Compressive Sensing and Sparse Recovery in Exploration Seismology

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Drivers

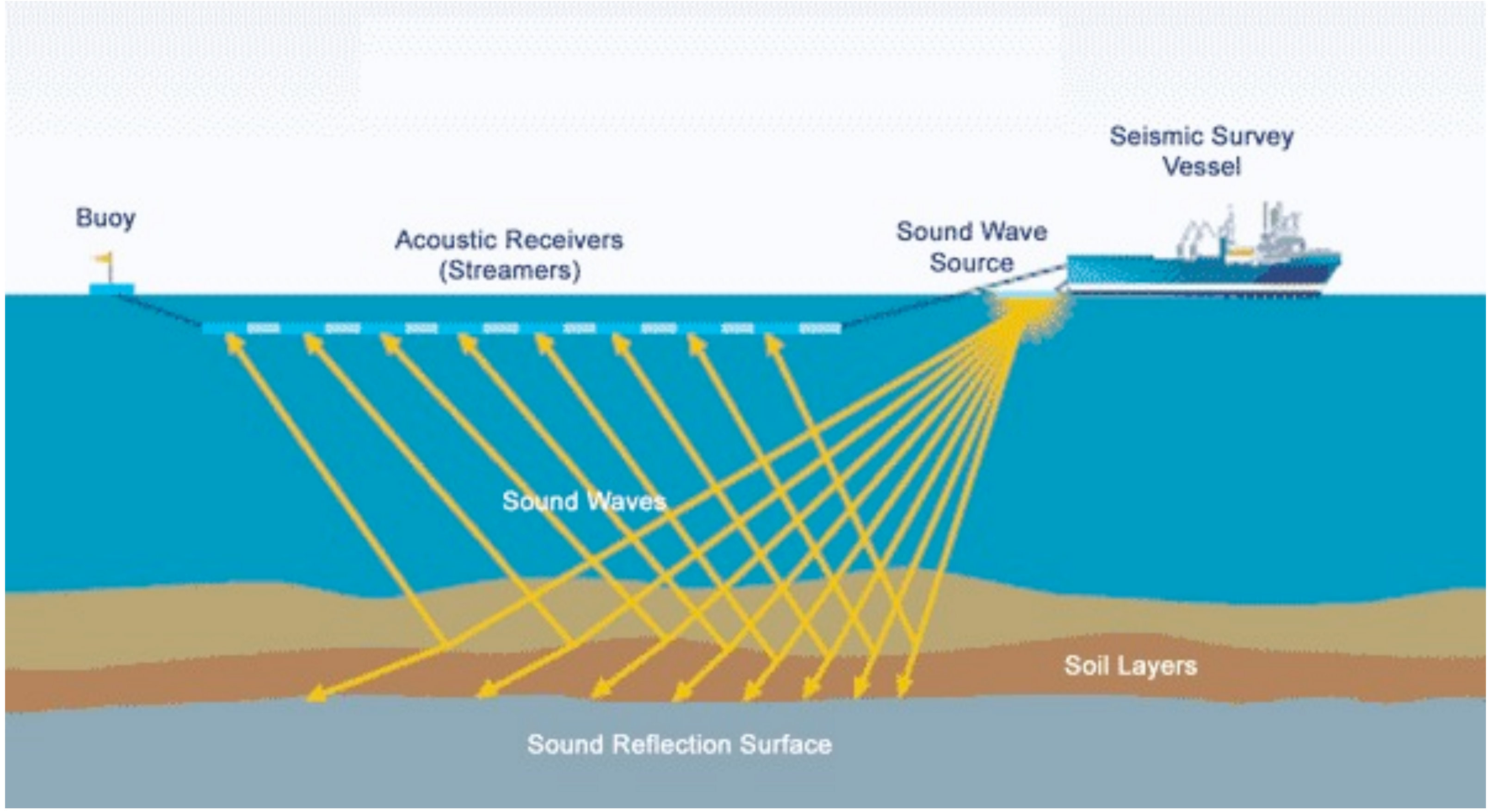
Our incessant

- demand for *hydrocarbons* while we are *no* longer finding oil...
- desire to understand the Earth's inner workings

Push for improved *seismic inversion* to

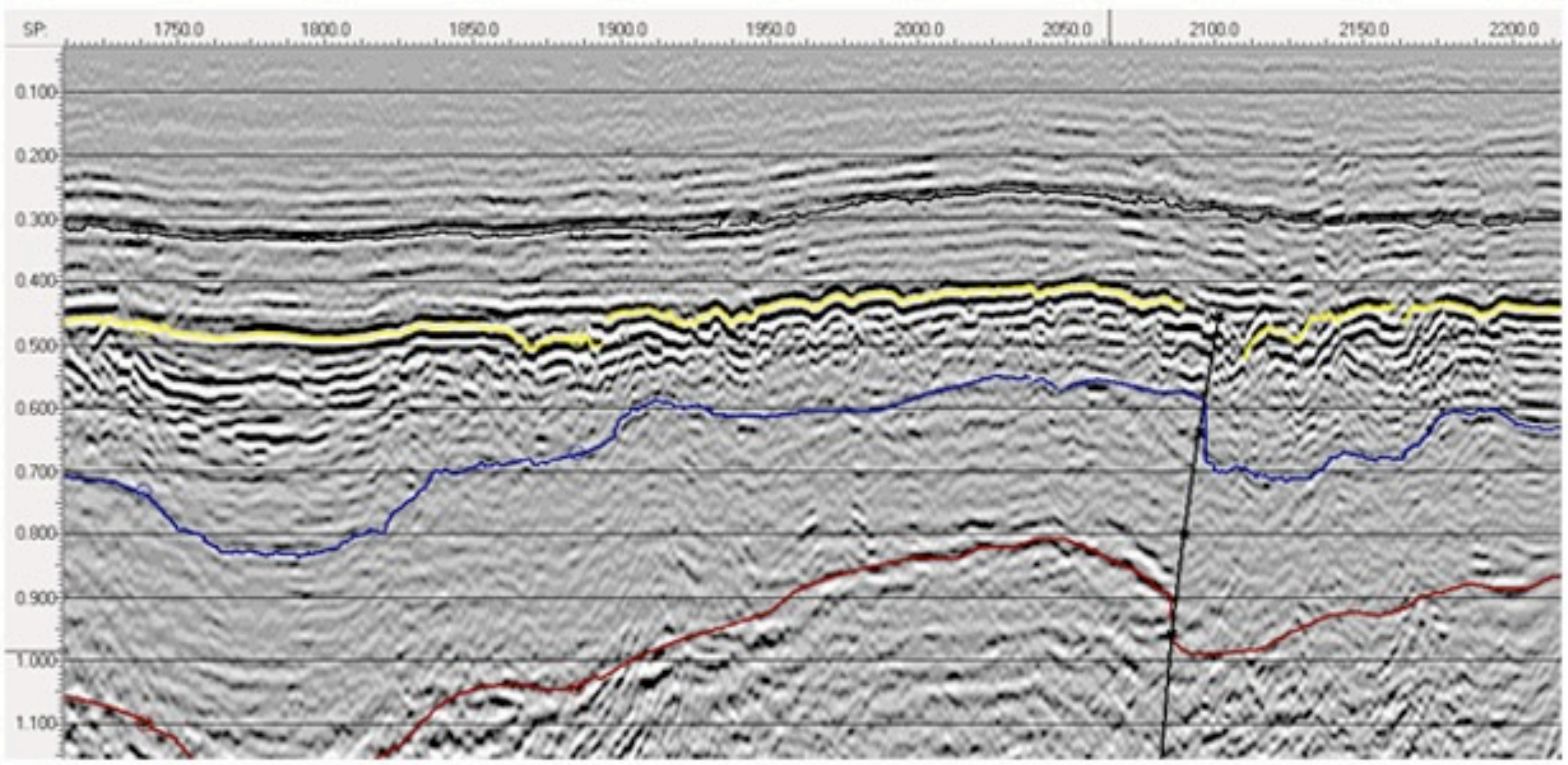
- create *more high-resolution* information
- from *more* and *more data*... (moving to 100k channel systems)

Seismic survey



http://fishsafe.eu/en/offshore-structures/seismic-surveys.aspx

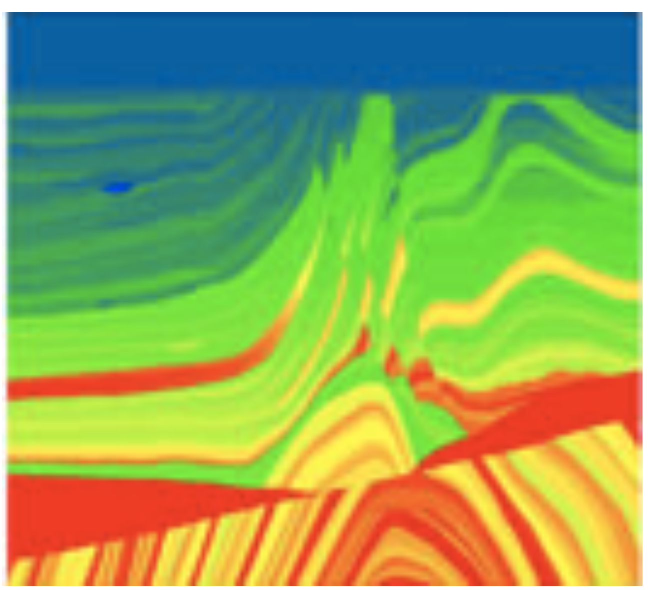
Seismic image



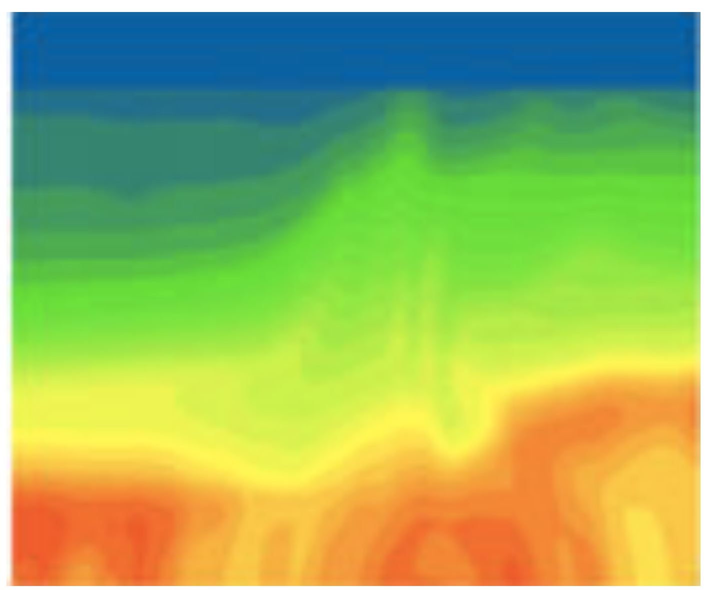
http://www.gentechintl.com/seismic.htm

Full-waveform inversion (FWI)

