Approximation of Points on Low-dimensional Manifolds via Compressive Measurements

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Thursday, March 10, 2011

Outline

The Problem – Manifold CS [Baraniuk, Wakin, . . .]

- We have an approximate representation for a compact d-dimensional Riemannian submanifold, $\mathcal{M}$, of $\mathbb{R}^D$
  - We expect to recover points, $\vec{x} \in \mathbb{R}^D$, nearly on $\mathcal{M}$
  - $d << D$
- We acquire compressed measurements of $\vec{x}$, $M\vec{x} \in \mathbb{R}^m$, via an $m \times D$ matrix M
  - M is a Johnson-Lindenstrauss embedding (also has RIP)
- Approximate the Optimal Representative for $\vec{x}$ on $\mathcal{M}$,

$$\vec{x}_{\text{opt}} = \arg\min_{\vec{y} \in \mathcal{M}} \left\| \vec{x} - \vec{y} \right\|.$$

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Geometric Wavelets [Allard, Chen, Maggioni]

Built in two stages:

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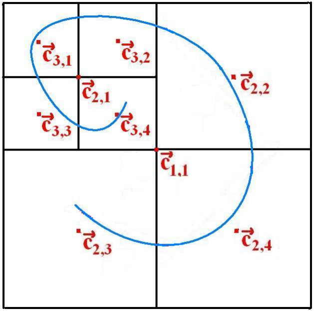

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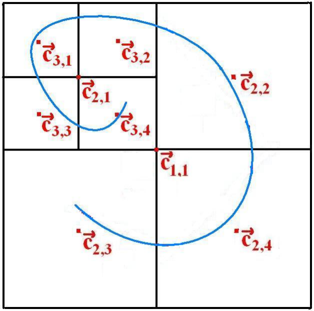

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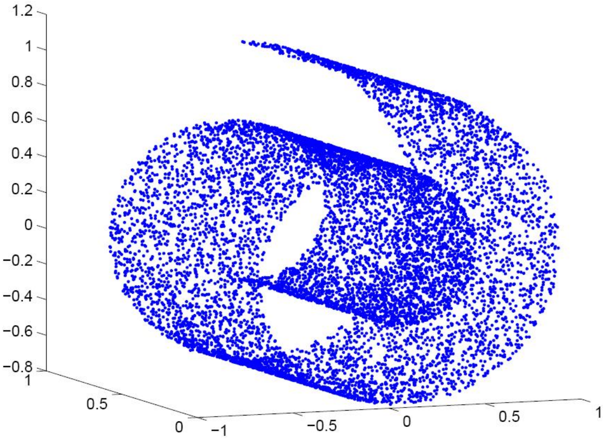
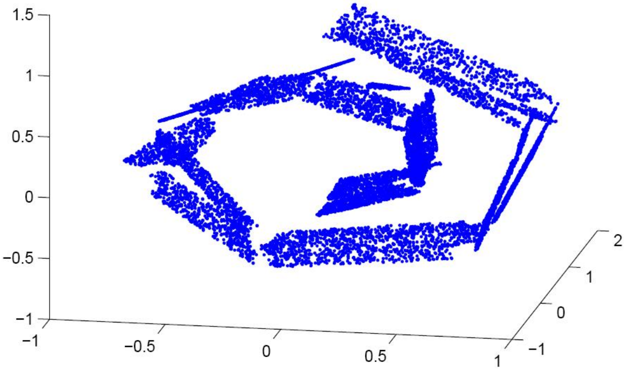
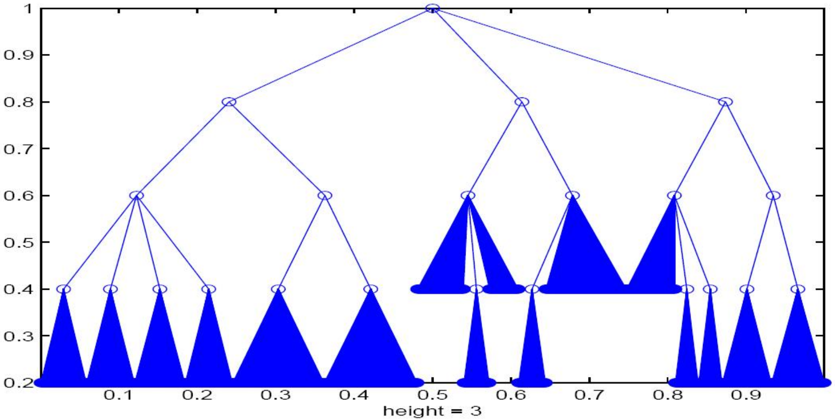
Example: Swiss Roll



Coarse Scale Approximation of Swiss Roll



Dyadic Structure for Swiss Roll Approximation



Geometric Wavelets Give Us . . .

