R_1 = H\,(\,X\,|\,Y\,)$ (Slepian-Wolf coding)



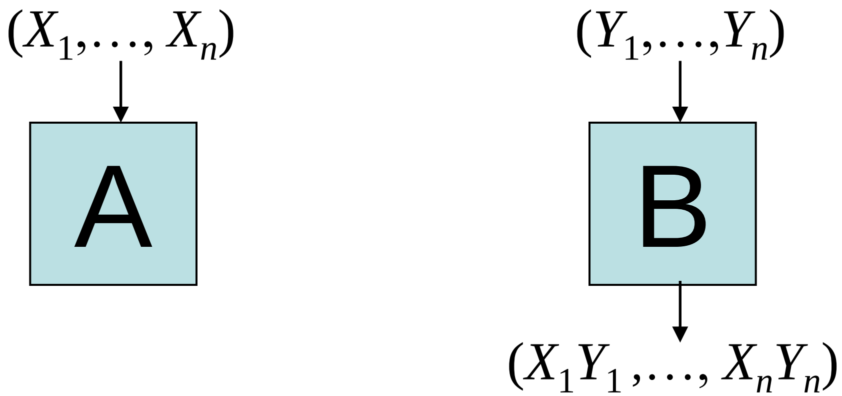
Multiple msgs ($t \geq 1$)

$R'_1+\ldots+R'_t \geq H\,(\,f_B(X,Y)\,|Y\,) = H\,(\,X\,|Y\,)$

- ***No benefit in multiple messages***
- If A also reprod. $\mathbf{Y}$, at least two msgs. No benefit to use $t > 2$ msgs
- *Caveat:* interaction still beneficial if $\Pr(error) = 0$ [Orlitsky, et al] or for faster rate of convergence and for *nonergodic* sources [Da-ke He, et al]

Example 2: Function computation

- Indep. sources: $X \sim$ Uniform$\{1, 2, \ldots, L\}$, $Y \sim$ Bernoulli (p)
- Only B reproduces XY



$f_B(x, y)$	$y = 0$	$y = 1$
$x = 1$	0	1
$x = 2$	0	2
…	…	…
$x = L$	0	L

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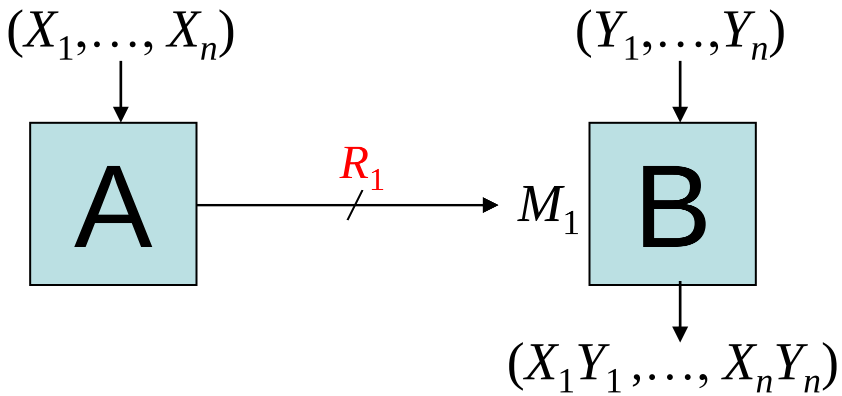

Single msg: $R_1 = ?$

$f_B(x, y)$	$y = 0$	$y = 1$
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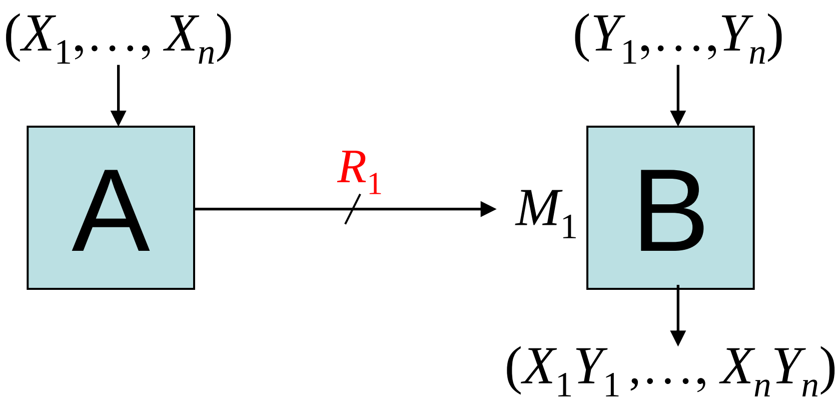

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…	…	…
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Han & Kobayashi (1987):

If for any (x, y), $p_{XY}(x, y) > 0$ and any two rows are different, then

$R_1 \geq H(X \mid Y)$ no better than sending $\mathbf{X}$ completely!

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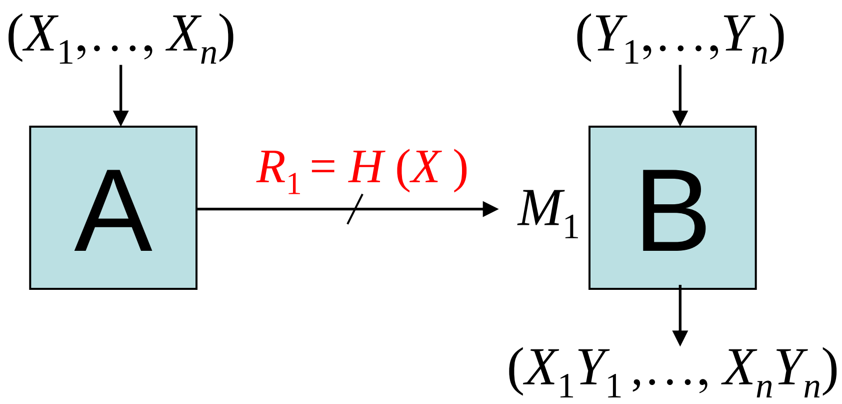

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[Han, Kobayashi, 1987]

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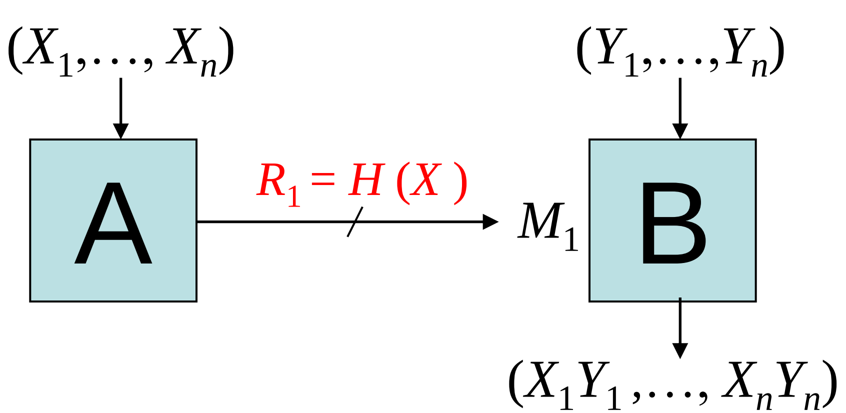

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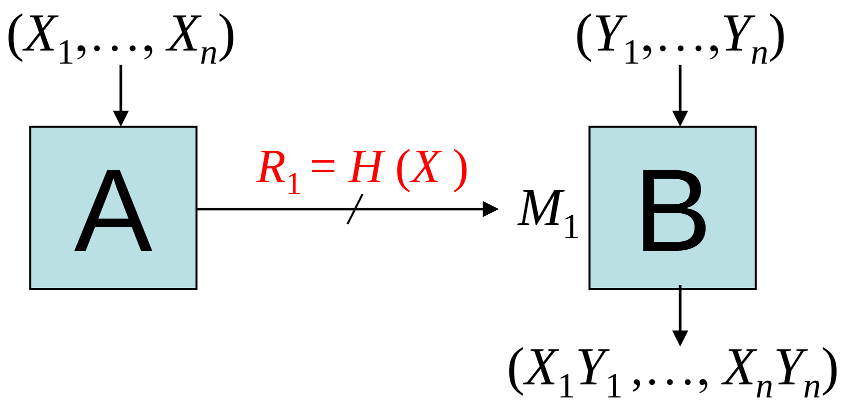

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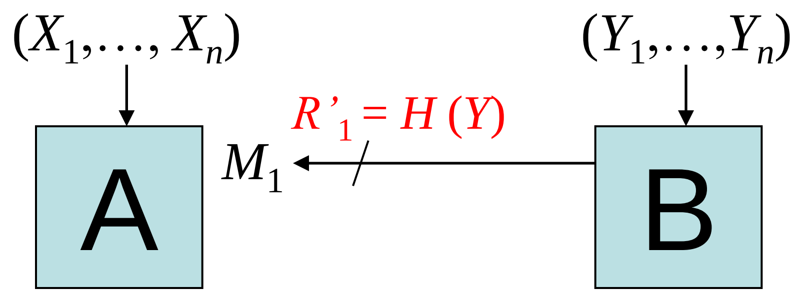

1st msg: compress **Y**

Example 2: Function computation

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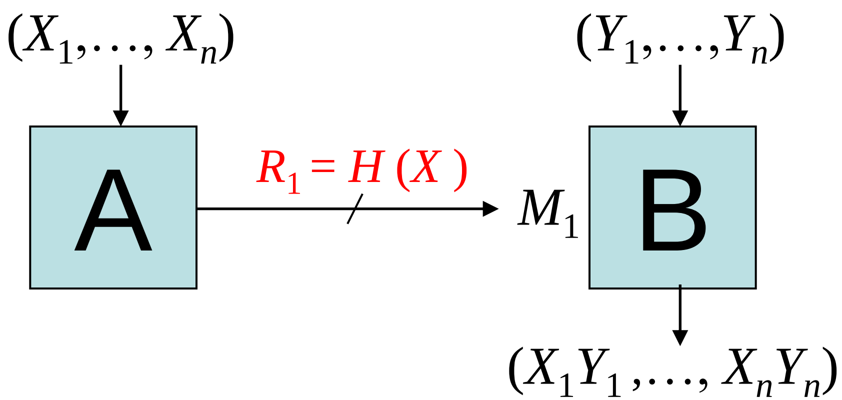

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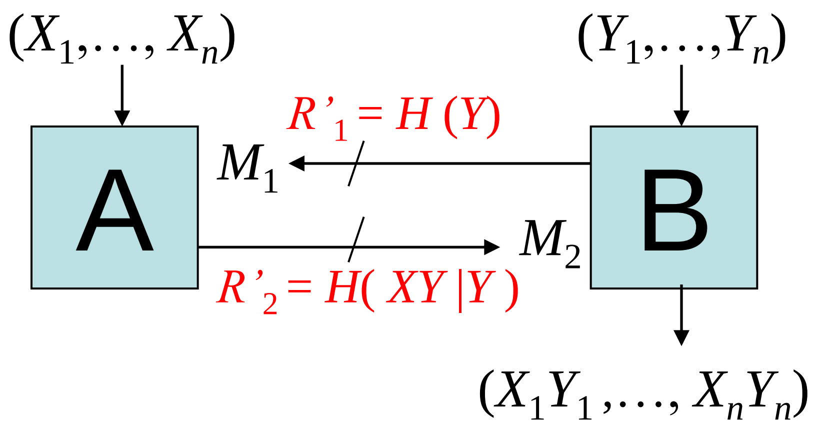

1st msg: compress **Y**

2nd msg: send **X** only if $Y = 1$

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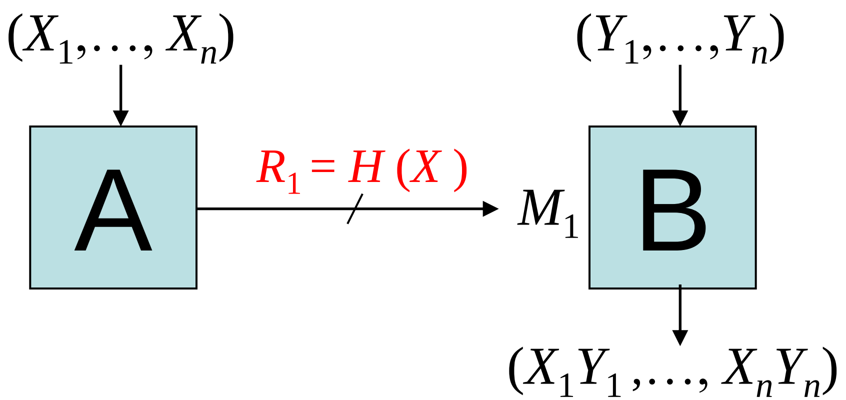

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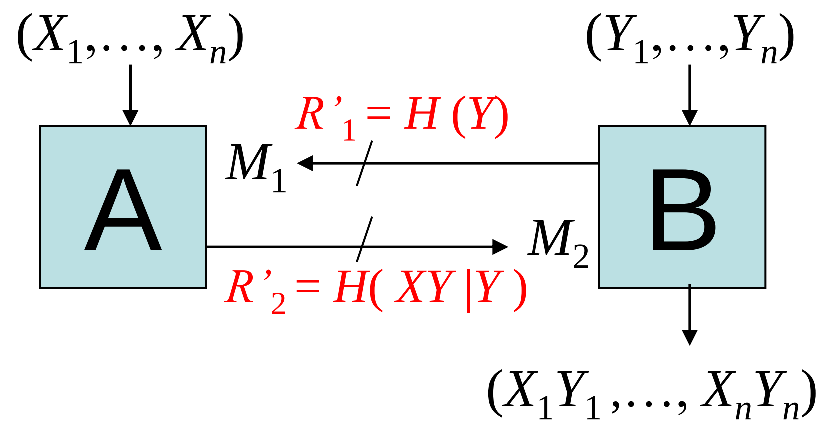