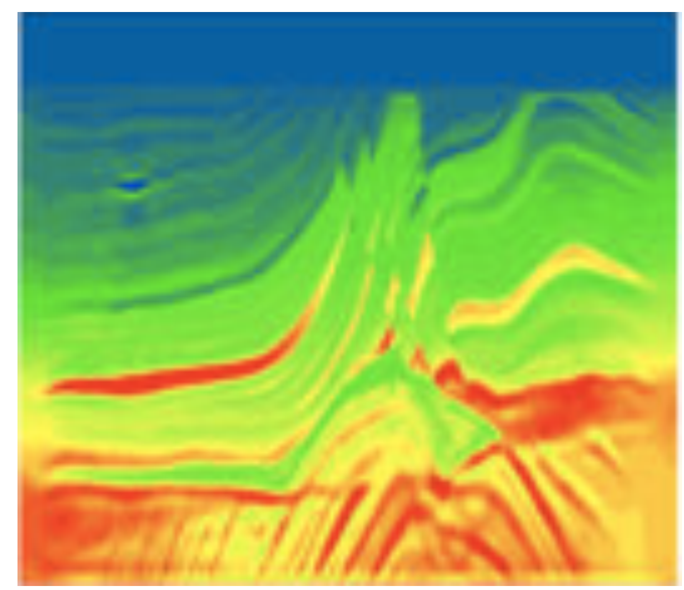
true model



starting model



inverted model



http://www.westerngeco.com/services/dp/omega/depth/tomoportfolio/fwi.aspx

Wish list

Inversion costs determined by *structure of data & complexity* of the *subsurface*

- sampling & computational *costs* that are *dictated by sparsity* and *not* by the *dimensionality* of the problem (e.g. *size* of the *discretization*)

Controllable error that depends on

- degree of *subsampling / dimensionality* reduction
- available computational resources

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[Tarantola, 84; Pratt, '98; Plessix, '06]

Problem statement

PDE-constrained optimization problem (*unconstrained form*):

$$\min_{\mathbf{m}} \frac{1}{2N} \sum_{j=1}^{n_f} \sum_{i=1}^{n_s} \|\mathbf{d}_{i,j} - \boldsymbol{\mathcal{F}}_{i,j}[\mathbf{m}, \mathbf{q}_{i,j}]\|_2^2 \quad \text{with} \quad \boldsymbol{\mathcal{F}}_{i,j}[\mathbf{m}; \mathbf{q}_{i,j}] := \mathbf{P}_i \mathbf{H}_j^{-1}[\mathbf{m}]\mathbf{q}_{i,j},$$

$$\begin{aligned}
\mathbf{d}_{i,j} &= \text{Monochromatic data from source } i \text{ and frequency } j \\
\mathbf{P}_i &= \text{Detection operator for source } i \\
\mathbf{H}_j^{-1} &= \text{Inverse of time-harmonic Helmholtz at frequency } j \\
\mathbf{q}_{i,j} &= \text{Seismic source } i \text{ at frequency } j \\
\mathbf{m} &= \text{Unknown model, e.g. } c^{-2}(x) \\
N &= n_s \times n_f \quad \text{('batch size')}
\end{aligned}$$

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[Tarantola, 84; Pratt, '98; Plessix, '06]

Simplification

Multiexperiment optimization problem:

$$\min_{\mathbf{m}\in\mathcal{M}} \frac{1}{2}\|\mathbf{D} - \mathcal{F}[\mathbf{m};\mathbf{Q}]\|_{2,2}^2 \quad \text{with} \quad \mathcal{F}[\mathbf{m};\mathbf{Q}] := \mathbf{P}\mathbf{H}^{-1}[\mathbf{m}]\mathbf{Q}$$

$$\begin{aligned} \mathbf{D} &= \text{Total multi-source and multi-frequency data volume} \\ \mathbf{P} &= \text{Single detection operator} \\ \mathbf{H}^{-1} &= \text{Inverse of time-harmonic Helmholtz} \\ \mathbf{Q} &= \text{Seismic sources} \\ \mathbf{m} &= \text{Unknown model, e.g. } c^{-2}(x) \end{aligned}$$

Properties

Multiexperiment optimization problem:

$$\min_{\mathbf{m}\in\mathcal{M}} \frac{1}{2}\|\mathbf{D}-\mathcal{F}[\mathbf{m};\mathbf{Q}]\|_{2,2}^2 \quad \text{and} \quad \mathbf{H}[\mathbf{m}]\cdot := [\omega^2 \mathrm{diag}\{\mathbf{m}\} + \nabla^2]\cdot$$

- hyperbolic PDE, non convex, 'over-' and 'underdetermined'
- wave-equation Hessian, $\nabla\mathcal{F}^H[\mathbf{m};\mathbf{Q}]\nabla\mathcal{F}[\mathbf{m};\mathbf{Q}]$, is *pseudo local*, i.e., 'preserves' singularities
- # PDE solves increases linearly with # of sources & frequencies
- *linear* in the *sources*

[Tarantola, 84; Pratt, '98; Plessix, '06; Symes '09]

Gauss-Newton

Algorithm 1: Gauss Newton

Result: Output estimate for the model $\mathbf{m}$

$\mathbf{m} \longleftarrow \mathbf{m}_0$; $k \longleftarrow 0$; `// initial model`

while not converged **do**

$\delta\mathbf{m}^k \longleftarrow \arg\min_{\delta\mathbf{m}} \frac{1}{2}\|\mathbf{D} - \mathcal{F}[\mathbf{m}^k;\mathbf{Q}] - \nabla\mathcal{F}[\mathbf{m}^k;\mathbf{Q}]\delta\mathbf{m}\|_{2,2}^2$

$\mathbf{m}^{k+1} \longleftarrow \mathbf{m}^k + \gamma^k\delta\mathbf{m}^k$; `// update with linesearch`

$k \longleftarrow k+1$;

end

Evaluation of $\nabla\mathcal{F}^H[\mathbf{m};\mathbf{Q}]$ and $\nabla\mathcal{F}[\mathbf{m};\mathbf{Q}]$ each require ***two*** PDE solves for *each* source & *angular* frequency

Involves inversion of a ***tall*** linear system of equations

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Related work

Approximations of the Hessian

- Matrix probing: a randomized preconditioner for the wave-equation Hessian [FJH et. all, '03,'09; Demanet '08-'10]
- accurate linearization & high-frequency asymptotics
- redone for each GN iteration

Randomized-dimensionality reduction

- Randomized Kaczmarz [Strohmer & Vershynen, '09; Eldar & Needell '10]
- Faster Least Squares Approximation [Drineas, Mahoney, Muthukrishnan, and Sarlos, '07]
- Blendenpik: supercharging LAPACK's LS-solver [Avron et.al., '10]
- full overdetermined *explicit* matrices

Our approach

Combine techniques from

- compressive sensing (fast phase encoders)
- stochastic optimization (stochastic approximation)

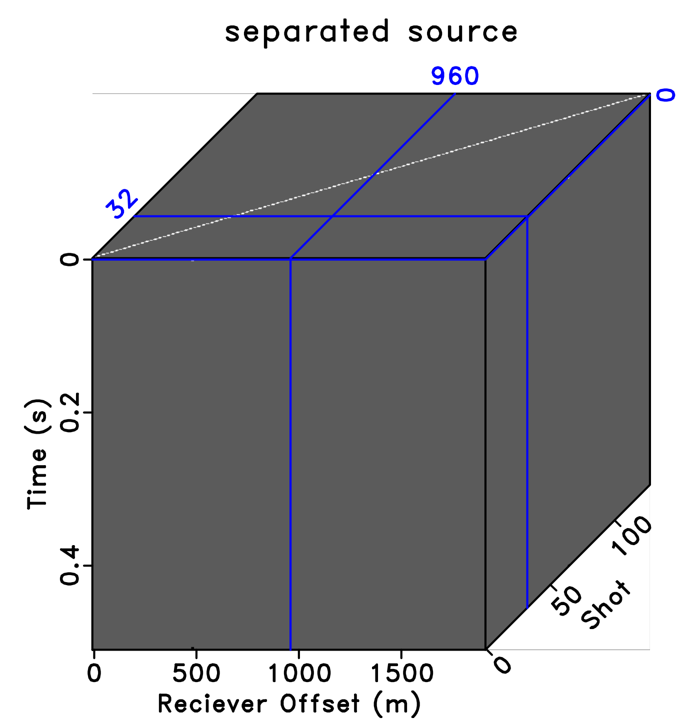
Exploit

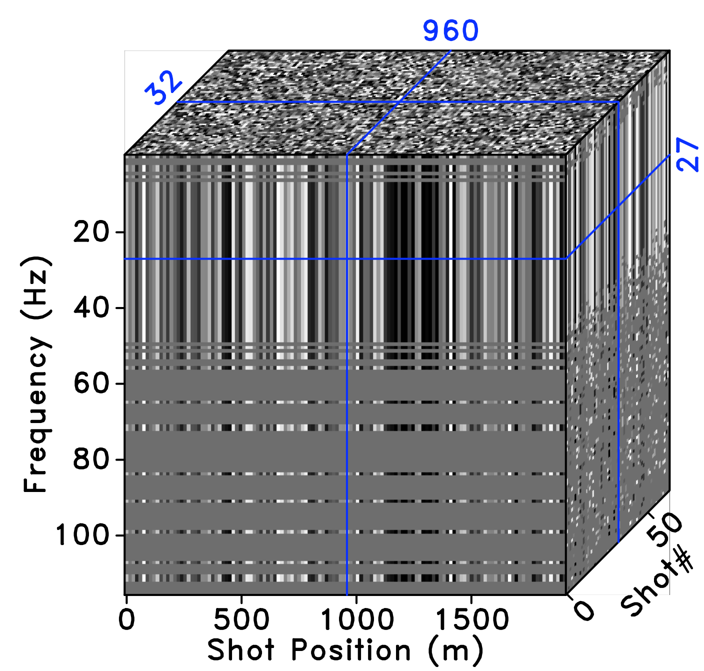
- block structure PDE-constrained optimization problem
- curvelet-domain sparsity
- convexity subproblems & properties Pareto curve

[FJH et. al. ’08-’10]