At each Scale $j \in [J] = \{1, \ldots, J\}$ we get:

- A set of dyadic "centers" denoted by $\vec{c}_{j,k}$ for $k \in [K_j]$
- A set of orthogonal $d \times D$ matrices, $\Phi_{j,k}$, for $k \in [K_j]$
- An affine projector, $\mathbb{P}_{j,k}(\vec{x})$, for $k \in [K_j]$ defined as

$$\mathbb{P}_{j,k}(\vec{x}) = \Phi_{j,k}^T \Phi_{j,k} (\vec{x} - \vec{c}_{j,k}) + \vec{c}_{j,k}$$

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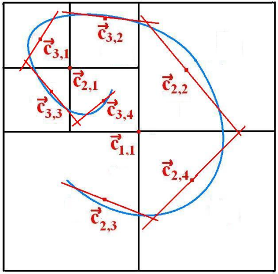

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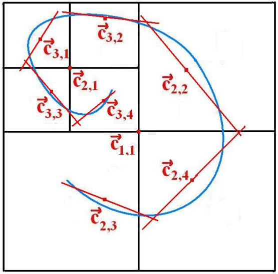

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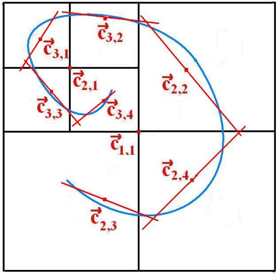

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- Recall that $\vec{x}_{\text{opt}} = \arg\min_{\vec{y} \in \mathcal{M}} \|\vec{x} - \vec{y}\|$.

Theorem

Let $\mathbb{P}_j$ be a scale j Geometric Wavelet representation for a compact smooth submanifold, $\mathcal{M}$, of $\mathbb{R}^D$. Then, for j sufficiently large, we will have

$$\left\| \vec{x} - \mathbb{P}_{j,k_j(\vec{x})}\left(\vec{x}\right) \right\| \leq 4 \left\| \vec{x} - \vec{x}_{\text{opt}} \right\| + O\left(2^{-j}\right)$$

for all $\vec{x} \in \mathbb{R}^D$.

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Approximate $\mathbb{P}_{j,k_j(\vec{x})}(\vec{x})$ using only $M\vec{x} \in \mathbb{R}^m$

1. **Input:** Measurements $M\vec{x} \in \mathbb{R}^m$, Approximation to manifold $\mathcal{M} \subset \mathbb{R}^D$, $\mathbb{P}_j = \{\mathbb{P}_{j,k} \mid k \in [K_j]\}$.
2. **Nearest Neighbors:** $k' \longleftarrow \arg\min_{k \in [K_j]} \left\| M\vec{x} - M\vec{c}_{j,k} \right\|$.
3. **Least Squares:** $\vec{u}\,' \longleftarrow \arg\min_{\vec{u} \in \mathbb{R}^d} \left\| M\Phi^{\mathrm{T}}_{j,k'}\vec{u} - M\vec{x} \right\|$.
4. $\mathcal{A}(M\vec{x}) \longleftarrow \Phi^{\mathrm{T}}_{j,k'}\vec{u}\,' - \Phi^{\mathrm{T}}_{j,k'}\Phi_{j,k'}\vec{c}_{j,k'} + \vec{c}_{j,k'}$.
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A Nonuniform Approximation Guarantee

Theorem

Let $\mathcal{M} \subset \mathbb{R}^D$ be a compact d-dimensional Riemannian submanifold of $\mathbb{R}^D$, $\vec{x} \in \mathbb{R}^D$, $\delta \in \mathbb{R}^+$, and $p \in (0,1)$. Then, the reconstruction algorithm on the last slide, $\mathcal{A} : \mathbb{R}^m \rightarrow \mathbb{R}^D$, is such that a random $m \times D$ measurement matrix, M, will satisfy

$$\left\|\vec{x} - \mathcal{A}\left(M\vec{x}\right)\right\| \;\leq\; 8 \cdot \left\|\vec{x} - \vec{x}_{\rm opt}\right\| + \delta$$

with probability at least $1 - p$ whenever m is $\Omega\left(d \log(d/p\delta)\right)$. The reconstruction algorithm's runtime will be $O(dmD)$.

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Uniform Approximation Guarantees

- Uniform recovery guarantees are worse, as expected [sparse recovery results, Wakin]
- Guarantees only hold when our measurement matrix has the RIP of order r so that $\frac{1}{\sqrt{r}} \left\| \vec{x} - \vec{x}_{\mathrm{opt}} \right\|_1$ is $O\left(\max\left\{\left\| \vec{x} - \vec{x}_{\mathrm{opt}} \right\|_2, 2^{-j}\right\}\right)$
  - This generally means $r = \Omega(D)$ if $\left\| \vec{x} - \vec{x}_{\mathrm{opt}} \right\|_2$ is large
  - Otherwise, $\left\| \vec{x} - \vec{x}_{\mathrm{opt}} \right\|_1$ must be $O\left(\sqrt{r} \cdot 2^{-j}\right)$
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Notes

- If $\mathcal{M}$ collectively spans only a small subspace of $\mathbb{R}^D$, Geometric Wavelets will reveal it. We can then reduce the effective extrinsic dimensionality.
- If we can adaptively measure $\vec{x} \in \mathbb{R}^D$ then we can approximate $\vec{x}_{\text{opt}} \in \mathcal{M}$ by...
  - Performing $O(dj \log d)$ half space tests to find the proper scale j dyadic center.
  - Projecting $\vec{x}$ onto the proper approximate tangent space
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  - Projecting $\vec{x}$ onto the proper approximate tangent space

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Thank You!

Questions?