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$R_1 = \log_2 L$ *strictly* > $R'_1 + R'_2 = h_2(p) + p \log_2 L$

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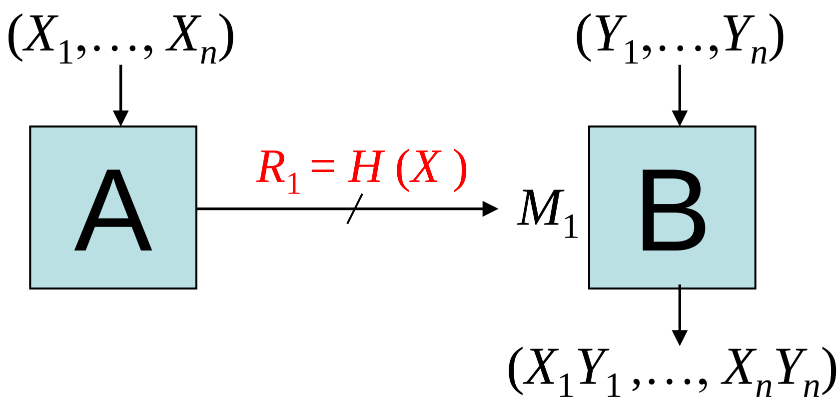

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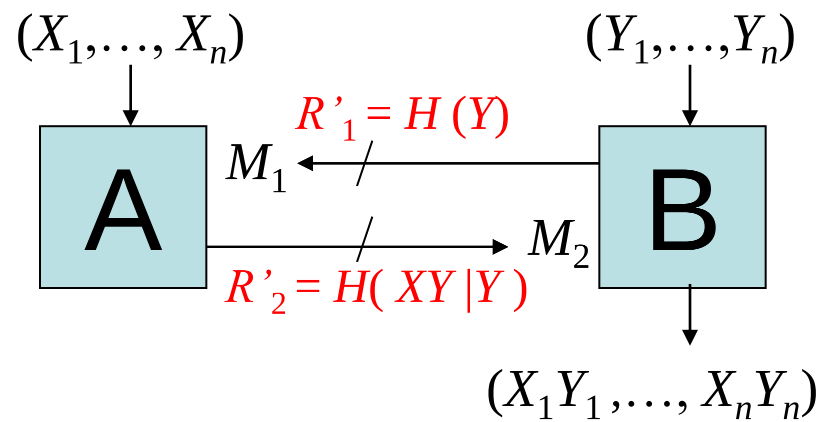

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$R_1 = \log_2 L$ *strictly* > $R'_1 + R'_2 = h_2(p) + p \log_2 L$

- ***Even for indep. sources, interaction gain can be arbitrarily large***

General two-terminal problem

- 2-component DMS source, 2 locations:

  $(X_i, Y_i) \sim \text{iid } p_{XY}$

- Samplewise function computation:

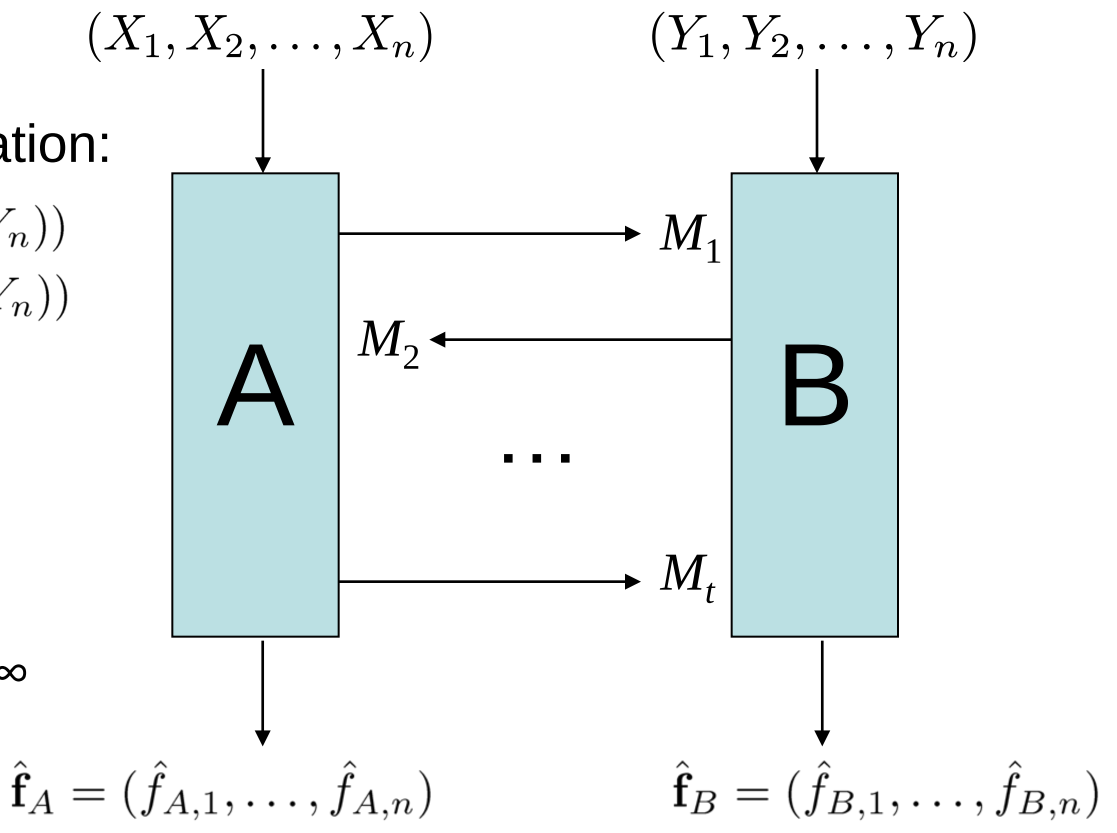
$$\begin{aligned}\mathbf{f}_A &= (f_A(X_1,Y_1),\ldots,f_A(X_n,Y_n)) \\ \mathbf{f}_B &= (f_B(X_1,Y_1),\ldots,f_B(X_n,Y_n))\end{aligned}$$

- t alternating messages

- In this talk, focus on:

  $\Pr$(comp. error) $\rightarrow 0$ as $n \rightarrow \infty$

- Can also handle coupled single-letter distortion



Two-terminal interaction

- Admissible rate-tuple $(R_1, \ldots, R_t)$:

  Exists a sequence of codes:

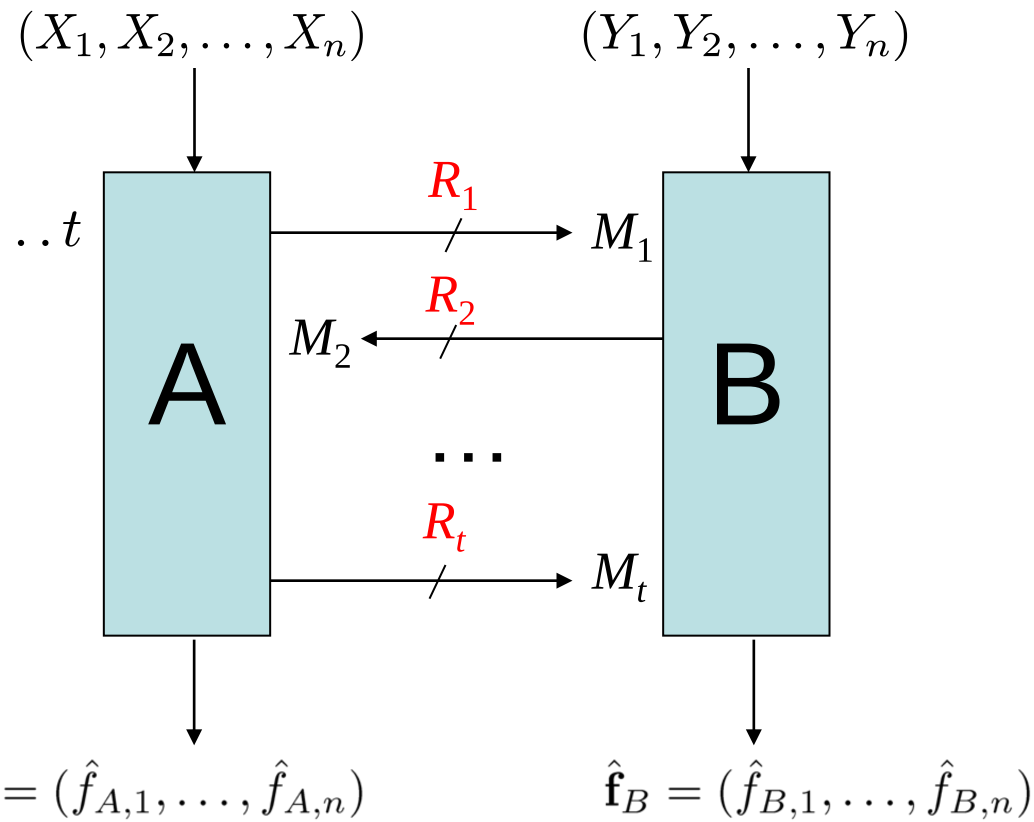
  as $n \to \infty$

$$(\#\text{ bits msg } j)/n \to R_j, \;\; j = 1 \ldots t$$

$$\Pr[(\mathbf{f}_A \neq \hat{\mathbf{f}}_A) \text{ or } (\mathbf{f}_B \neq \hat{\mathbf{f}}_B)] \to 0$$

- Rate region $\mathcal{R}_t^A$:

  set of admissible rate-tuples



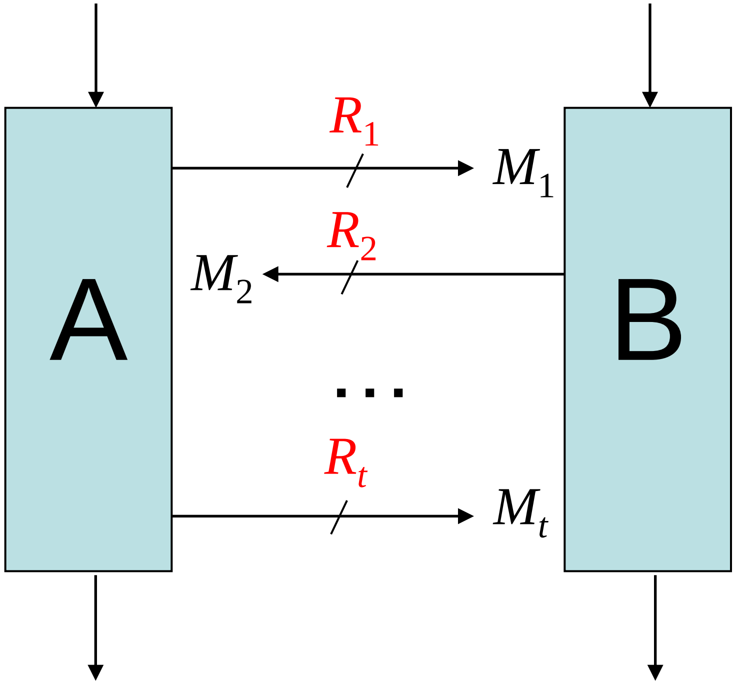
Need:

$$\mathbf{f}_A = (f_A(X_1, Y_1), \ldots, f_A(X_n, Y_n)), \quad \mathbf{f}_B = (f_B(X_1, Y_1), \ldots, f_B(X_n, Y_n))$$

M_1

Two-terminal interaction

$$(X_1, X_2, \ldots, X_n) \qquad (Y_1, Y_2, \ldots, Y_n)$$



$$\hat{\mathbf{f}}_A = (\hat{f}_{A,1}, \ldots, \hat{f}_{A,n}) \qquad \hat{\mathbf{f}}_B = (\hat{f}_{B,1}, \ldots, \hat{f}_{B,n})$$

<u>Goals:</u>

1) Obtain a computable characterization of the rate region (limit-free and independent of n)

2) Understand the benefit of interaction for different sources and functions

Need:

$$\mathbf{f}_A = (f_A(X_1, Y_1), \ldots, f_A(X_n, Y_n)), \quad \mathbf{f}_B = (f_B(X_1, Y_1), \ldots, f_B(X_n, Y_n))$$

Information-theoretic rate region

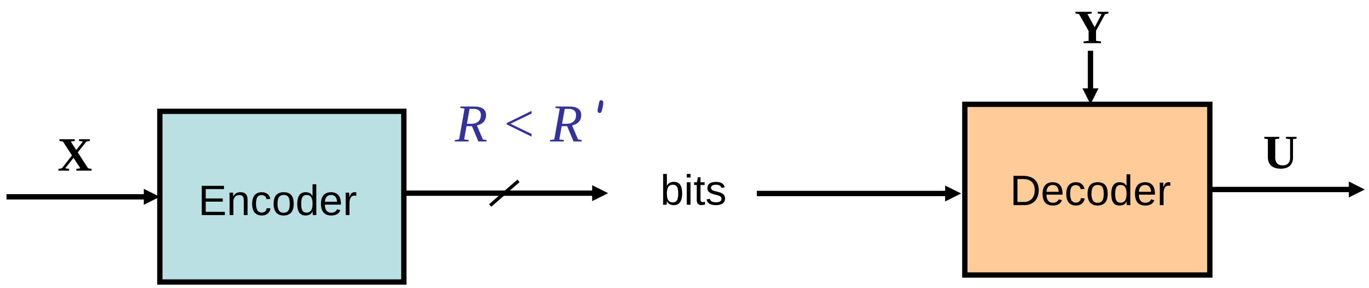
[Nan Ma & PI: ISIT'08, IT'11]