CS experiment

adapted from FJH et. al. ,09

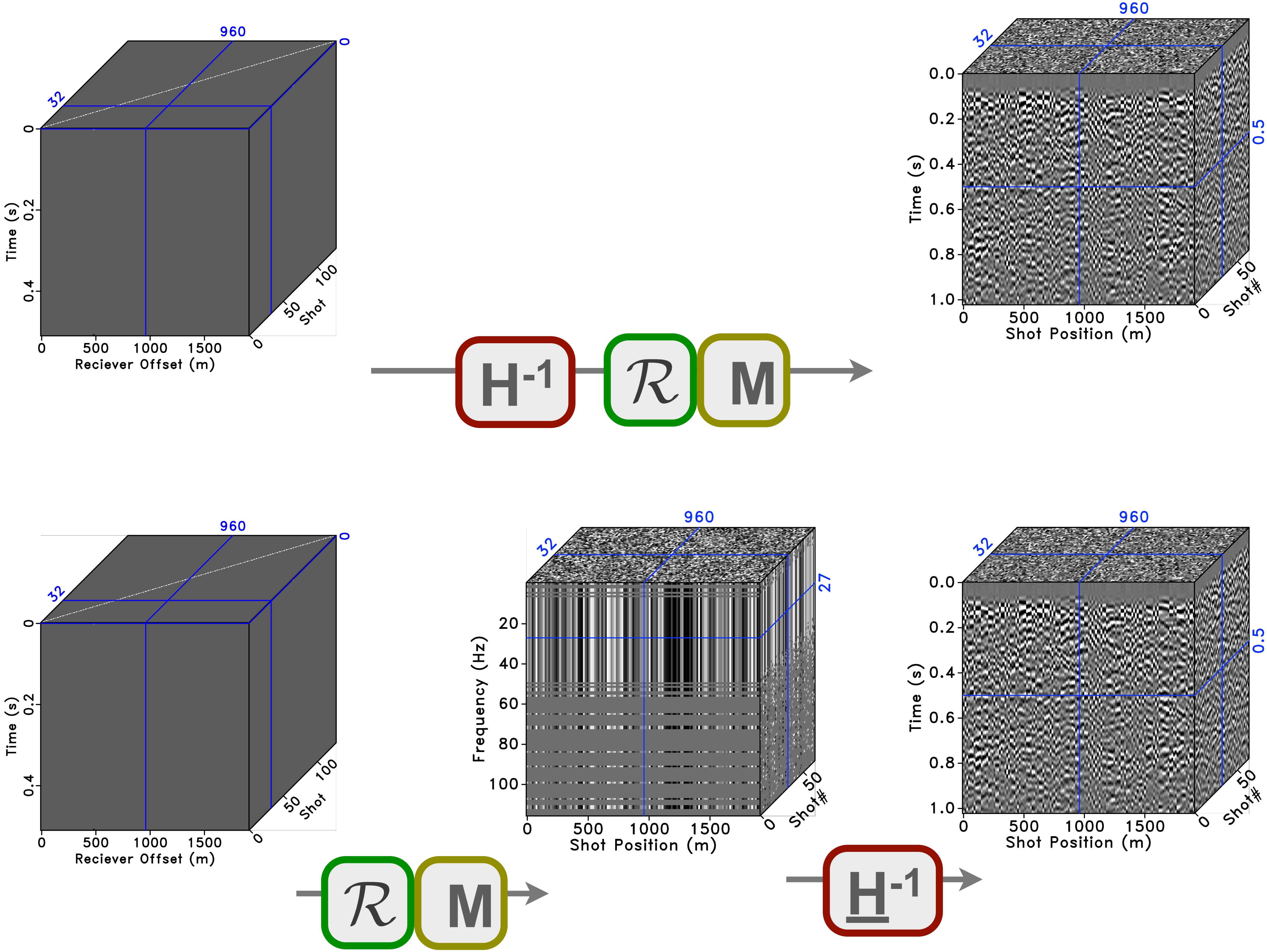




$$\mathbf{Q}$$

$$\underline{\mathbf{Q}} = \mathbf{RMQ}$$

- collection of K *simultaneous-source* experiments (*supershots*)
- $K = n'_f \times n'_s \ll n_f \times n_s$



960
32
0
Time (s)
0.2
0.4
100
50
Shot
0
500
1000
1500
Reciever Offset (m)
H-1
M
0.0
0.6
0.8
1.0
0.5
Shot#
Shot Position (m)
Frequency (Hz)
20
40
60
80
100
27

Math [Romberg, '07, FJH, '08-'10]

Fast ($n \log n$) compressive-sampling operator

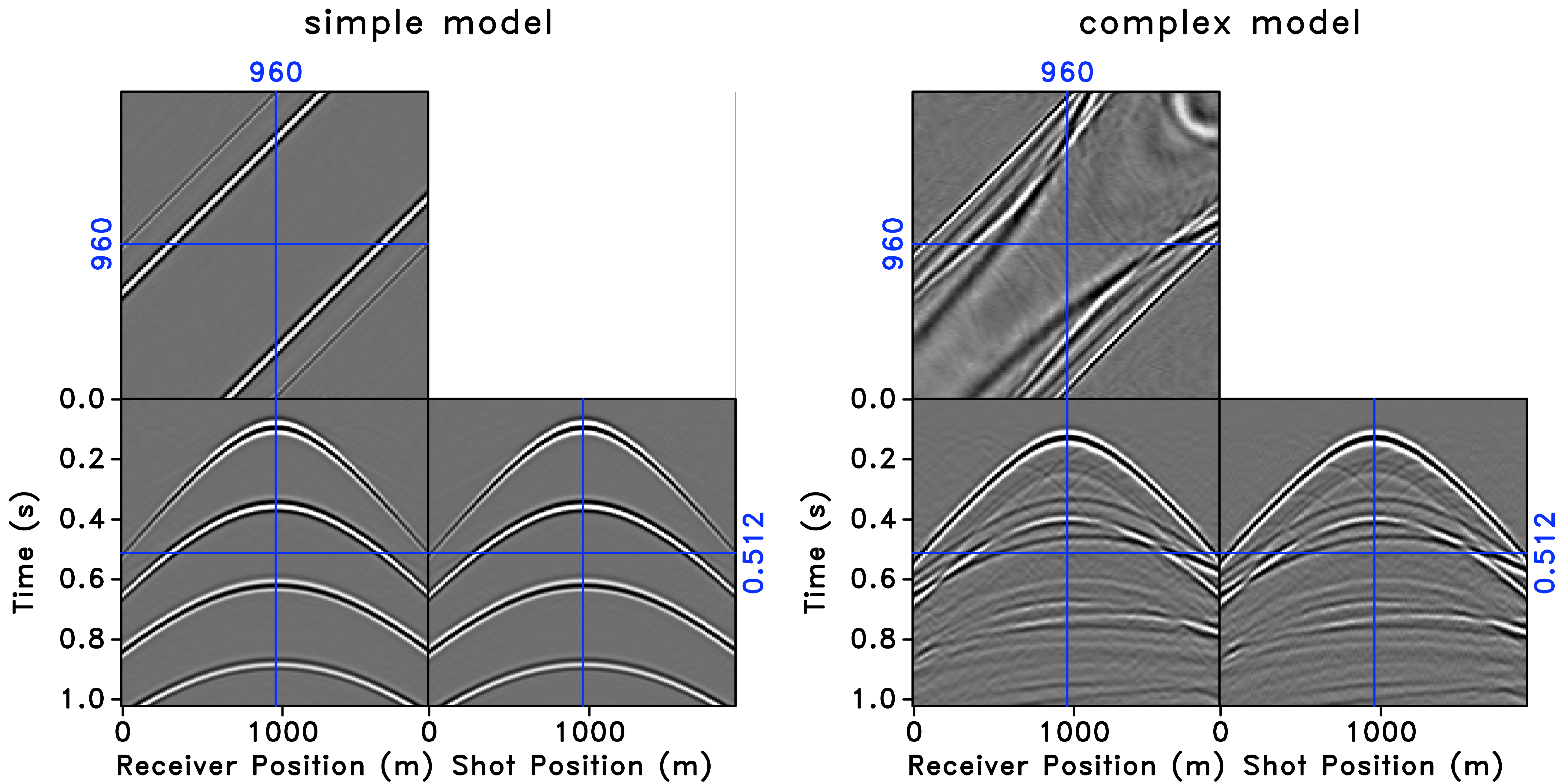
$$\mathbf{RM} = \mathrm{vec}^{-1}\left[(\mathbf{RM})_{1\cdots n_s'}\right]\mathrm{vec}$$

with
$$(\mathbf{RM})_k = (\mathbf{R^\Sigma}_k \mathbf{M^\Sigma} \otimes \mathbf{I} \otimes \mathbf{R^\Omega}_k)$$

and
$$\mathbf{M^\Sigma} = \overbrace{\mathrm{sign}(\eta) \odot \mathbf{F_\Sigma}^H e^{j\theta}\mathbf{F_\Sigma}}^{\text{'Gaussian matrix'}}$$

where $\theta \in \mathrm{Uniform}(-\pi,\, \pi]$, and $\eta \in \mathrm{Normal}(0, 1)$

Recovered Green's functions



28.1dB

18.2dB

300 SPGL1 iteration

Bottom line

Computational cost for the ℓ_1-solver is less $(\mathcal{O}(n^3 \log n)$ vs. $\mathcal{O}(n^4))$ than the cost of solving Helmholtz...

Problem:

- data space too large in 3D acquisition (1000^5 - $100k^5$)
- have to resimulate for each gradient update...

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Reduced FWI formulation

Multiexperiment simultaneous-source optimization problem:

$$\min_{\mathbf{m}\in\mathcal{M}} \frac{1}{2}\|\underline{\mathbf{D}} - \mathcal{F}[\mathbf{m};\underline{\mathbf{Q}}]\|_{2,2}^2 \quad \text{with} \quad \mathcal{F}[\mathbf{m};\underline{\mathbf{Q}}] := \mathbf{P}\underline{\mathbf{H}}^{-1}\underline{\mathbf{Q}}$$

- requires *smaller* number of PDE solves
- explores *linearity* in the *sources* & *block-diagonal* structure of the *Helmholtz system*
- uses *randomized* frequency *selection* and *phase encoding*

Interpretations

Consider *randomized*-dimensionality reduction as instances of

- *stochastic optimization* [Haber, Chung, and FJH, '10; van Leeuwen, Aravkin, FJH, '10]
  - *random*-trace estimates [Hutchinson, '90, Avron & Toledo, '10]
  - *stochastic* gradient descent [Bertsekas,' '96; Nemirovski, '09]
- *"compressive sensing"* [FJH et. al, '08-'10]

Stochastic optimization

Replace *deterministic*-optimization problem

$$\min_{\mathbf{m}\in\boldsymbol{\mathcal{M}}} f(\mathbf{m}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\|\mathbf{d}_i - \mathcal{F}[\mathbf{m};\mathbf{q}_i]\|_2^2$$

with *sum* cycling over *different sources & corresponding monochromatic shot records* (columns of D & Q)

[Natterer, '01]

Stochastic *average* approximation [Haber, Chung, and FJH, '10]

by *a stochastic-optimization* problem

$$\min_{\mathbf{m}\in\mathcal{M}} \mathbf{E_w}\{f(\mathbf{m},\mathbf{w})\} = \frac{1}{2}\|\mathbf{Dw}-\mathcal{F}[\mathbf{m};\mathbf{Qw}]\|_2^2\}$$

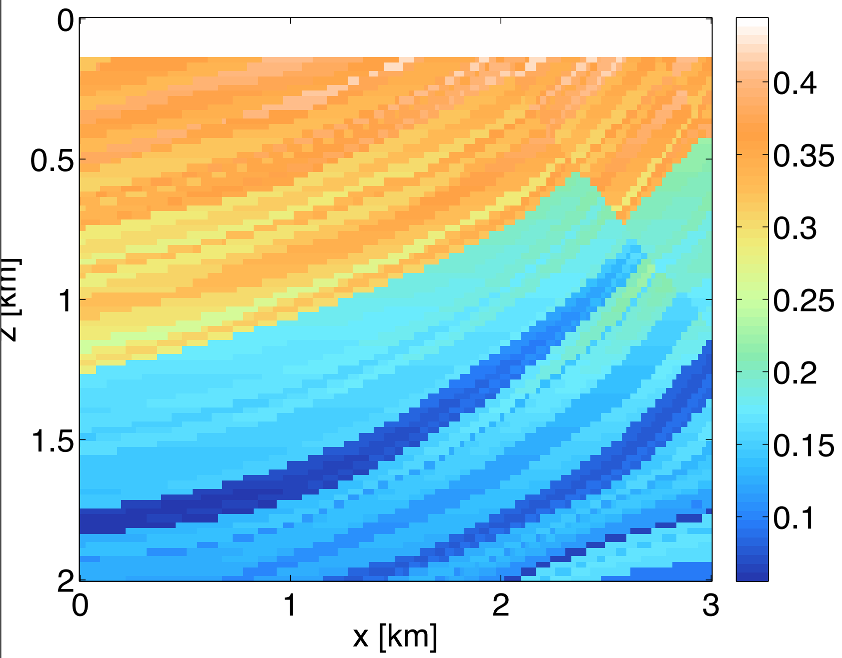
$$\approx \frac{1}{K}\sum_{j=1}^{K}\frac{1}{2}\|\underline{\mathbf{d}}_j-\mathcal{F}[\mathbf{m};\underline{\mathbf{q}}_j]\|_2^2$$

with $\mathbf{w}\in N(0,1)$ and $\mathbf{E_w}\{\mathbf{w}\mathbf{w}^H\}=\mathbf{I}$

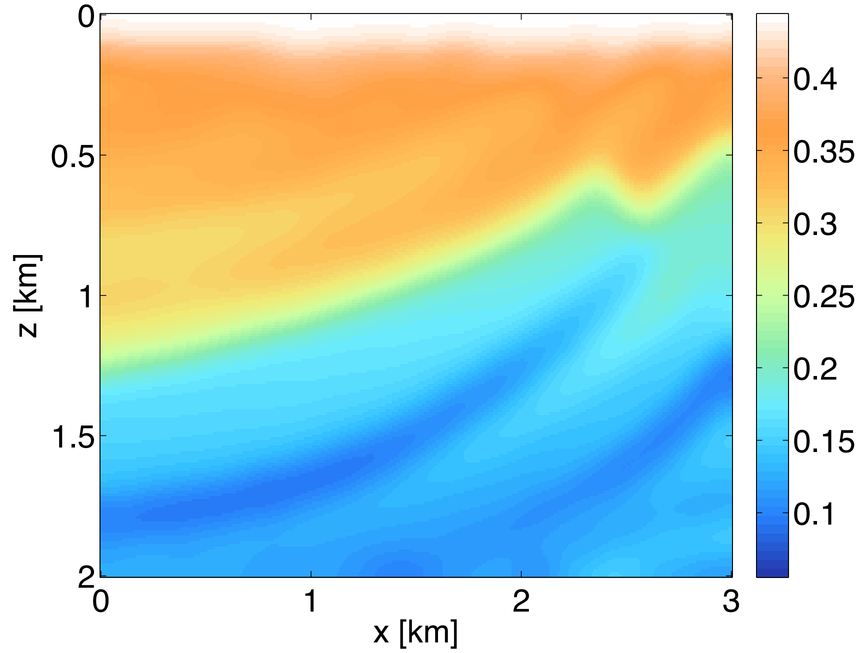
and $\underline{\mathbf{d}}_j=\mathbf{Dw}_j,\ \underline{\mathbf{q}}_j=\mathbf{Qw}_j$

Stylized example

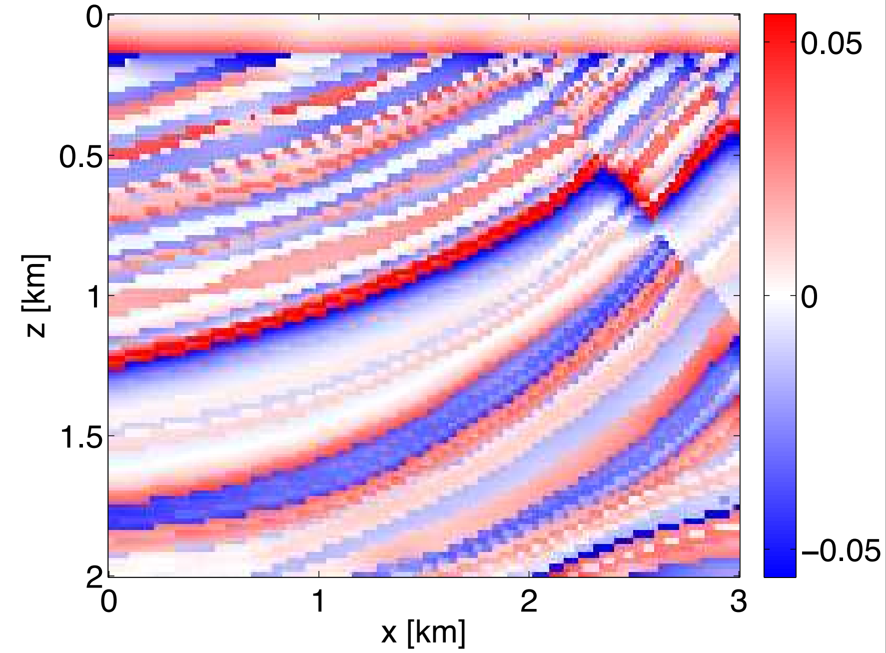
true model



starting model



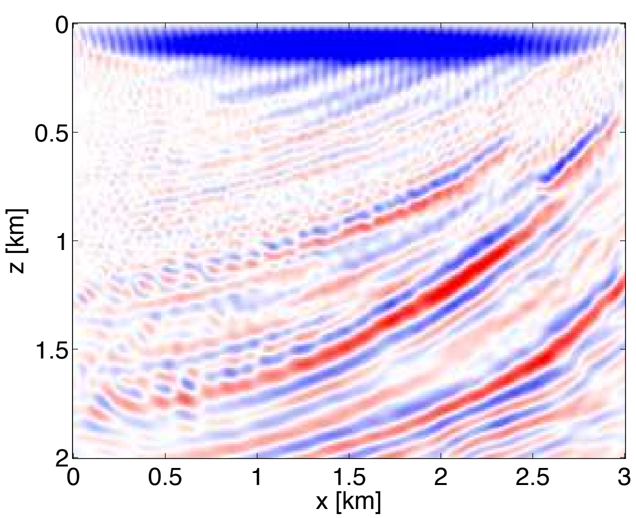
‘reflectivity’



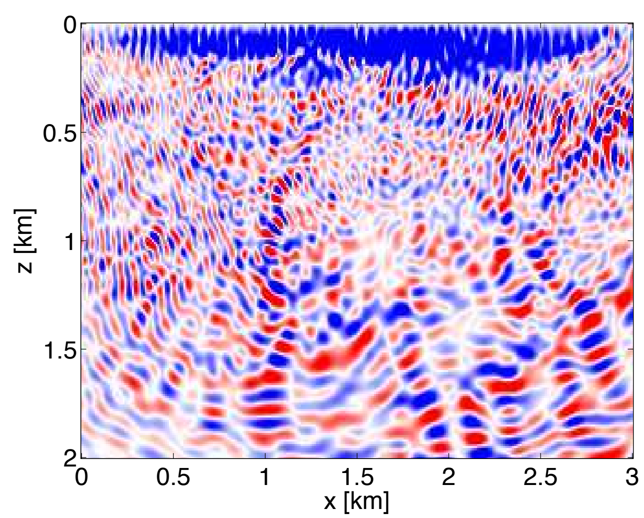
Gradients

Search direction for *increasing* batch size K:

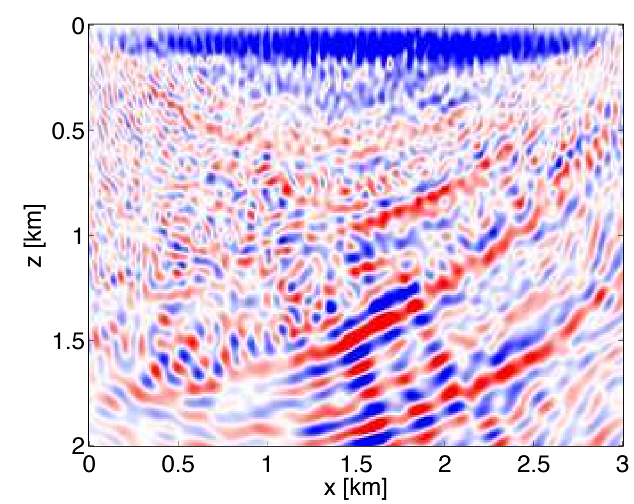
$$\mathbf{g}_K \approx \frac{1}{K}\sum_{j=1}^{K} \nabla\mathcal{F}^*[\mathbf{m};\underline{\mathbf{q}}_j]\delta\underline{\mathbf{d}}_j$$