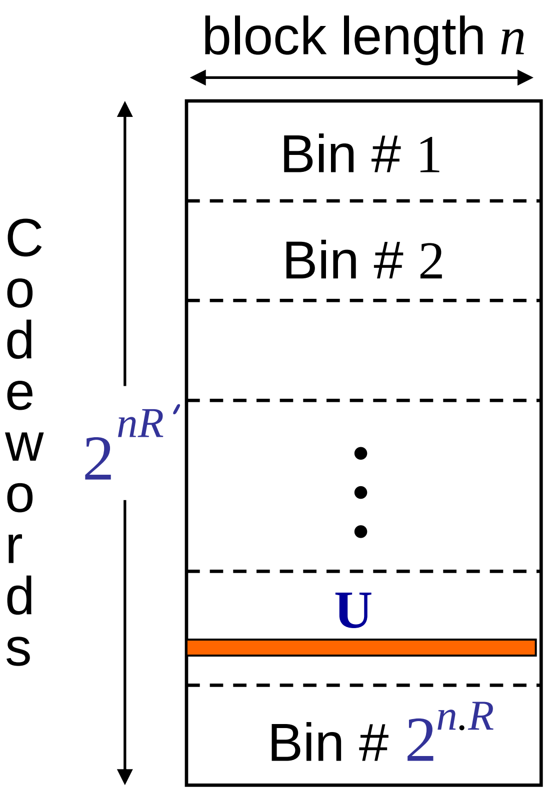
$$\mathcal{R}_t^A = \{ \ (R_1 \ldots R_t) \mid \exists \, U^t, s.t. \ |\mathcal{U}_i| \cdot \ \ g_i(|\mathcal{X}|, |\mathcal{Y}|),$$

$$R_i \geq \begin{cases} I(X;U_i|Y,U^{i-1}), \ U_i - (X,U^{i-1}) - Y, & i \text{ odd} \\ I(Y;U_i|X,U^{i-1}), \ U_i - (Y,U^{i-1}) - X, & i \text{ even} \end{cases}$$

$$H(f_A(X,Y)|X,U^t) = 0, \ H(f_B(X,Y)|Y,U^t) = 0 \ \}$$

Wyner-Ziv coding





$R' \geq I(X;U)$ $\qquad$ $R \geq I(X;U \mid Y)$, U - Y - X

- $2^{nR'}$ codewords split into 2^{nR} bins
- Encoder sends only bin-index => rate reduced from R' to R but extra confusion for decoder
- Decoder uses **Y** to disambiguate entries in bin
- Encoding and decoding: find statistically most consistent codeword
- Notion of decoding failure

Information-theoretic rate region

$$\mathcal{R}_t^A = \{\ (R_1 \ldots R_t) \mid \exists\, U^t, s.t.\ |\mathcal{U}_i| \cdot \quad g_i(|\mathcal{X}|, |\mathcal{Y}|),$$

$$R_i \geq \begin{cases} I(X; U_i | Y, U^{i-1}),\ U_i - (X, U^{i-1}) - Y, & i \text{ odd} \\ I(Y; U_i | X, U^{i-1}),\ U_i - (Y, U^{i-1}) - X, & i \text{ even} \end{cases}$$

$$H(f_A(X,Y)|X, U^t) = 0,\ H(f_B(X,Y)|Y, U^t) = 0\ \}$$

Achievability (sequence of Wyner-Ziv codes):

- 1st msg: Quantizes $\mathbf{X}$ to $\mathbf{U}_1$ with side info $\mathbf{Y}$

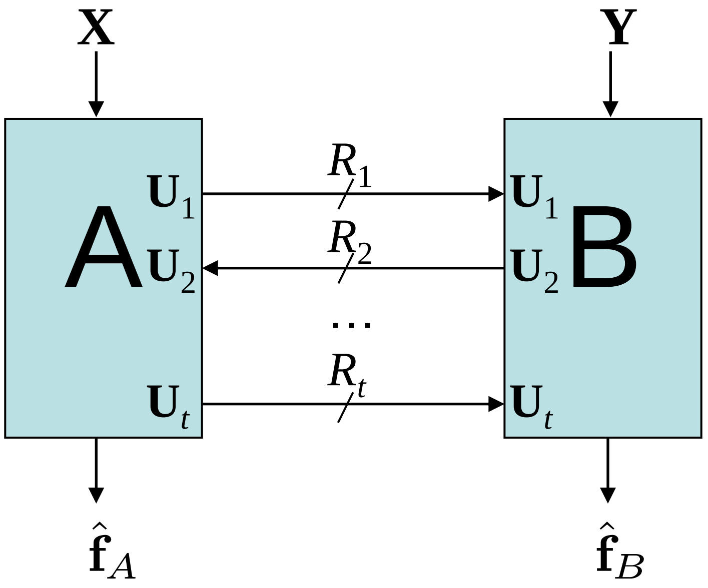
  $$R_1 = I(X; U_1|Y), \quad U_1 - X - Y$$

- 2nd msg: Quantizes $(\mathbf{Y}, \mathbf{U}_1)$ to $\mathbf{U}_2$ with side info $(\mathbf{X}, \mathbf{U}_1)$

  $$R_2 = I(Y; U_2|X, U_1), \quad U_2 - (Y, U_1) - X$$

  ....

- Recover $\mathbf{f}_A$ based on $(\mathbf{X}, \mathbf{U}_1 \ldots \mathbf{U}_t)$: $H(f_A|X, U_1 \ldots U_t) = 0$
- Recover $\mathbf{f}_B$ based on $(\mathbf{Y}, \mathbf{U}_1 \ldots \mathbf{U}_t)$: $H(f_B|Y, U_1 \ldots U_t) = 0$



Information-theoretic rate region

$$\mathcal{R}_t^A = \{ \ (R_1 \dots R_t) \mid \exists\, U^t, s.t.\ |\mathcal{U}_i| \cdot \ \ g_i(|\mathcal{X}|, |\mathcal{Y}|),$$
$$R_i \geq \begin{cases} I(X; U_i | Y, U^{i-1}),\ U_i - (X, U^{i-1}) - Y, & i \text{ odd} \\ I(Y; U_i | X, U^{i-1}),\ U_i - (Y, U^{i-1}) - X, & i \text{ even} \end{cases}$$
$$H(f_A(X,Y)|X,U^t) = 0,\ H(f_B(X,Y)|Y,U^t) = 0 \ \}$$

Converse (impossible to do better):

- Standard information inequalities
- Auxiliary random variables

$$U_{1,i} = (M_1, X_1, \dots, X_{i-1}, Y_{i+1}, \dots, Y_n)$$
$$U_2 = M_2,\ \dots,\ U_t = M_t$$

- Cardinality bounds on alphabets of auxiliary random variables

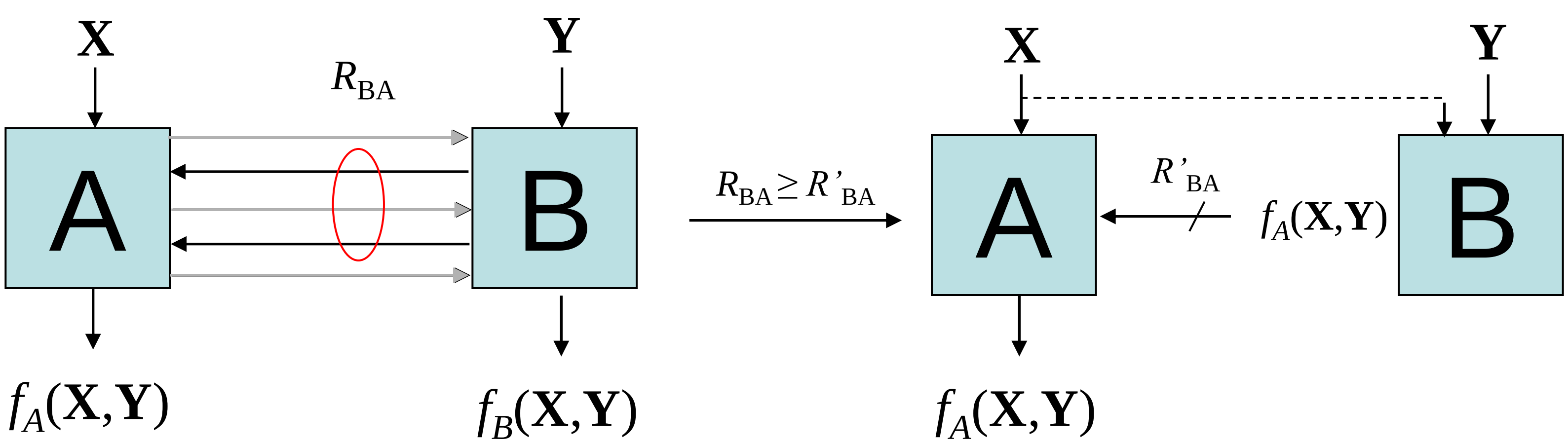
Minimum sum-rate

- *t*-msg min sum-rate: $R_{sum,t} = \min I(X;U^t|Y) + I(Y;U^t|X)$

aux. r. v. subject to

$$\begin{aligned} & U_i - (X, U^{i-1}) - Y, \ \ i \text{ odd} \\ & U_i - (Y, U^{i-1}) - X, \ \ i \text{ even} \\ & H(f_A(X,Y)|X,U^t) = 0 \\ & H(f_B(X,Y)|Y,U^t) = 0 \end{aligned}$$

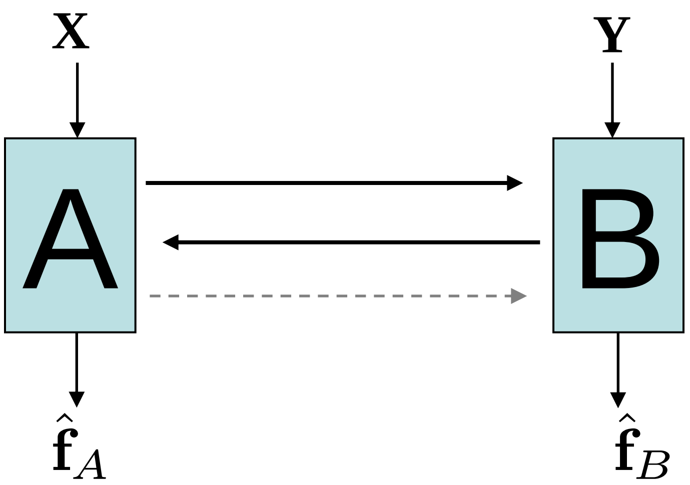
- <u>*Genie lower-bound:*</u> $R_{sum,t} \geq H(f_A(X,Y)|X) + H(f_B(X,Y)|Y)$



$$R_{BA} \geq R'_{BA} \geq H(f_A(X,Y)|X)$$

Minimum sum-rate

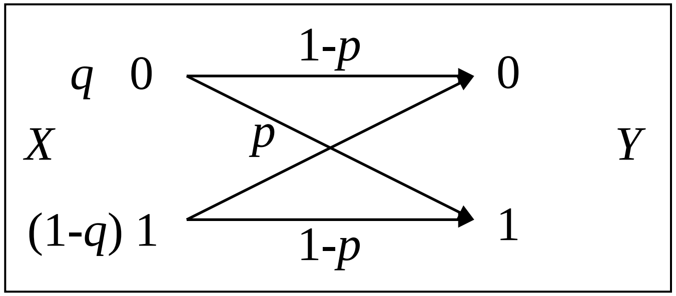
- $R_{sum,1} \geq R_{sum,2} \geq R_{sum,3} \geq \ldots \geq R_{sum,\infty}$
  - Each message could be a null message

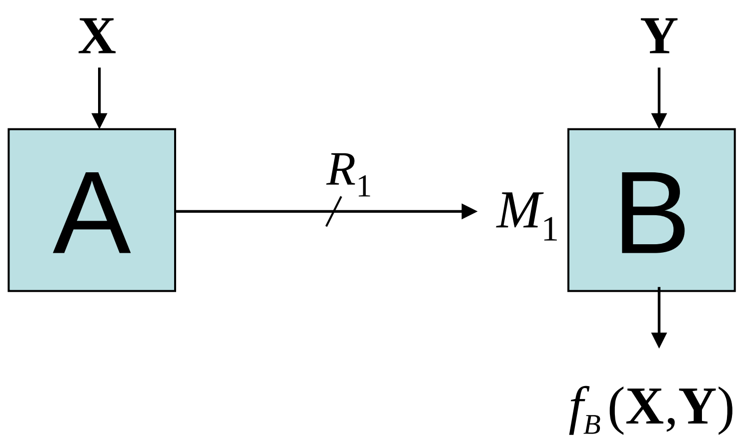


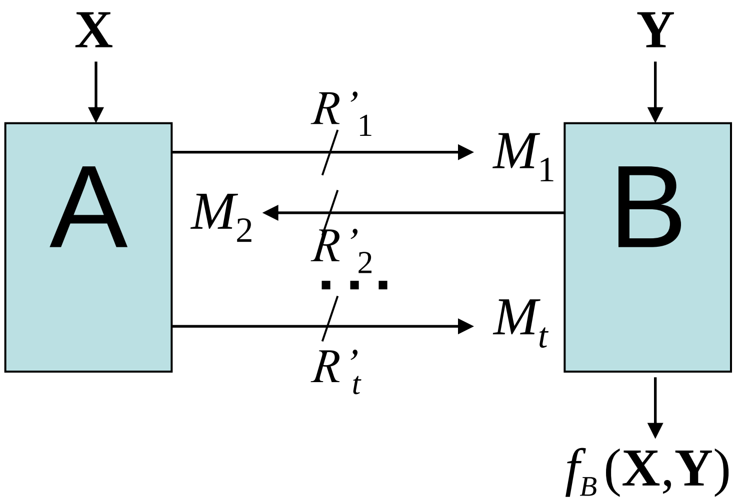
- For all finite t, $R_{sum,t}$ computable; $R_{sum,\infty}$ not.
- *Recent result:* a new functional characterization of $R_{sum,\infty}$
- *Opens new dimension of investigation:* message asymptotics with infinitesimal rate messages

Example 3: Effect of Distribution

- Correlated binary sources:
- Only B reprod. samplewise $f_B(\mathbf{X},\mathbf{Y})$







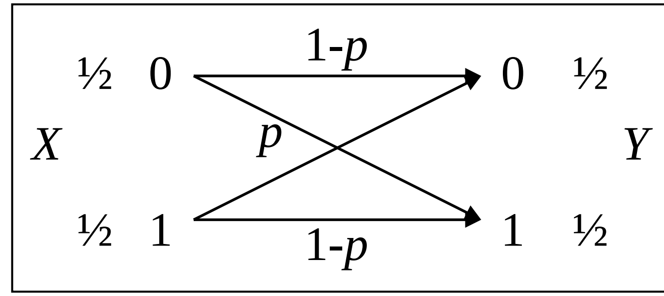
<u>Theorem:</u> *for <u>any</u> function* $f_B(x, y)$:

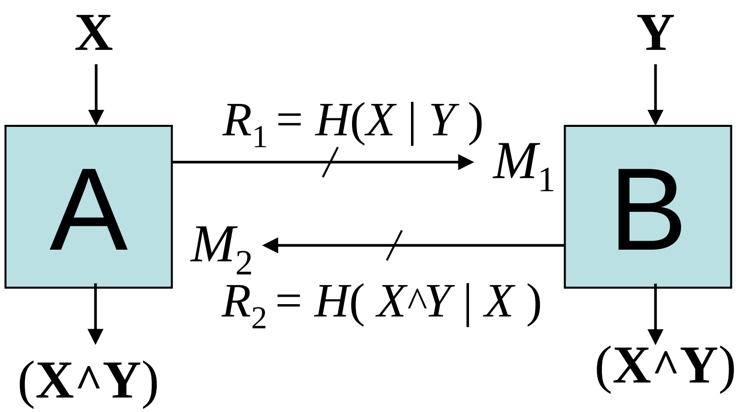
$$\min R_1 \quad = \quad \min\,(R'_1 + \ldots + R'_t)$$

- ***<u>Even ∞-msg interaction is not better than one-msg. comm.</u>***

Example 4: Effect of Demand

- Doubly symmetric binary sources ($q = 1/2$)
- Both sides reproduce **X^Y** (Boolean AND)

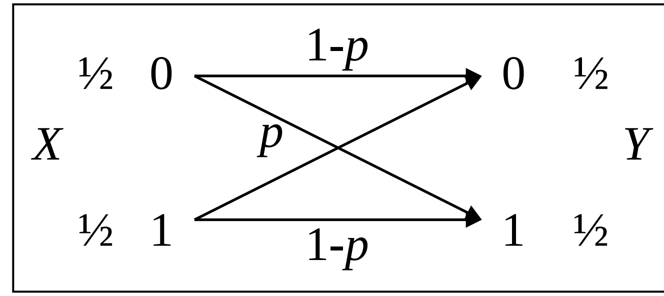


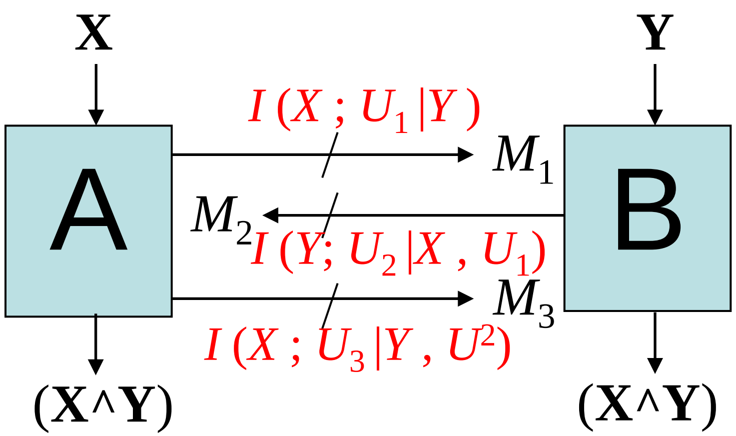


2 msgs

Example 4: Effect of Demand

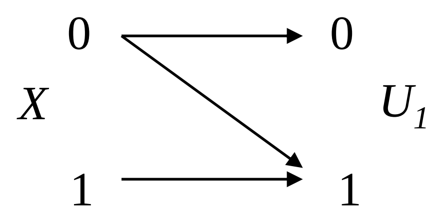
- Doubly symmetric binary sources ($q = 1/2$)
- Both sides reproduce **X^Y** (Boolean AND)





3 msgs

$\mathbf{U_1}$: part of zeros of **X**



$\mathbf{U_2}$: all of zeros of **Y**

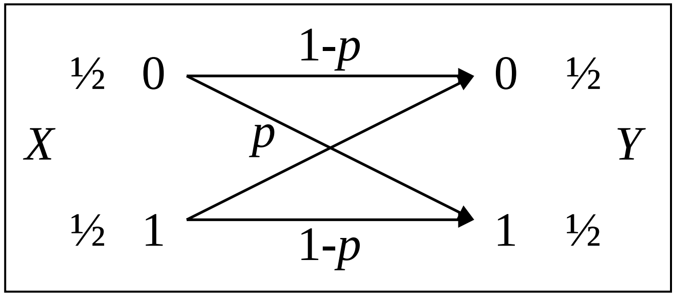
$$U_2 = Y \wedge U_1$$

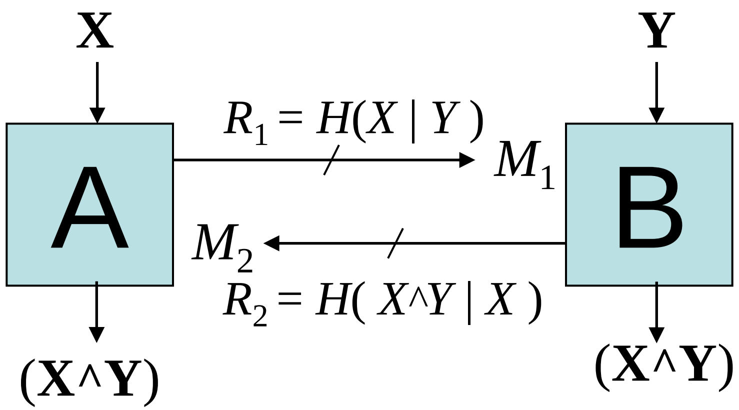
$\mathbf{U_3}$: **X^Y**

$$U_3 = X \wedge U_2 = X \wedge Y$$

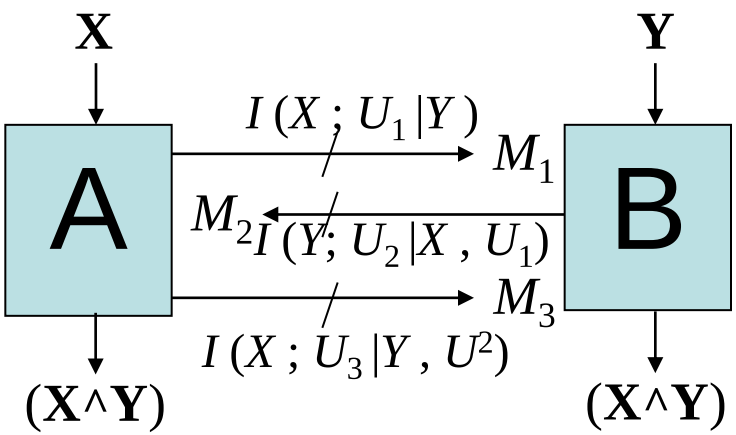
Example 4: Effect of Demand

- Doubly symmetric binary sources ($q = 1/2$)
- Both sides reproduce **X^Y** (Boolean AND)





2 msgs



3 msgs: strictly better than 2 msgs

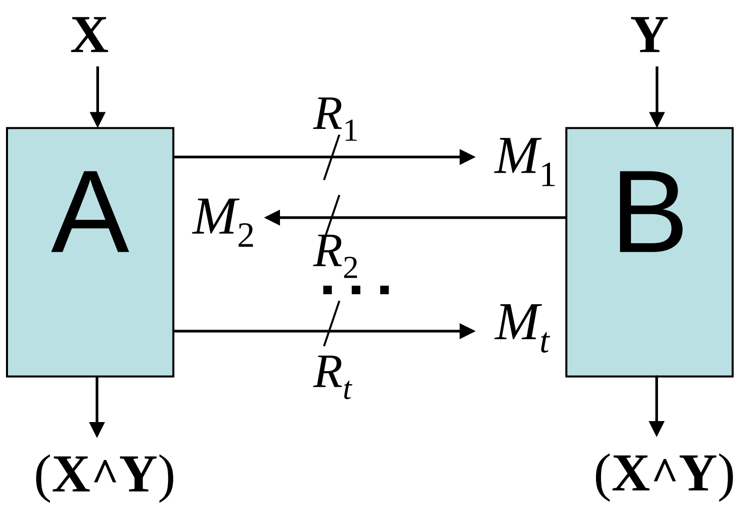
- E.g, $p = ½$, X, $Y \sim$ iid Ber (½), 2-msg: 1.5 vs 3-msg: 1.406
- ***<u>3 messages are better than 2 (interaction does help here)</u>***

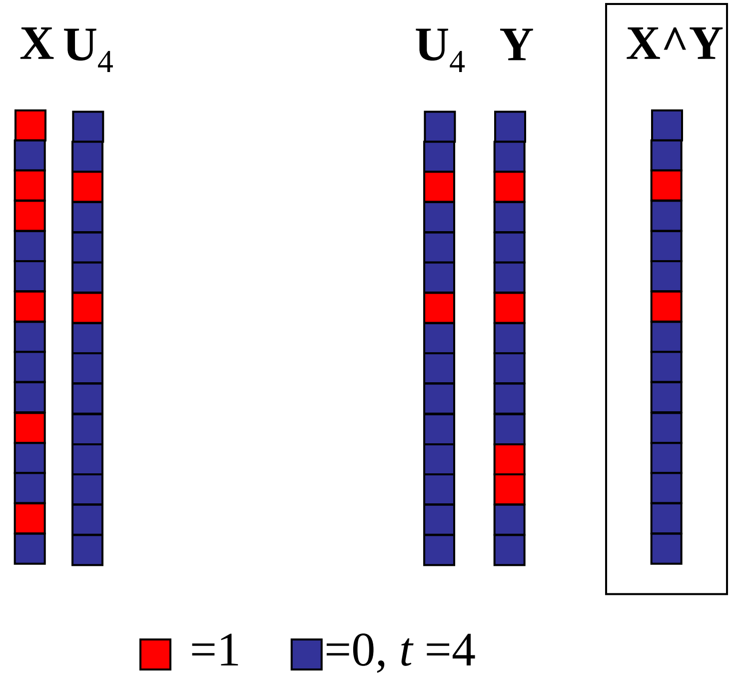
Example 5: ∞-msg interaction

- Independent (X , Y), $X \sim$ Ber (p), $Y \sim$ Ber (q)
- Both sides reproduce **X^Y**
- ∞-msg minimum sum-rate:

For $p = q = ½$, X, $Y \sim$ iid Ber(½),
∞-msg: 1.36 vs. 2-msg: 1.5 and 3-msg: 1.41

$$h_2(p) + p\, h_2(q) + p \log_2 q + p\,(1-q) \log_2 e \qquad (q > p)$$





Outline

- Introduction
- General two-terminal problem
- Co-located network with independent sources
- General multi-terminal problem: some observations