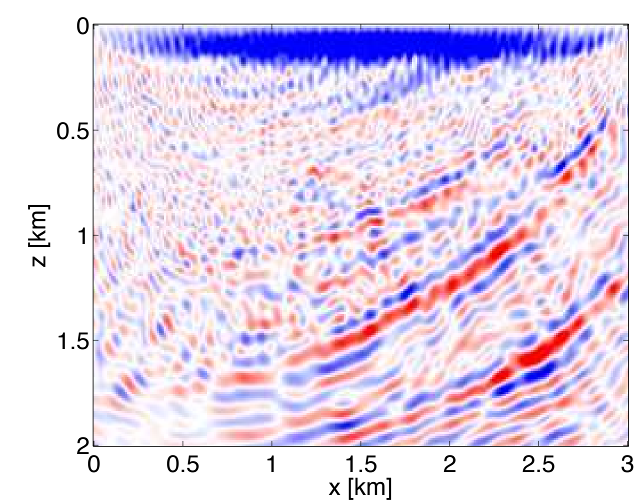
full



K=1

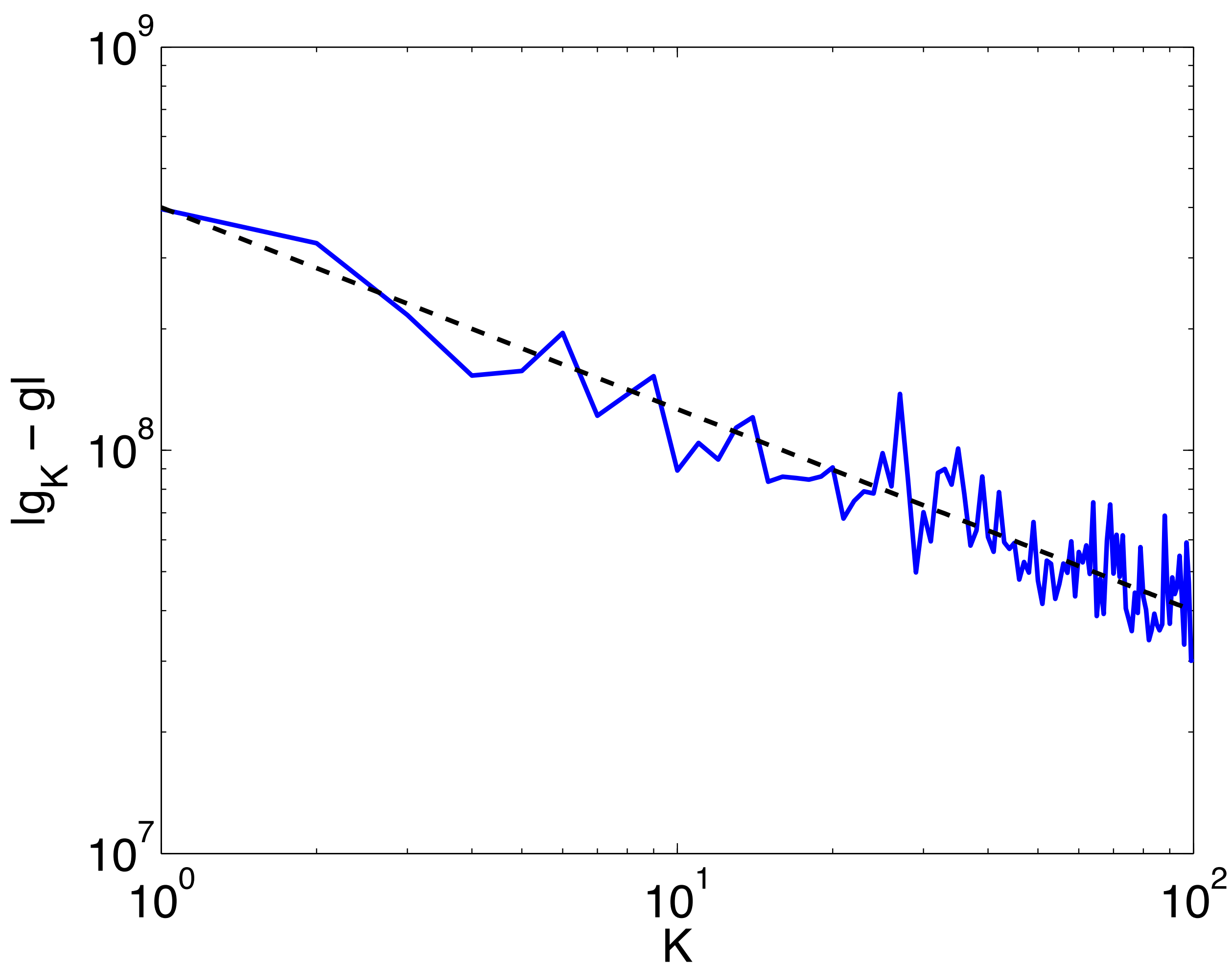


K=5



K=10

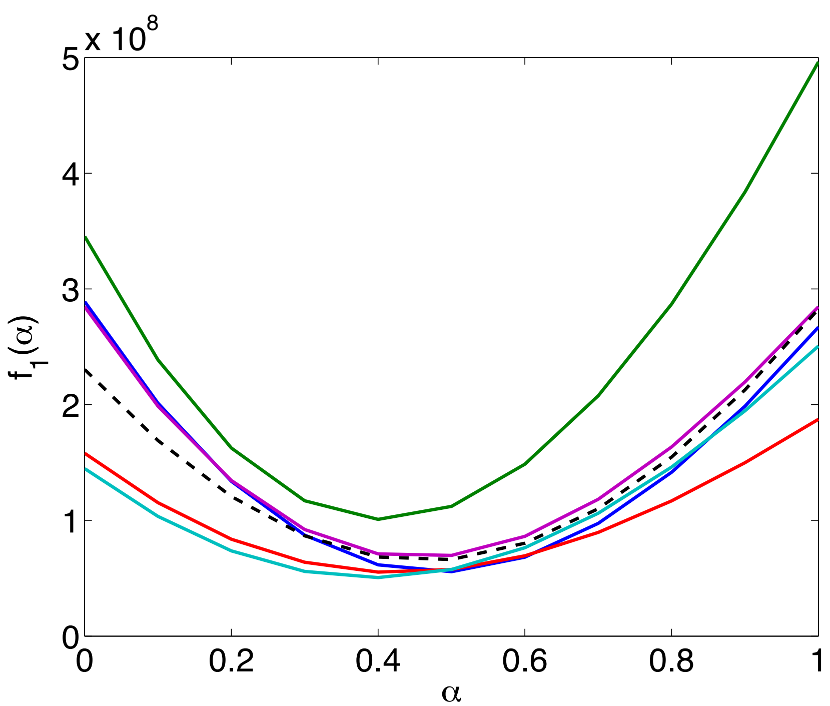
Decay



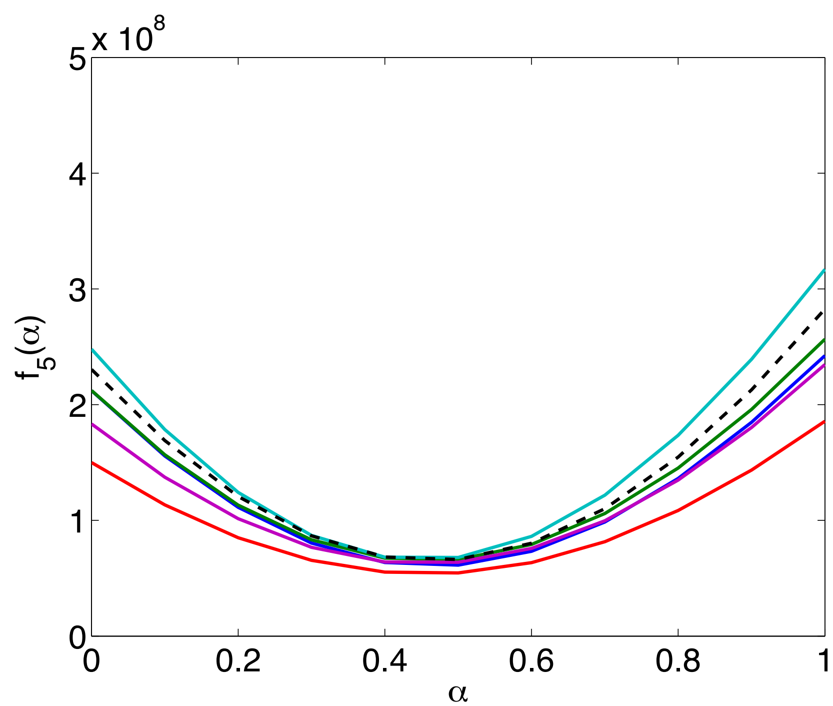
error between full and sampled gradient

Misfit functional

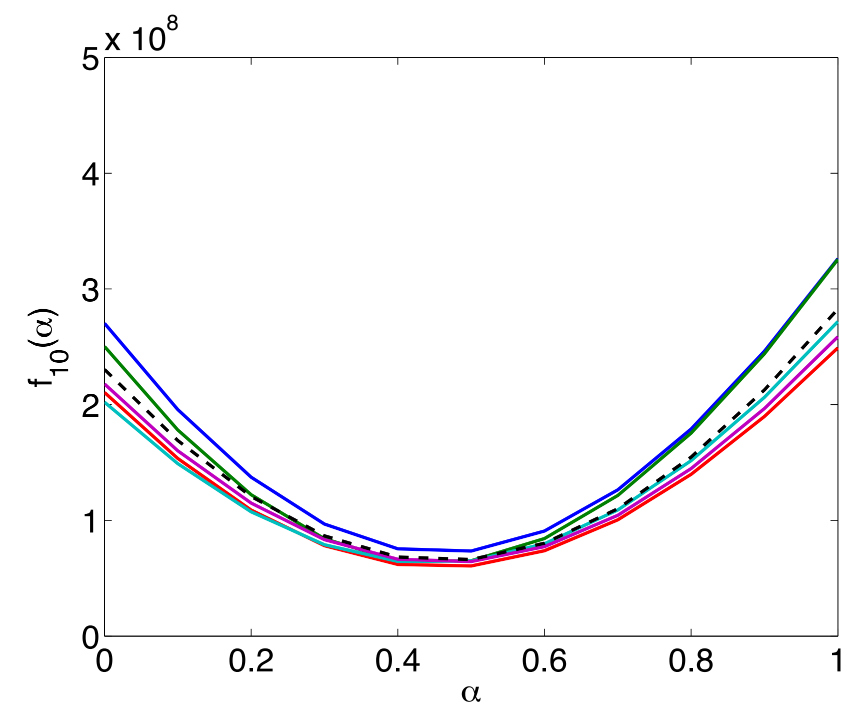
$$f_K(\mathbf{g}_K) = \frac{1}{K}\sum_{j=1}^{K}\frac{1}{2}\|\underline{\mathbf{d}}_j - \mathcal{F}[\mathbf{m}+\alpha\mathbf{g}_K;\underline{\mathbf{q}}_j]\|_2^2$$



K=1



K=5



K=10

[Haber, Chung, and FJH, '10; van Leeuwen, Aravkin, FJH, '10]

Stochastic *average* approximation

In the *limit* $K \to \infty$, *stochastic* & *deterministic* formulations are *identical*

We *gain* as long as $K \ll N$ ...

But the error in *Monte-Carlo* methods decays only slowly $(\mathcal{O}(K^{-1/2}))$

Stochastic approximation [Bertsekas,' '96; Nemirovski, '09]

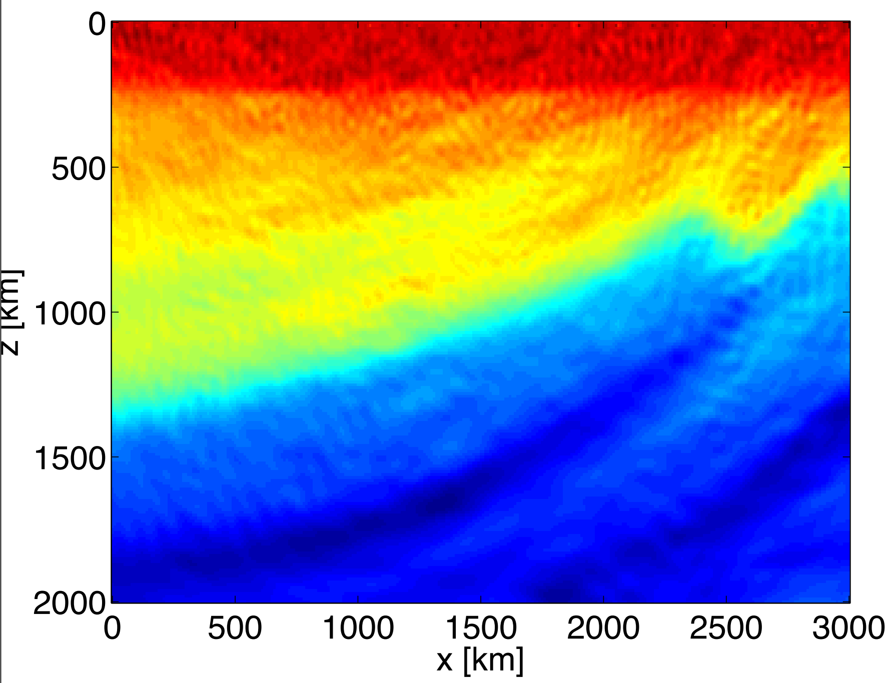
Use *different* simultaneous shots for each *subproblem*, i.e.,

$$\underline{\mathbf{Q}} \quad \mapsto \quad \underline{\mathbf{Q}}^k$$

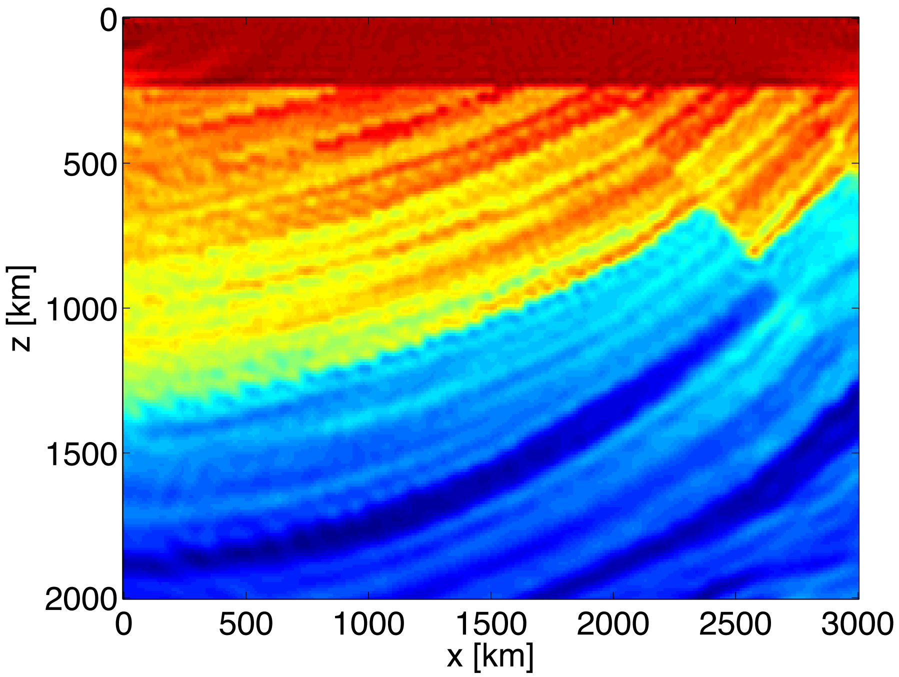
Requires *fewer* PDE solves for each *subproblem*...