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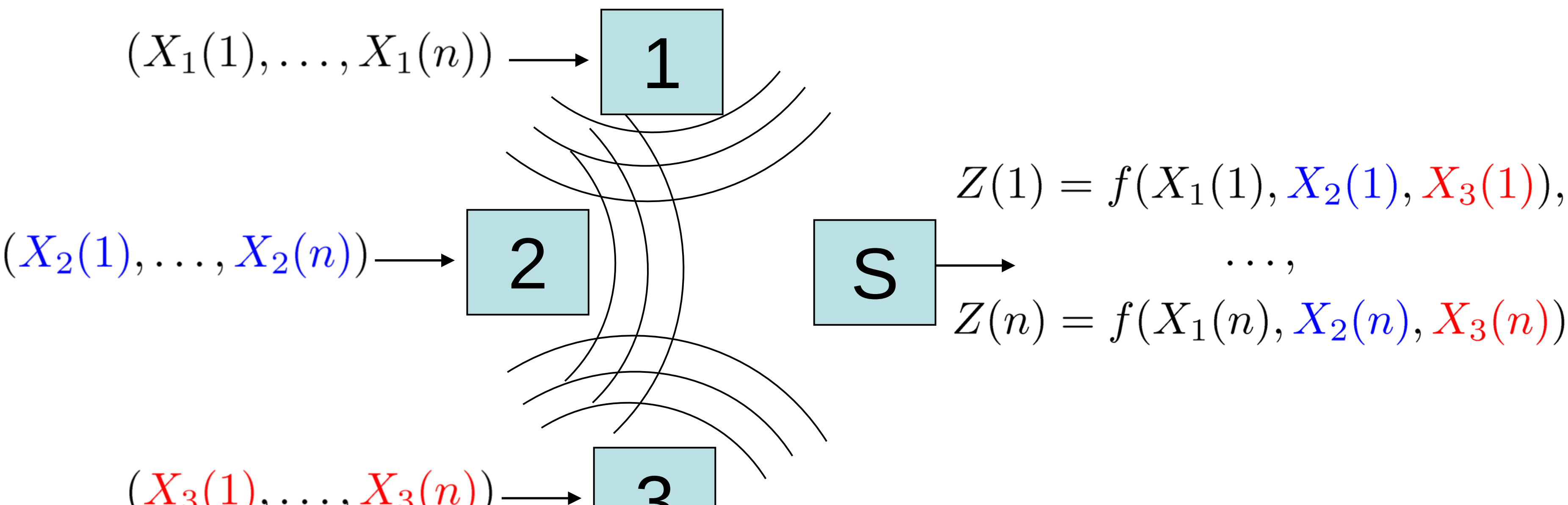
Collocated network

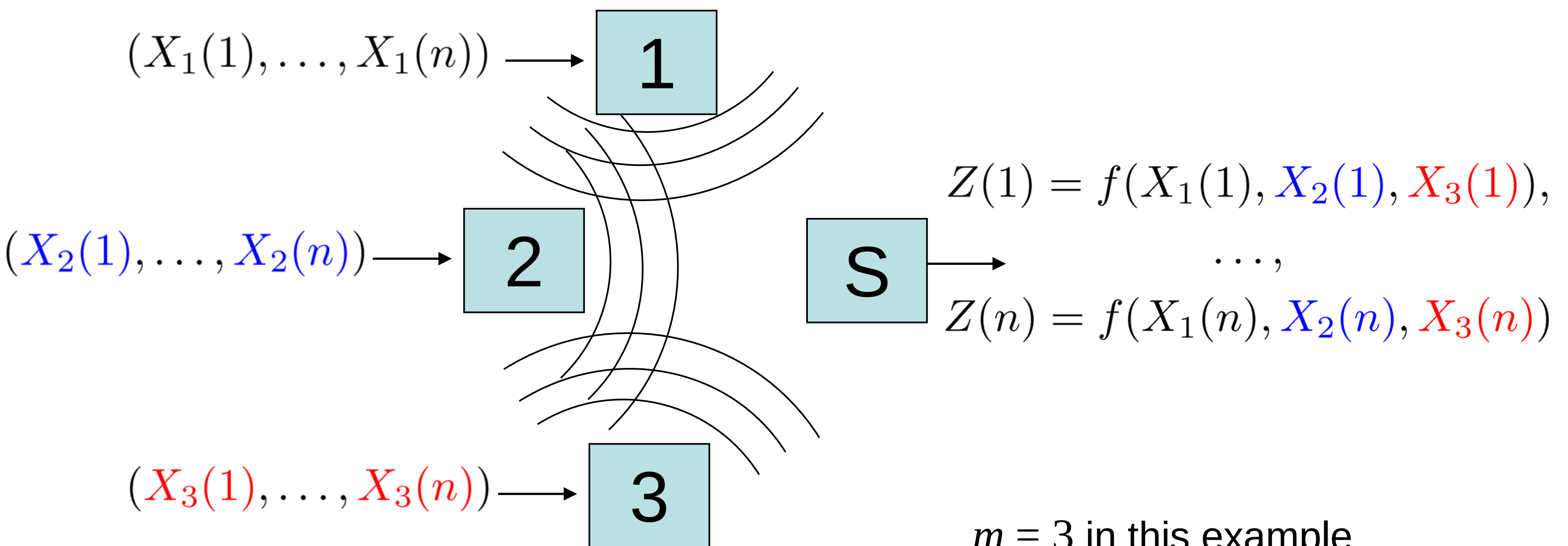
- Consider:
  - m sensors. Each observes n samples of a source
  - A sink needs to compute a samplewise function
  - A sequence of (noiseless) broadcasts for r rounds
  - How many bits/sample for each message? In total?



m = 3 in this example

Collocated network

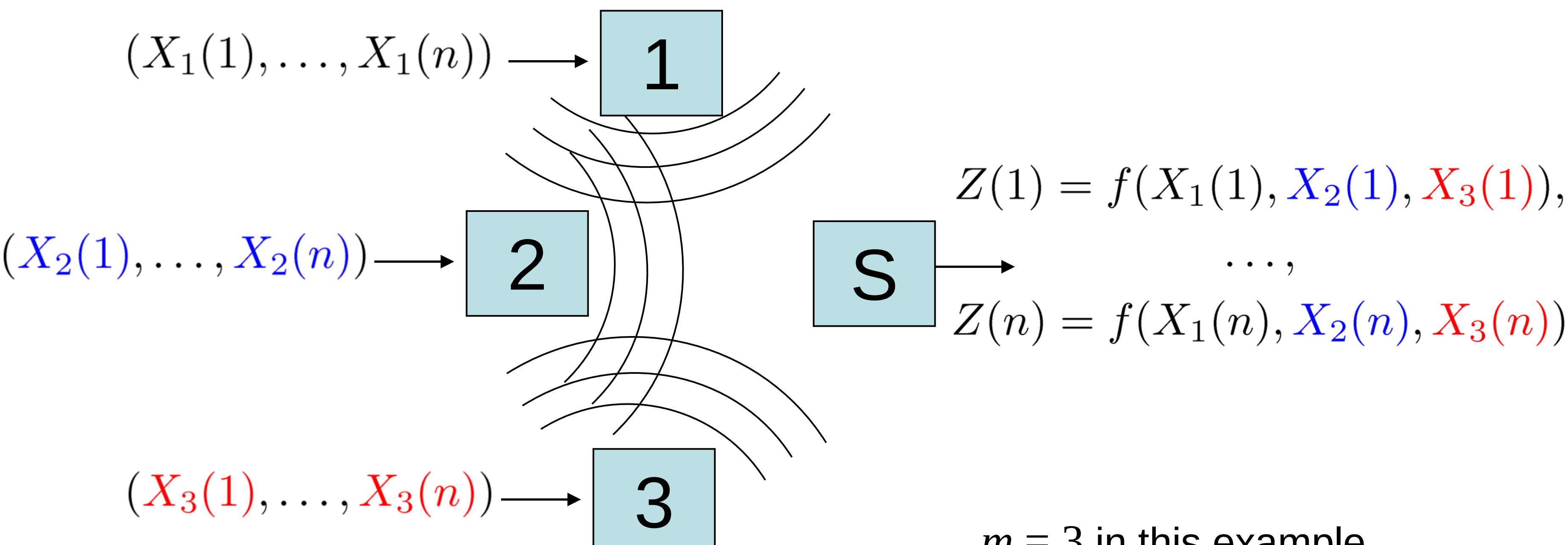
- Sources: iid across samples ~ p_{X_i}, independent across sensors
- Samplewise function: $Z(i) = f(X_1(i), \ldots, X_m(i)),\ i = 1, \ldots, n$
- Broadcasting for r rounds, $t = mr$ msgs: $(1,\ldots,m,\ \ 1,\ldots,m,\ \ \ldots,\ \ 1,\ldots,m)$
- Pr(computation result vector ≠ correct function vector) → 0 as $n \to \infty$



$m = 3$ in this example

Collocated network

- Operational rate R_i (in bits/sample): (# bits msg. i)$/n \to R_i$, as $n \to \infty$
- Rate region $\mathcal{R}_r$: set of all operational $(R_1, R_2, \ldots, R_t)$
- Minimum sum-rate: $R_{sum,r} = \min \left(R_1 + \ldots + R_t\right)$

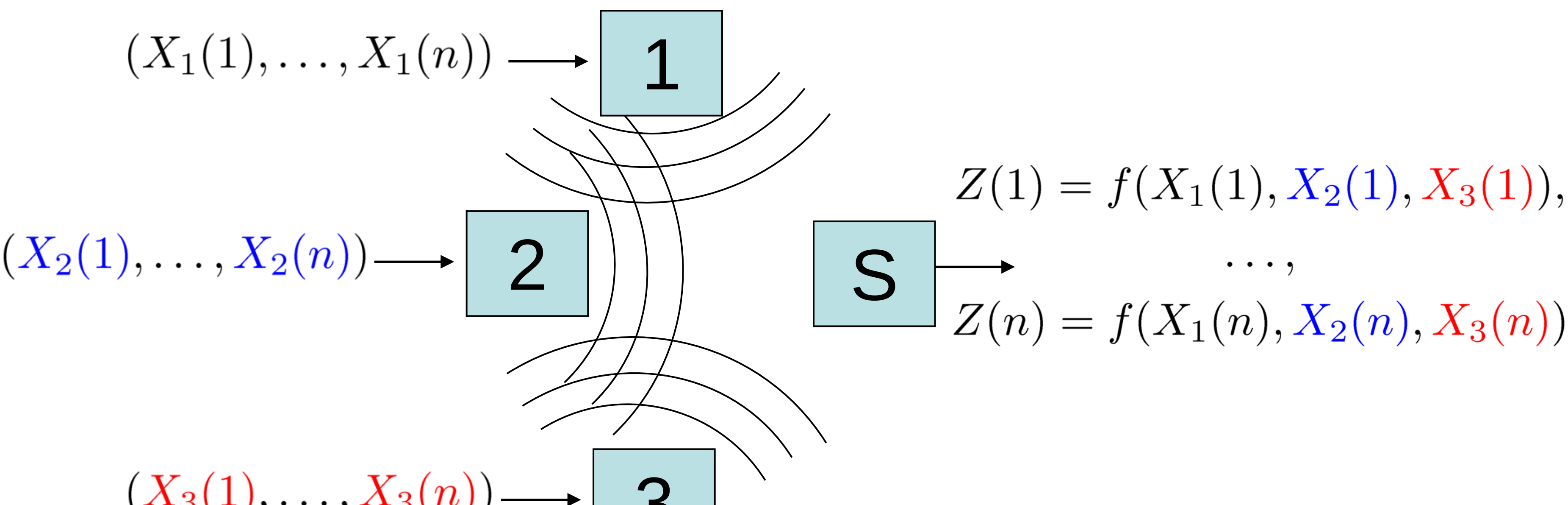


m = 3 in this example

Collocated network

Goals:

- Obtain a computable characterization of $\mathcal{R}_r$ (independent of *n*)
- Scaling behavior of $R_{sum,r}$ w.r.t. *m* (# sensors)
- Understand the benefit of interaction for different sources and functions



m = 3 in this example

Information-theoretic rate region

A characterization independent of n :

$\{1, 2, \ldots, t\}$

$$
\begin{aligned}
\mathcal{R}_r \quad = \quad & \{ \ (R_1 \ldots R_t) \mid \exists \ U^t, s.t. \ \forall j \in [1, t], k = (j \bmod m), \\
& R_j \geq I(X_k; U_j | U^{j-1}), \\
& U_j - (U^{j-1}, X_k) - (X^{k-1}, X_{k+1}^m), \\
& H(f(X^m)|U^t) = 0 \ \}
\end{aligned}
$$

[Nan Ma, PI, and P.Gupta: ISIT'09]

Information-theoretic rate region

$$
\begin{aligned}
\mathcal{R}_r \quad = \quad & \{\ (R_1 \ldots R_t) \mid \exists\, U^t, s.t.\ \forall j \in [1,t], k = (j \bmod m), \\
& R_j \geq I(X_k; U_j | U^{j-1}), \\
& U_j - (U^{j-1}, X_k) - (X^{k-1}, X_{k+1}^m), \\
& H(f(X^m)|U^t) = 0\ \}
\end{aligned}
$$

Achievability:

- 1st msg: Sensor-1 quantizes $\mathbf{X}_1$ to $\mathbf{U}_1$ and broadcasts $\mathbf{U}_1$

$$R_1 = I(X_1; U_1), \quad U_1 - X_1 - X_2^m$$

- 2nd msg: Sensor-2 quantizes $\mathbf{X}_2$ to $\mathbf{U}_2$ with side info $\mathbf{U}_1$ available to every node, and broadcasts $\mathbf{U}_2$ (conditional coding)

$$R_2 = I(X_2; U_2 | U_1), \quad U_2 - (U_1, X_2) - (X_1, X_3^m)$$

....

- Recover $\mathbf{Z}$ based on $(\mathbf{U}_1 \ldots \mathbf{U}_t)$: $\quad H(f(X^m)|U_1 \ldots U_t) = 0$

Information-theoretic rate region

$$
\begin{aligned}
\mathcal{R}_r \quad = \quad & \{\ (R_1 \ldots R_t) \mid \exists\ U^t, s.t.\ \forall j \in [1,t], k = (j \bmod m), \\
& R_j \geq I(X_k; U_j | U^{j-1}), \\
& U_j - (U^{j-1}, X_k) - (X^{k-1}, X_{k+1}^m), \\
& H(f(X^m)|U^t) = 0\ \}
\end{aligned}
$$

Converse *(impossible to do better)*:

- Standard information inequalities
- Auxiliary random variables

$$U_1(i) = (M_1, X_1(1), \ldots, X_1(i-1), \ldots, X_m(1), \ldots, X_m(i-1))$$
$$U_2 = M_2,\ \ldots,\ U_t = M_t$$

- Cardinality bounds on alphabets of auxiliary random variables

Minimum sum-rate

$$\mathcal{R}_r = \{ (R_1 \dots R_t) \mid \exists\, U^t, s.t.\ \forall j \in [1,t], k = (j \bmod m),$$
$$R_j \geq I(X_k; U_j | U^{j-1}),$$
$$U_j - (U^{j-1}, X_k) - (X^{k-1}, X_{k+1}^m),$$
$$H(f(X^m)|U^t) = 0 \}$$

Minimum sum-rate:

$$R_{sum,r} = \min_{U^t} I(X^m; U^t)$$

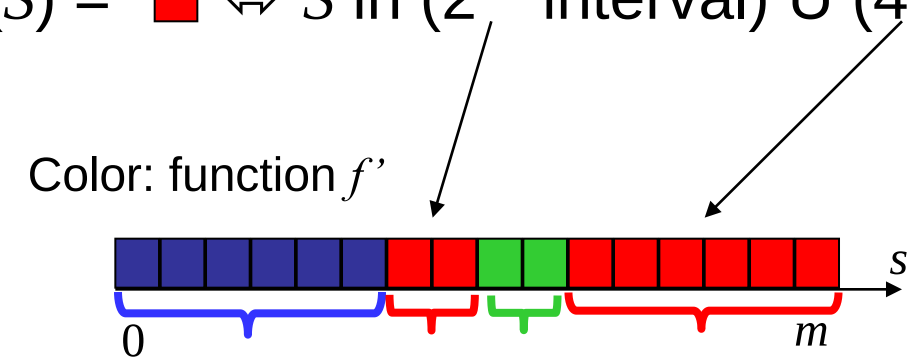
aux. r.v. subject to

$$\forall j \in [1,t], k = (j \bmod m),$$
$$U_j - (U^{j-1}, X_k) - (X^{k-1}, X_{k+1}^m),$$
$$H(f(X^m)|U^t) = 0 \}$$

Computing symmetric functions of binary sources

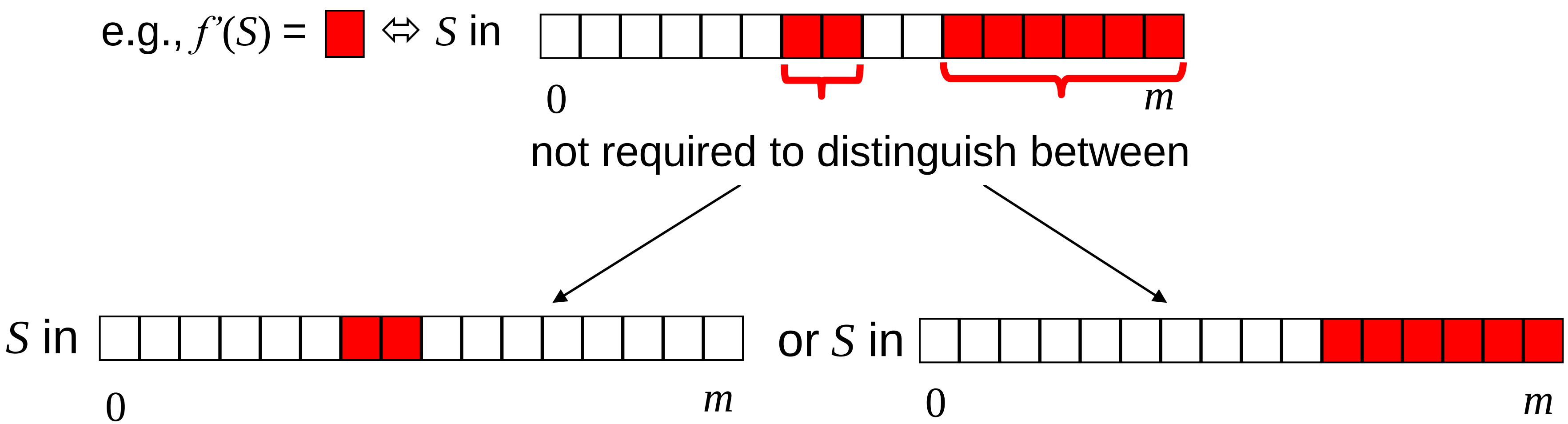
- Indep. Bernoulli sources: $Pr(X_i = 1) = p_i \in (0,1), Pr(X_i = 0) = 1 - p_i$
- Symmetric functions:
  - Invariant to permutations of arguments
  - Functions of $S = \sum_{i=1}^{m} X_i$ for binary sources $f(X^m) = f'(S)$
  - Maximal f'-monochromatic intervals: $\{ [a, b] \}$
  - Computing f ⇔ Locating S in a union of max f'-monochromatic intervals

e.g. $f'(S)$ = [red] ⇔ S in (2nd interval) U (4th interval)



Computing symmetric functions of binary sources

Computing f ⇔ Locating S in a union of several max f'-monochromatic intervals



However, due to the structure of the multiround code,
can inevitably distinguish between these cases

Lemma 2(i):

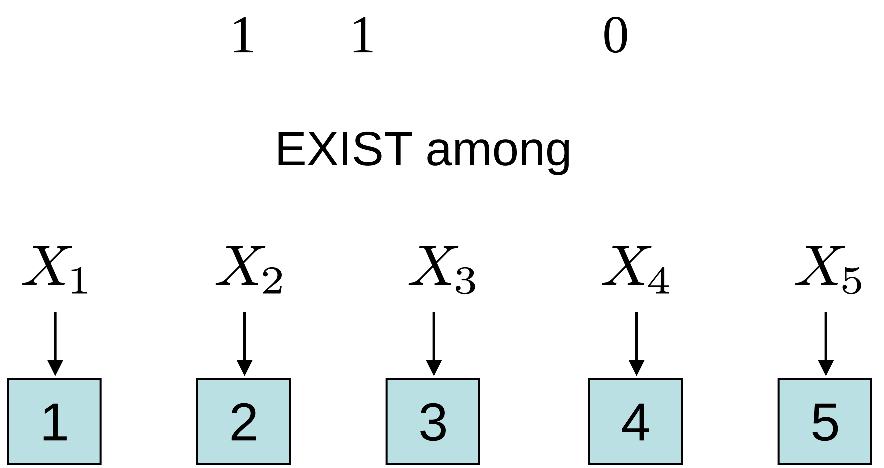
Given U^t, with probability one, there exists a single max f'-monochromatic interval to which S belongs.

Computing symmetric functions of binary sources

S in $[a, b]$ ⇔ existence of a 1's and $(m\text{-}b)$ 0's in X^m

e.g., if $m = 5$, then S in $[2, 4]$ ⇔ at least two 1's and one 0 in X^5

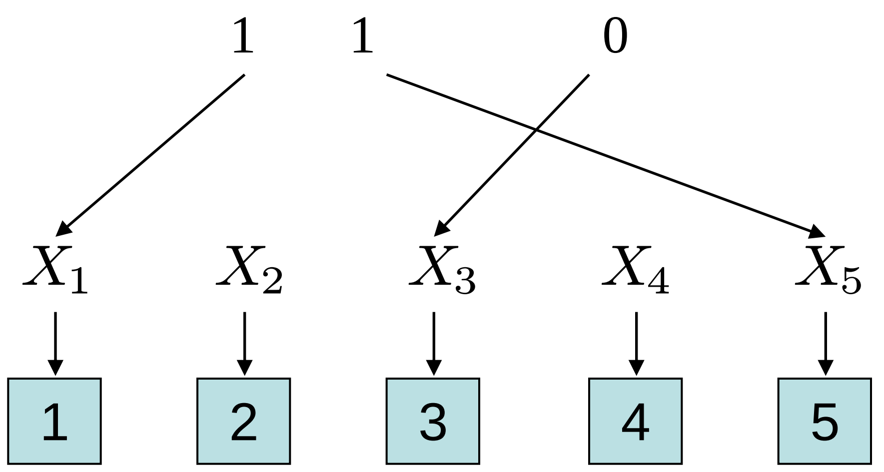
Not required to learn which X's are 1's and which are 0's



Computing symmetric functions of binary sources

S in $[a, b]$ ⇔ existence of a 1's and $(m\text{-}b)$ 0's in X^m

e.g., if $m = 5$, then S in $[2, 4]$ ⇔ at least two 1's and one 0 in X^5



However, due to the structure of the multiround code, will inevitably learn a X's which are 1 and $(m\text{-}b)$ X's which are 0.

Lemma 2(ii):

Given U^t, with probability one, can identify a X's which are 1 and $(m\text{-}b)$ X's which are 0.

Computing symmetric functions of binary sources

Lemma for single-letter characterization

(holds with Prob = 1)

Lemma 2: *Given* U^t,

(i) S *is in a single interval* $[a, b]$,

(ii) Can identify a X*'s which are* 1 *and* $(m\text{-}b)$ X*'s which are* 0.

Operational block-coding counterpart

(holds with Prob > 1 – Pr(blk. error))

Lemma 3: *Given any message sequence, for each sample* i,

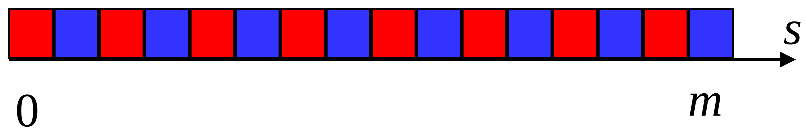
(i) Sink can identify $S(i)$ *within a single interval* $[a_i, b_i]$,

(ii) Sink can identify a_i *sensors observing* 1 *and* $(m\text{-}b_i)$ *sensors observing* 0.

Example: PARITY

m max monochromatic intervals $\{ [0, 0], [1, 1], \ldots [m, m] \}$

Color: function f'



For any zero-error code (Pr(blk. error) = 0), for each sample i:

1. Given the messages, the sink can identify $S(i)$ within a single monochromatic interval ⇔ The sink knows $S(i)$ exactly

2. If $S(i)$ in $[a_i , a_i]$, the sink knows that a_i sensors observe 1's and $(m\text{-}a_i)$ sensors observe 0's ⇔ ***The sink has to learn all the sources, in order to compute their PARITY!***

Other Implications

- Lemma 2 leads to a new lower bound for the minimum sum-rate

  "Colocated Lower Bound":

$$R_{sum,r} \geq mh(p) - \sum_{v=1, a_v \neq b_v}^{v_{\max}} (b_v - a_v) h\left(\frac{E(S|S \in [a_v, b_v]) - a_v}{b_v - a_v}\right) Pr(S \in [a_v, b_v])$$

- For any symm fn of iid Ber(1/2) srcs, $\frac{1}{2} R_{sum,1} \leq R_{sum,r} \leq R_{sum,1}$

- For any type-threshold function (e.g., MIN, MAX) of iid Ber(p) sources $R_{sum,r}(m) = \Theta(1)$ (for zero-error computation $R_{sum,r}(m) = \Theta(\log m)$)

- *"Colocated Lower Bounds" for $R_{sum,r}$ could be order-wise better than cut-set bounds*, e.g., for MIN, iid Ber(1/2), cut-set bound → 0 but new bound = Θ(1) (tight-scaling)

- Implications for secure multi-party computation

Outline

- Introduction
- General two-terminal problem
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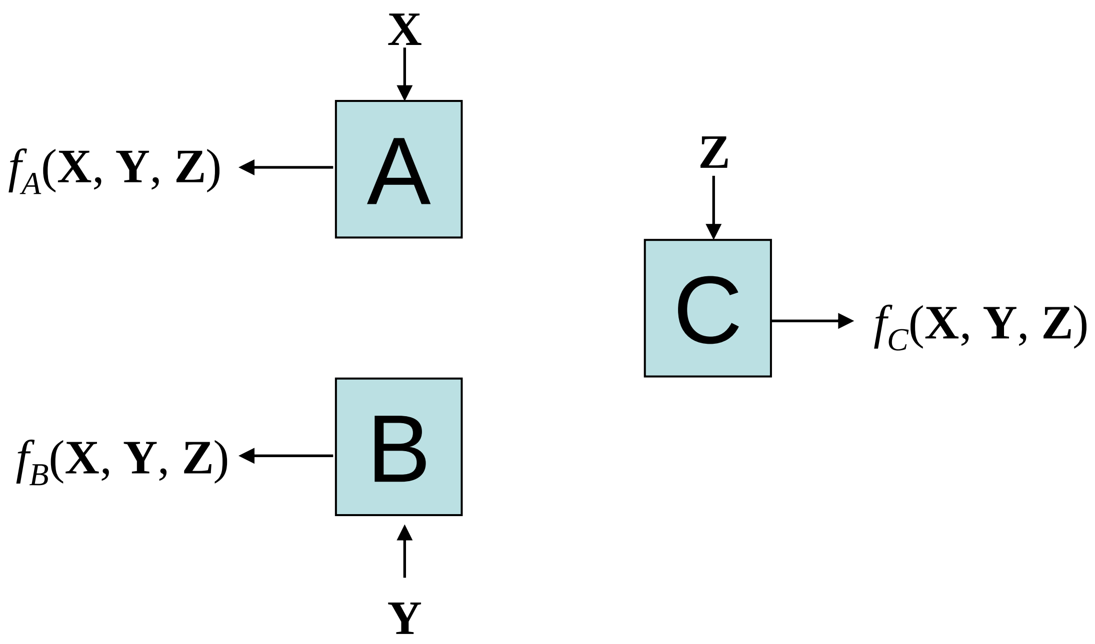
Multiterminal interaction