- corresponds to the *stochastic approximation*
- related to Kaczmarz ('37) method applied by Natterer, '01
- *supersedes ad hoc* approach by Krebs *et.al.*, '09

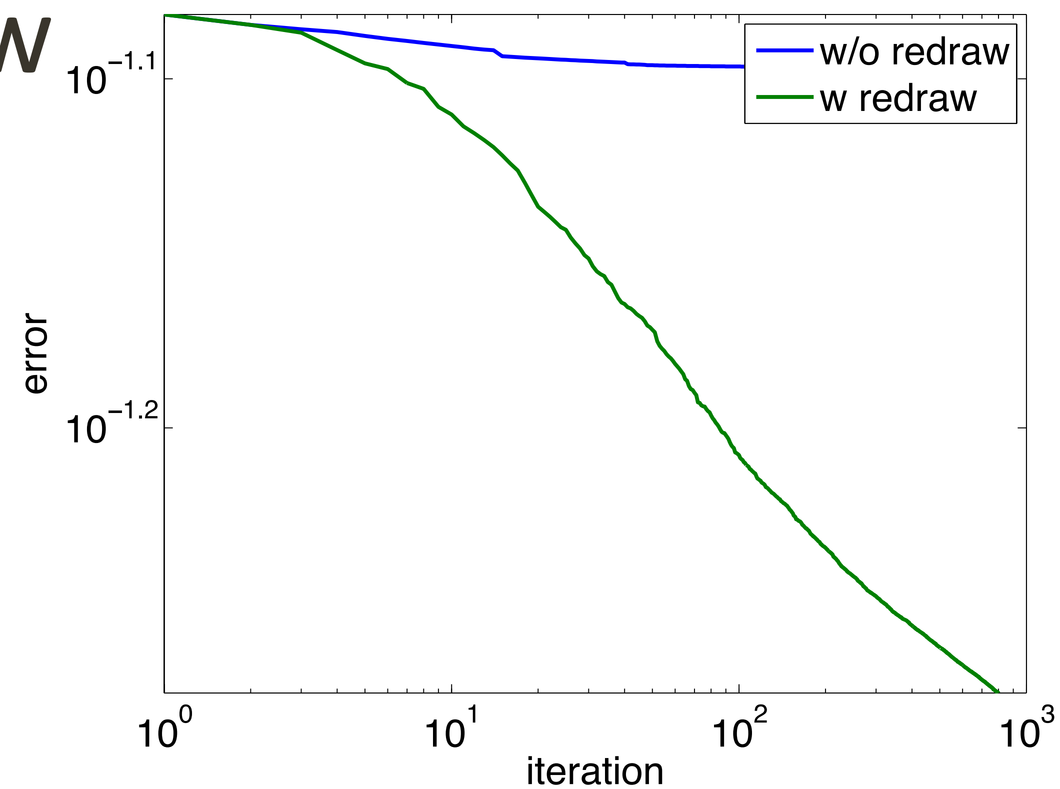
K=1 w and w/o redraw [noise-free case]



w/o redraw



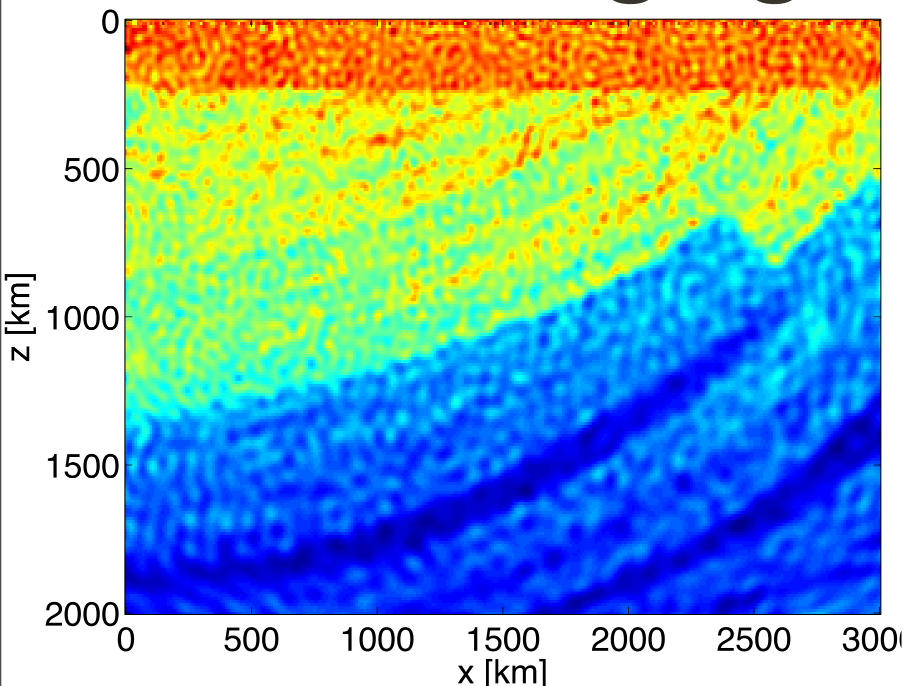
w redraw



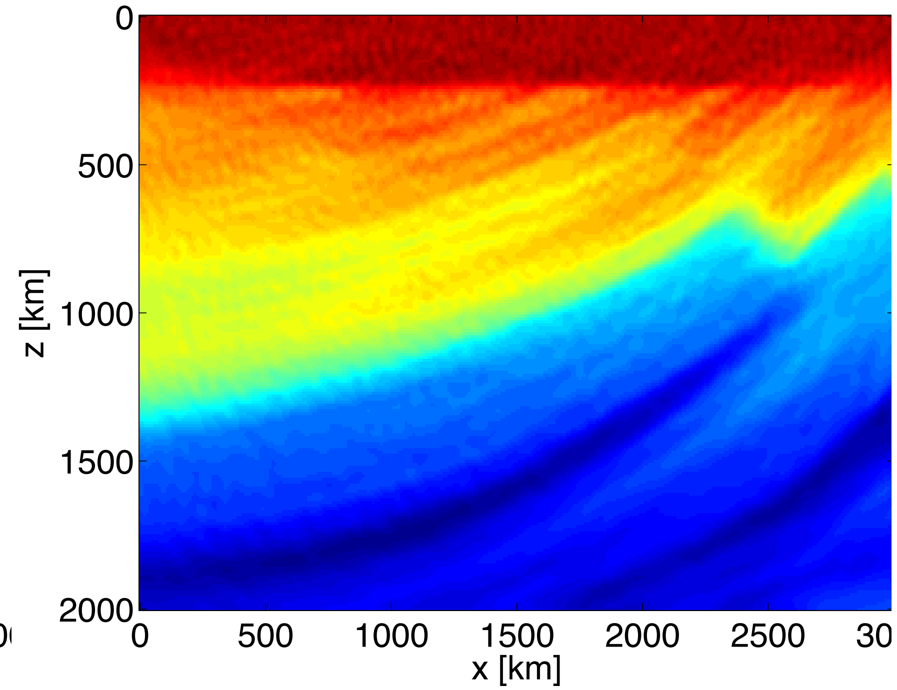
model error K=1

K=1 [noisy case]

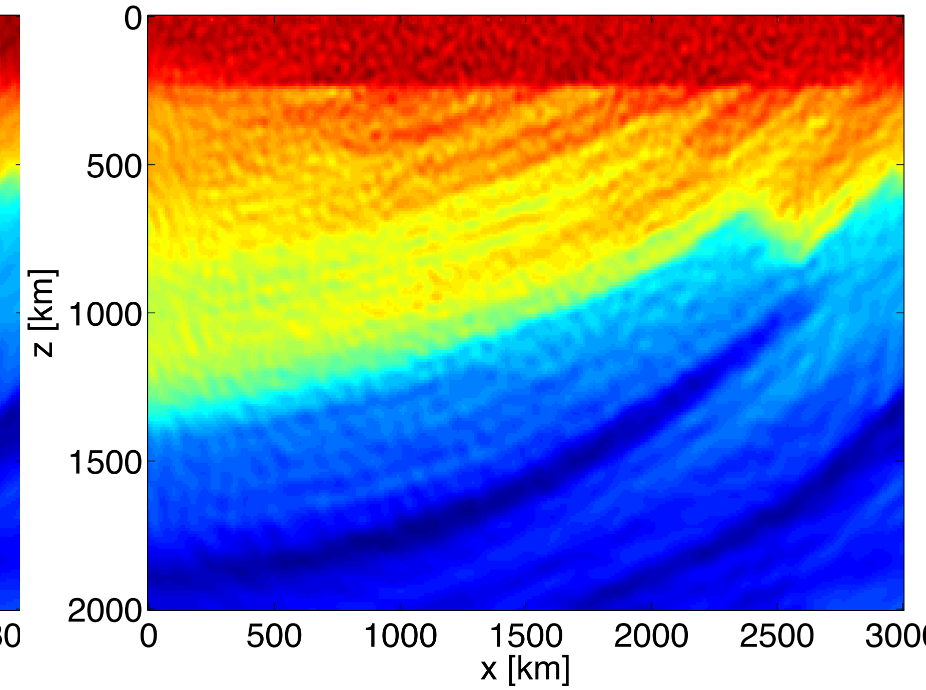
w/o averaging

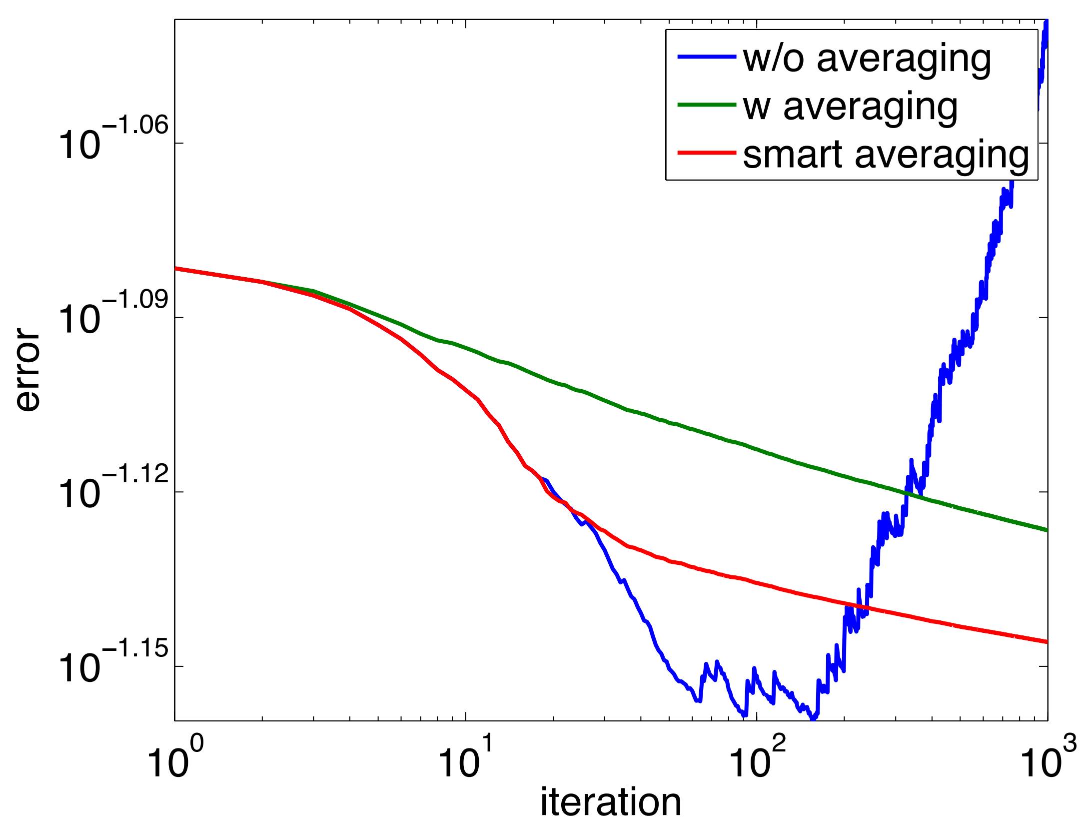


w averaging



smart averaging





Observations

SAA:

- Error decays slowly with batch size K
- becomes worse when noisy

SA

- Renewals improve convergence *significantly*
- Requires *averaging* to remove noise *instability*, which is *detrimental* to the *convergence*

Dimensionality reduction gives '*noisy*' results ... Sounds *familiar?*

Combined approach

Leverage findings from *sparse recovery* & *compressive sensing*

- consider *phase-encoded* Gauss-Newton updates as separate "*compressive-sensing* /$\ell_1$regularized experiments"
- remove *interferences* by *curvelet-domain sparsity* promotion
- exploit properties of Pareto curves in combination with stochastic optimization
- turn 'overdetermined' problems with large matrix-setup costs into 'undetermined' problems via *randomization*

Rationale [Smith, '97; Candes & Demanet, '03]

Wavefields are *compressible* in curvelet *frames*

- *correlations* between *source & residual wavefields* are *compressible*
- *velocity* distributions of *sedimentary basins* are *also compressible*

Linearized subproblems are *convex*

Assume proximity Pareto *curves* amongst *successive GN iterations*

Modified Gauss-Newton

- Objective:

$$\underline{f}(\mathbf{m}) := \|\underline{\mathbf{D}} - \boldsymbol{\mathcal{F}}[\mathbf{m}; \underline{\mathbf{Q}}]\|_F^2$$

- Iterative algorithm:

$$\mathbf{m}^{\nu+1} = \mathbf{m}^{\nu} + \gamma_\nu \boldsymbol{\mathcal{C}}^* \overline{\boldsymbol{\delta}\mathbf{x}}$$

- Direction $\overline{\boldsymbol{\delta}\mathbf{x}}$ solves

$$\min_{\boldsymbol{\delta}\mathbf{x}} \quad \|\underline{\mathbf{D}} - \boldsymbol{\mathcal{F}}[\mathbf{m}^\nu; \underline{\mathbf{Q}}] - \nabla \boldsymbol{\mathcal{F}}[\mathbf{m}^\nu; \underline{\mathbf{Q}}]\boldsymbol{\mathcal{C}}^* \boldsymbol{\delta}\mathbf{x}\|_F^2$$
$$\text{s.t.} \quad \|\boldsymbol{\delta}\mathbf{x}\|_1 \leq \tau$$

- The subproblem for $\overline{\boldsymbol{\delta}\mathbf{x}}$ is convex, and $\boldsymbol{\mathcal{C}}^* \overline{\boldsymbol{\delta}\mathbf{x}}$ is a ***descent*** direction:

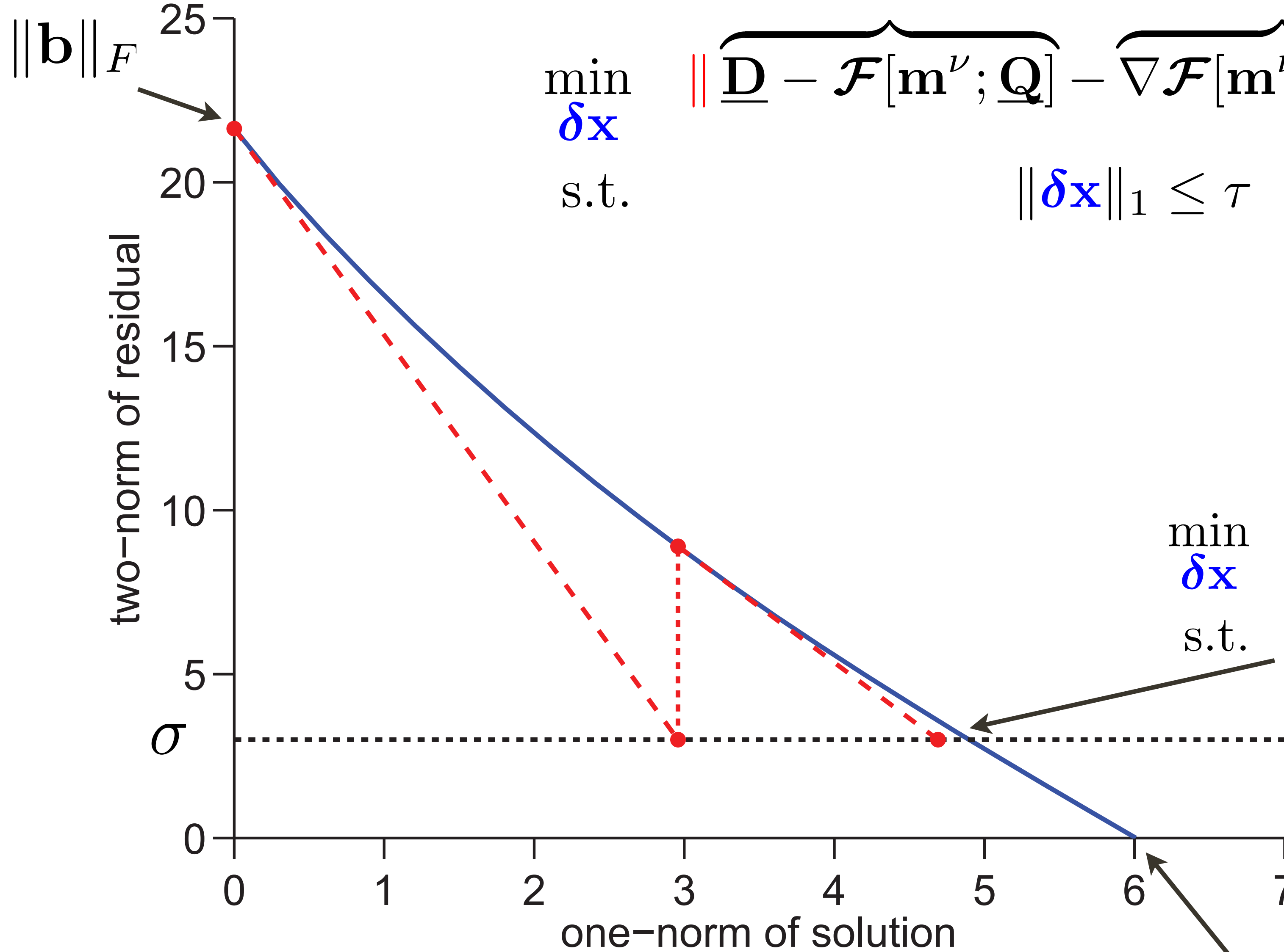
$$\underline{f}'(\mathbf{m}^\nu; \boldsymbol{\mathcal{C}}^* \overline{\boldsymbol{\delta}\mathbf{x}}) \leq \underline{f}(\mathbf{m}^\nu) - \| \underbrace{\underline{\mathbf{D}} - \boldsymbol{\mathcal{F}}[\mathbf{m}^\nu; \mathbf{Q}]}_{\underline{f}(\mathbf{m}^\nu)} - \nabla \boldsymbol{\mathcal{F}}[\mathbf{m}; \mathbf{Q}]\boldsymbol{\mathcal{C}}^* \overline{\boldsymbol{\delta}\mathbf{x}}\|_F^2 < 0$$

[Burke ’89, Burke ’92]

Picking Lasso Parameter

$$\min_{\boldsymbol{\delta}\mathbf{x}} \quad \| \overbrace{\underline{\mathbf{D}} - \mathcal{F}[\mathbf{m}^{\nu}; \underline{\mathbf{Q}}]}^{\mathbf{b}} - \overbrace{\nabla\mathcal{F}[\mathbf{m}^{\nu}; \underline{\mathbf{Q}}]\mathcal{C}^{*}}^{\mathbf{A}} \boldsymbol{\delta}\mathbf{x} \|_F^2 \quad \text{s.t.} \quad \|\boldsymbol{\delta}\mathbf{x}\|_1 \leq \tau$$

Lasso



$$\min_{\boldsymbol{\delta}\mathbf{x}} \quad \|\boldsymbol{\delta}\mathbf{x}\|_1 \quad \text{s.t.} \quad \|\mathbf{b} - \mathbf{A}\boldsymbol{\delta}\mathbf{x}\|_F \leq \sigma$$

BPDN

$$\min_{\boldsymbol{\delta}\mathbf{x}} \quad \|\boldsymbol{\delta}\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathbf{b} = \mathbf{A}\boldsymbol{\delta}\mathbf{x}$$

BP

[van den Berg '08]

Modified GN with renewals

Algorithm 1: Modified Gauss-Newton with renewals

Result: Output estimate for the model $\mathbf{m}$

$\mathbf{m} \longleftarrow \mathbf{m}_0$; $k \longleftarrow 0$; $\overline{\delta\mathbf{x}} \longleftarrow 0$; `// initial model`

for $j = 1 : M$ **do**

Obtain frequency band j, corresponding data slice $\mathbf{D}$ and operator $\boldsymbol{\mathcal{F}}$

while *not converged* **do**

Randomly subsample to obtain $\underline{\mathbf{D}}^k, \underline{\mathbf{Q}}^k$.

Solve with warm start $\overline{\delta\mathbf{x}}$

$$\overline{\delta\mathbf{x}} \longleftarrow \begin{cases} \underset{\delta\mathbf{x}}{\arg\min} \quad \|\underline{\mathbf{D}}^k - \boldsymbol{\mathcal{F}}[\mathbf{m}^k; \underline{\mathbf{Q}}^k] - \nabla\boldsymbol{\mathcal{F}}[\mathbf{m}^k; \underline{\mathbf{Q}}^k]\boldsymbol{\mathcal{C}}^*\delta\mathbf{x}\|_F \\ \qquad\qquad \text{subject to} \quad \|\delta\mathbf{x}\|_1 \leq \tau^k \end{cases}$$

$\mathbf{m}^{k+1} \longleftarrow \mathbf{m}^k + \gamma^k \boldsymbol{\mathcal{C}}^* \overline{\delta\mathbf{x}}$; `// update with linesearch`