- *<u>m-terminal problem:</u>* m sources, m samplewise functions
- *<u>message exchanges:</u>* t rounds
- *<u>each round:</u>* concurrent message transfers.
- can switch among many non-interactive configurations



Multiterminal interaction

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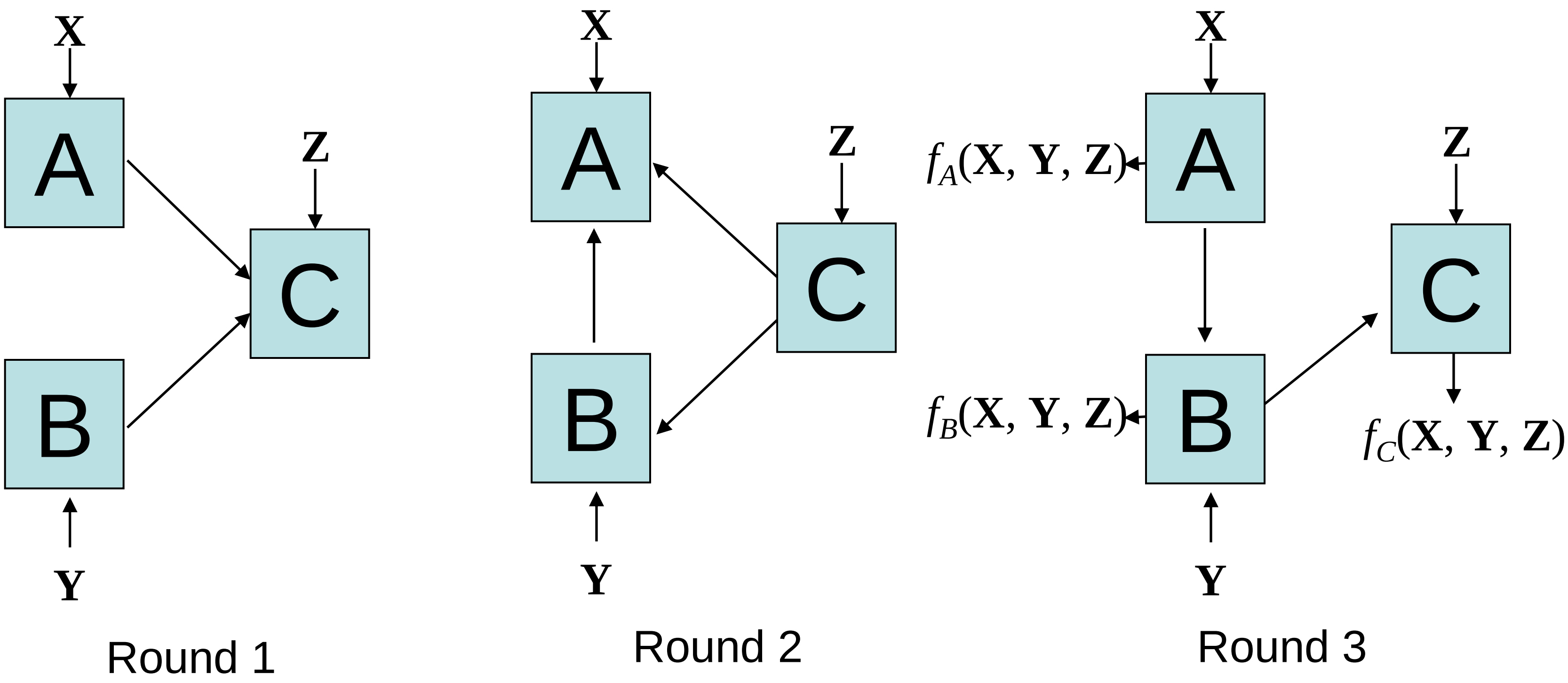
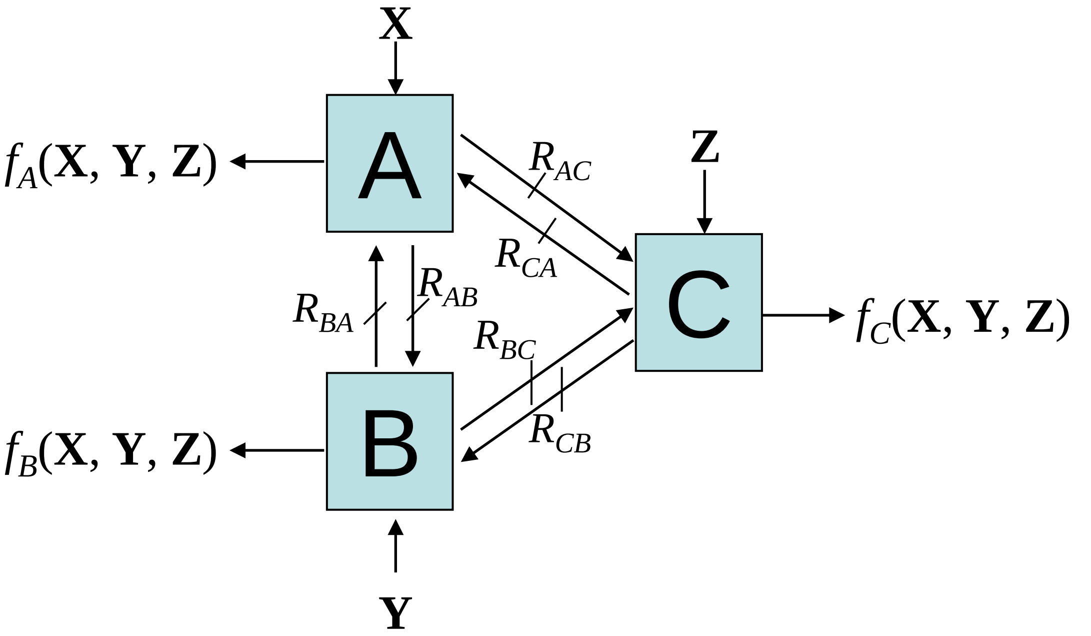


Multiterminal interaction

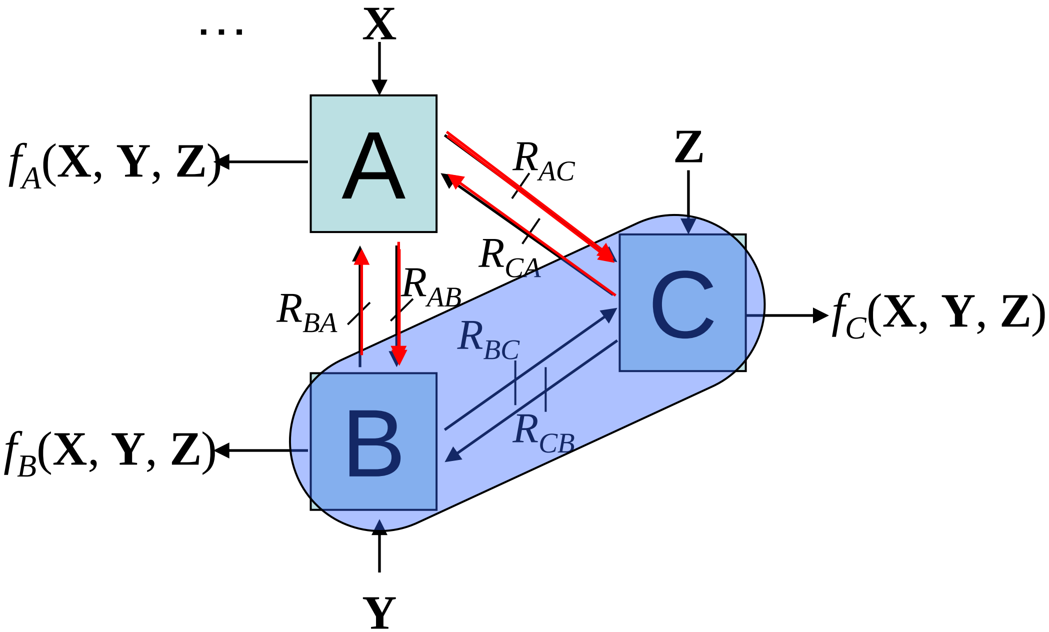
- *Complete rate region:* # bits/sample in each link, each round
- *Sum-rate region:* # bits/sample in each link, sum over all rounds
  - Region of admissible tuples $(R_{AB}, R_{BA}, R_{AC}, R_{BC}, R_{CA}, R_{CB})$
- *Minimum sum-rate:* min # bits/sample, sum over all links & all rounds
  - $R_{sum} = \min R_{AB} + R_{BA} + R_{AC} + R_{BC} + R_{CA} + R_{CB}$
- Does interaction help?

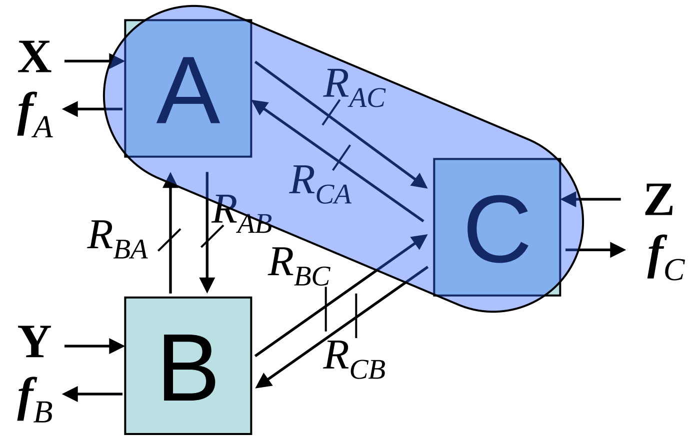


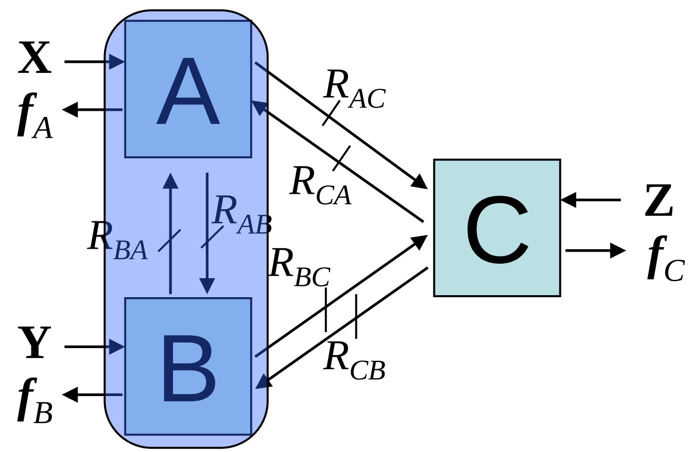
Cut-set bounds

- *Simple cut-set bounds:*
  - $R_{AB} + R_{BA} + R_{AC} + R_{CA} \geq$ 2-term min sum-rate
  - $R_{AB} + R_{AC} \geq H(f_B(X, Y, Z), f_C(X, Y, Z) | Y, Z)$

…







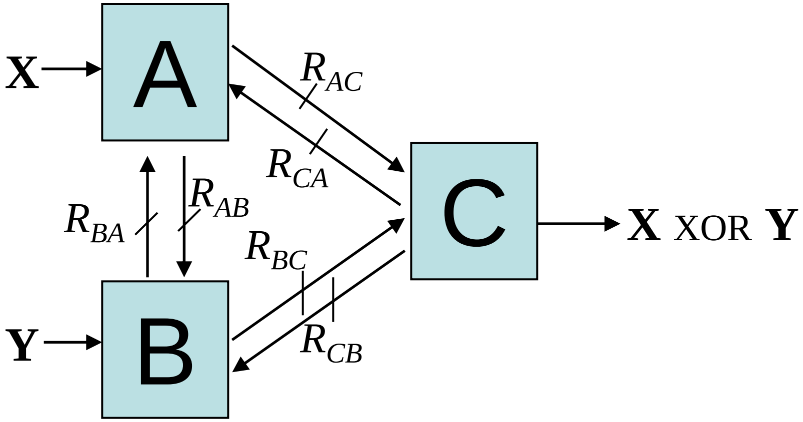
Goal: min $(R_{AB} + R_{BA} + R_{AC} + R_{BC} + R_{CA} + R_{CB})$ s.t.

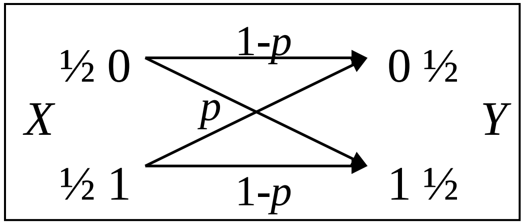
a) all sum rate lower bounds for each cut: *Linear program*

b) rates consistent with rate-regions for each cut: *Convex program*

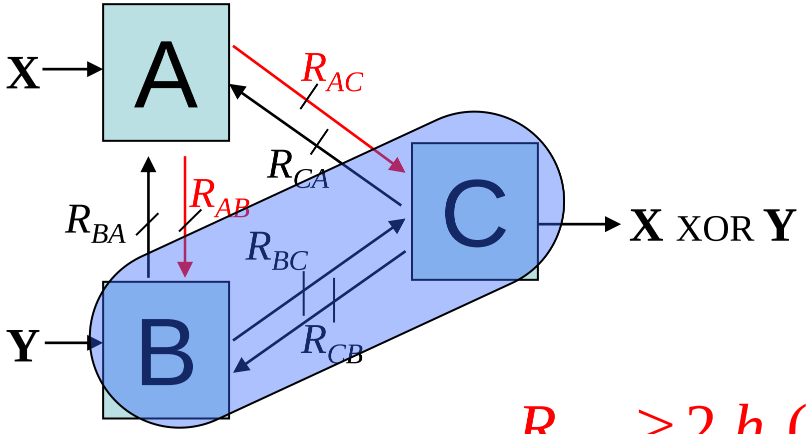
Example 1: interactive Körner-Marton

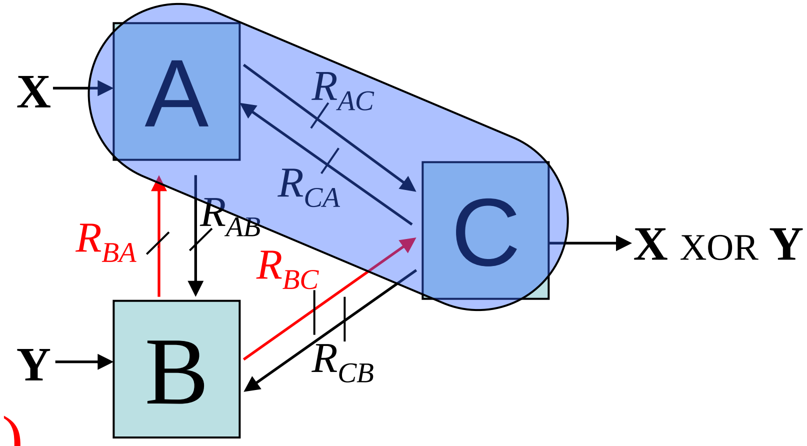
- Interactive communication allowing all possible links:





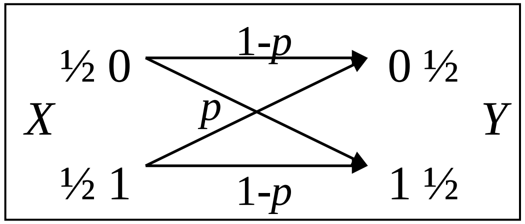
- $R_{AB} + R_{AC} \geq H(f_C(X, Y)\,|Y) = h_2(p)$ $\qquad R_{BA} + R_{BC} \geq H(f_C(X, Y)\,|\,X) = h_2(p)$



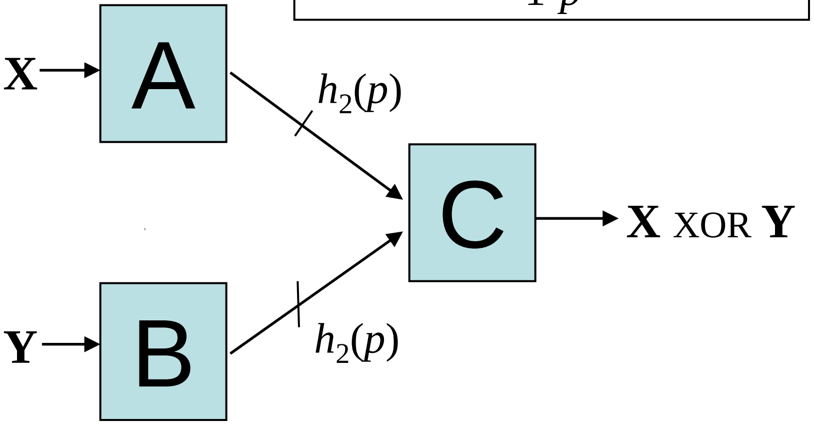


$R_{\text{sum}} \geq 2\, h_2(p)$

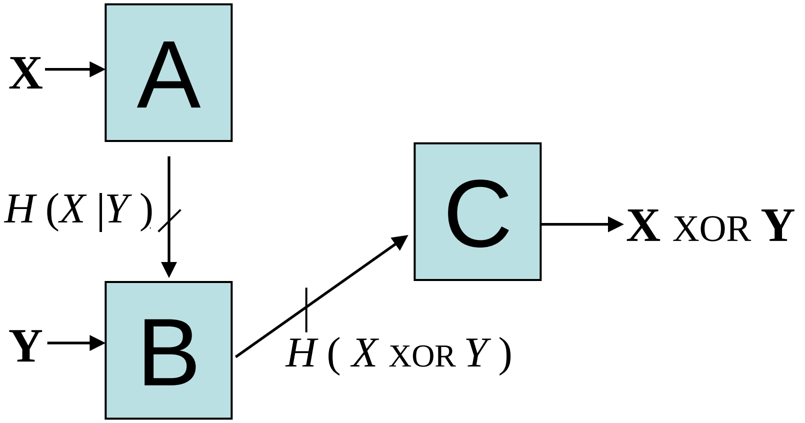
Example 1: interactive Körner-Marton



- *Non-interactive Körner-Marton:*
  - $(X, Y) \sim$ DSBS(p); $f_C(x, y) = x$ XOR y
  - Many-to-one scheme
  - $R_{AC} = R_{BC} = h_2(p)$ by linear codes
  - $R_{\text{sum}} = 2\, h_2(p)$



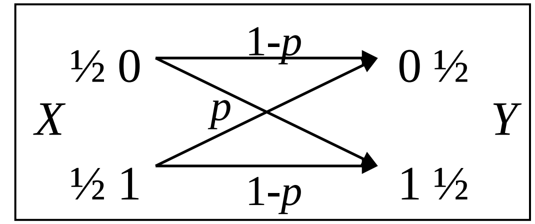
- *Relay scheme:*
  - $R_{AB} = H(X \mid Y) = h_2(p)$
  - $R_{BC} = H(f_C(X, Y)) = h_2(p)$
  - $R_{\text{sum}} = 2\, h_2(p)$

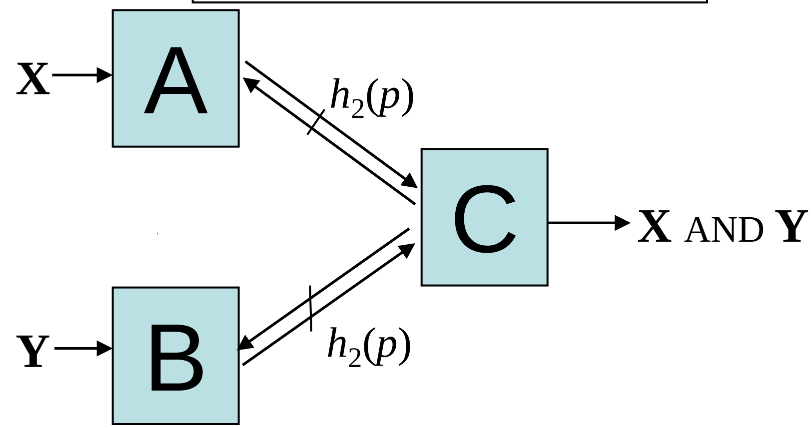


May be possible to "bypass" difficult configurations

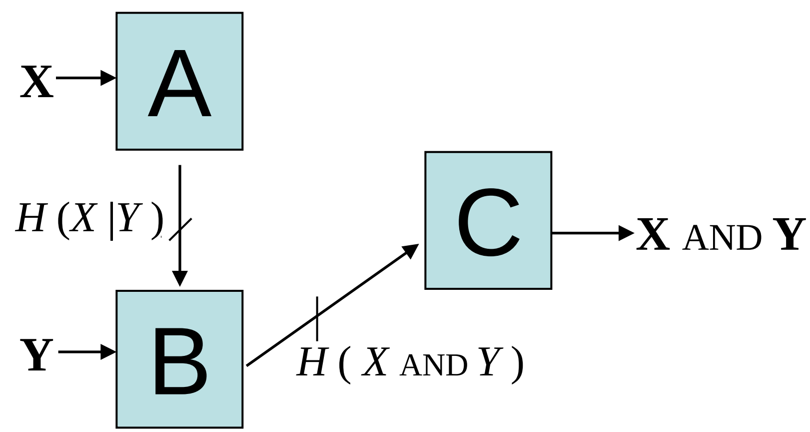
Example 2: Körner-Marton "AND"

- ***Körner-Marton problem (interactive):***
  - $(X, Y) \sim$ DSBS(p); $f_C(x, y) = x$ AND y
  - Many-to-one interactive scheme
  - $R_{\text{sum}} \geq h_2(p) + h_2(p)$
  - Min rate is unknown



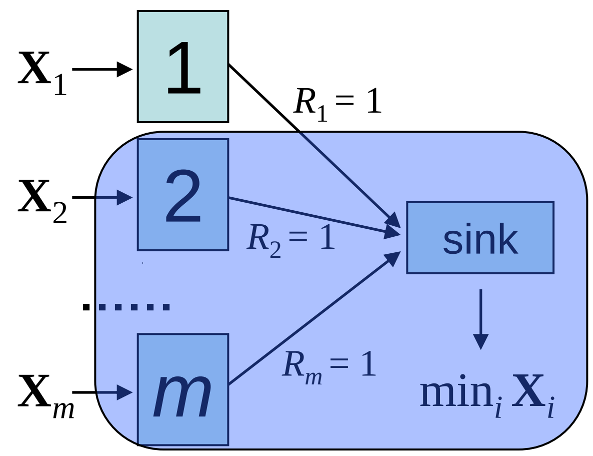


- ***Relay scheme (noninteractive)***
  - $R_{AB} = H(X \mid Y) = h_2(p)$
  - $R_{BC} = H(X \text{ AND } Y) = h_2(0.5(1\text{-}p))$
  - $R_{\text{sum}} = h_2(p) + h_2(0.5(1\text{-}p))$
  - $< 2\, h_2(p)$ for $p > 1/3$
  - Can compare configs. even if optimum is unknown

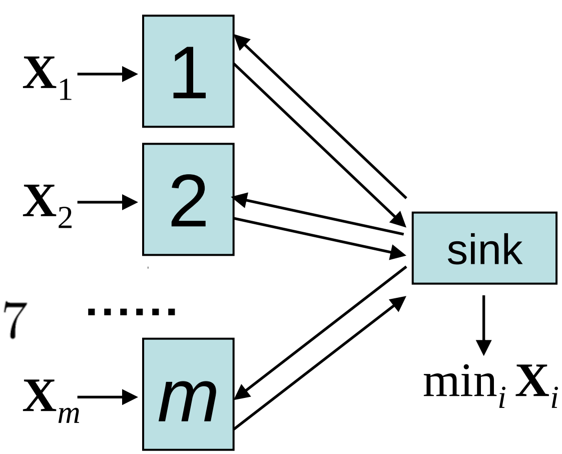


Example 3: Star networks

- $X_i \sim$ iid Ber(1/2), $f(x^m) = \min_i x_i$
- Noninteractive star network
  - Cut-set bounds: by using [Han & Kobayashi]
    Each rate ≥ 1 bit/sample
  - $R_{sum}(m) = m$



- Interactive star network
  - 1→s: send $\mathbf{X}_1$: $h_2(1/2)$
  - s→2: send $\mathbf{X}_1$: $h_2(1/2)$
  - 2→s: send $\min\{\mathbf{X}_1, \mathbf{X}_2\}$: $h_2(1/4)$
  - s→3: send $\min\{\mathbf{X}_1, \mathbf{X}_2\}$: $h_2(1/4)$
  - ……
  - Sum-rate = $2h_2(1/2) + 2\,h_2(1/4) + 2\,h_2(1/8) + \ldots < 7$
  - Using colocated lower bound: $1 \leq R_{sum}(m) < 7$
- ***Interaction changes scaling law!***



Concluding remarks

- General two-terminal problem:
  - “completely solved”
  - no benefit of interaction for data downloading;
  - benefit can be huge for computing non-trivial functions;
  - benefit depends on the structure of the functions and correlation
  - new unexplored dimension: infinite, infinitesimal-rate messages

- Colocated networks:
  - “completely solved” for independent sources
  - comm. structure reveals more information than demanded
  - cut-set bounds can be order-wise loose
  - “colocated lower bounds” order-wise tight

Questions

- Is it possible to *“bypass”* open problems in multiterminal “non-interactive” source coding by enlarging the space of strategies to include interactive ones?

- Are structured codes needed for interactive source coding?

- What are the channel coding duals of interactive source coding?

- How do distortion structure, distribution structure, and network structure influence efficiency limits in interactive source coding problems?

Thank you!