$k \longleftarrow k + 1$

end

end

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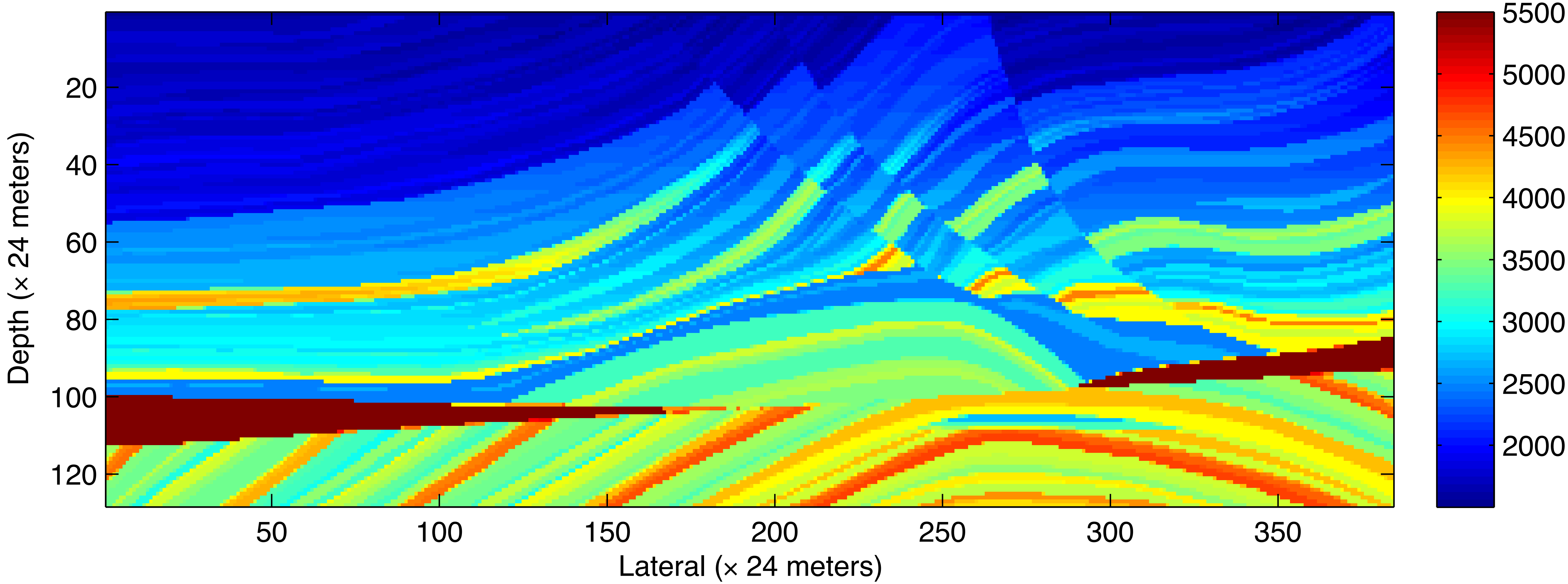
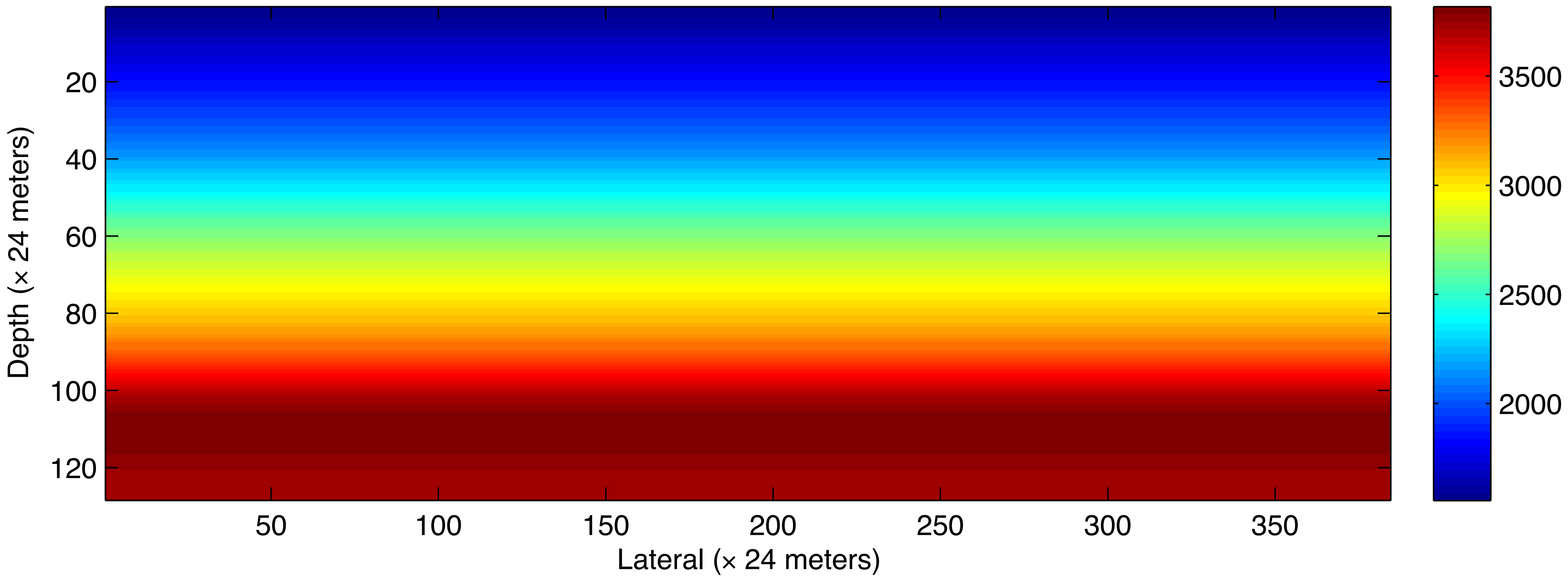
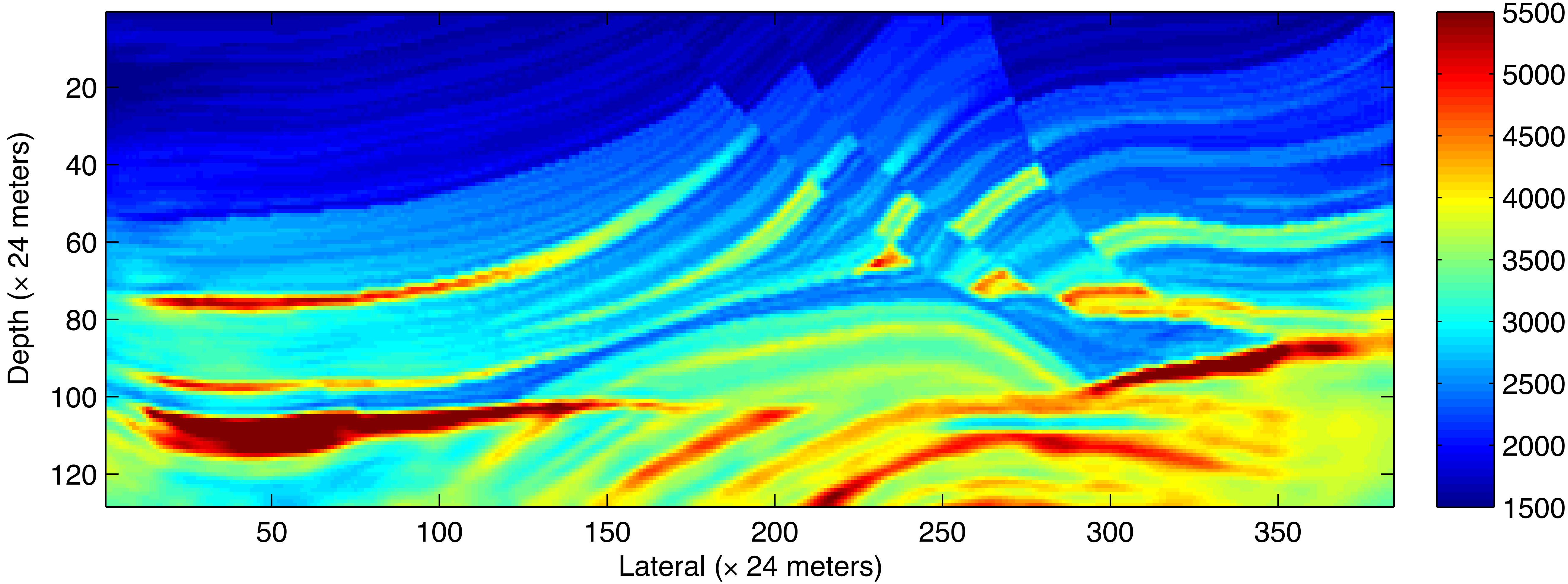
True model



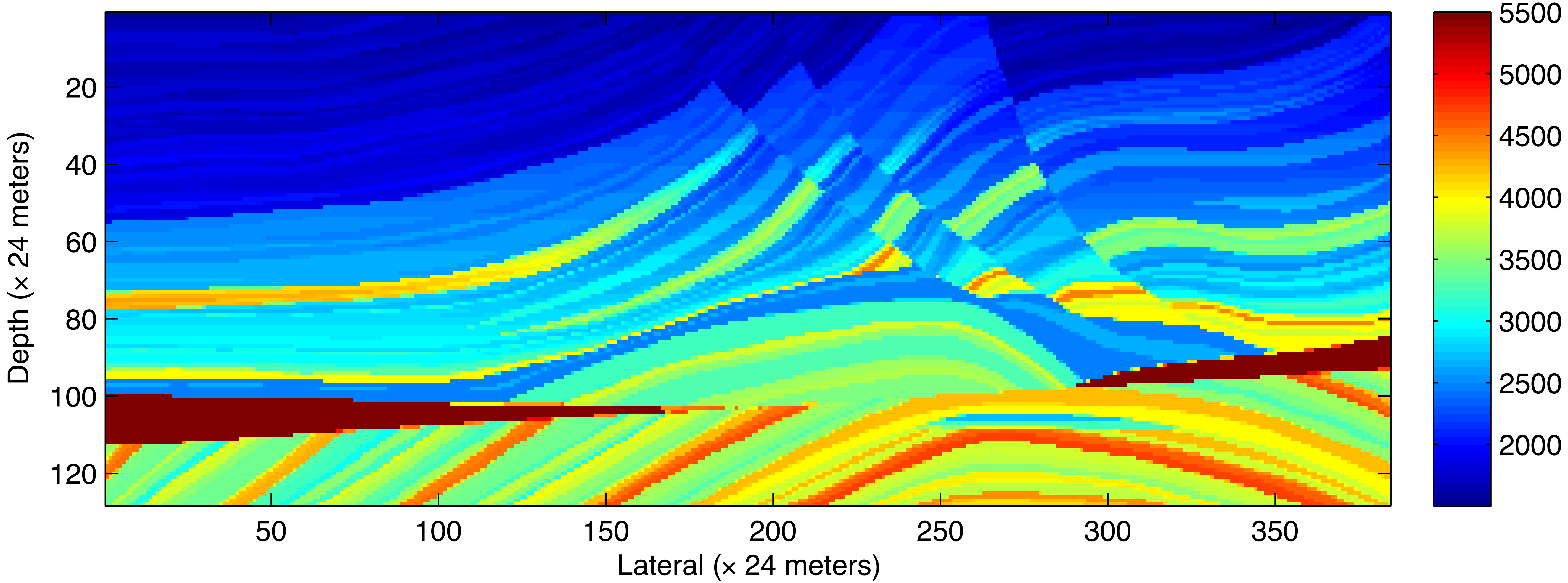
Initial model



Inverted model



True model


Depth (× 24 meters)
20
40
60
80
100
120
50
100
150
200
250
300
350
Lateral (× 24 meters)
5500
5000
4500
4000
3500
3000
2500
2000

Performance

Remember per *subproblem*

$$n_{PDE}^{\ell_1} \times K \ll n_{PDE}^{\ell_2} \times n_f \times n_s$$

$$\begin{aligned} n_{PDE}^{\ell_1} &\approx 200 \\ K &= 150 \end{aligned}$$

versus

$$\begin{aligned} n_{PDE}^{\ell_2} &\approx 10 \\ K &= 38400 \end{aligned}$$

SPEEDUP of 13 X

Conclusions

Leveraged

- curvelet-domain sparsity on the model
- invariance under solution operators <=> preservation of sparsity

Indications that *compressive sensing* supersedes the *stochastic approximation by sparse recovery of dimensionality reduced* subproblems

Extension to 3D (5D data) will lead to larger *improvements*...

Open problems
[some of them]

Preconditioner for *indirect* Helmholtz solvers in 3D

Extension to incomplete data, i,e, $\mathbf{P} \mapsto \mathbf{P}_i$ (Hadamard product)

Analysis of performance of the proposed algorithm

- extension to nonlinear problems
- behavior Pareto curves etc.

Non-convexity of FWI

- *'ad-hoc' multiscale* continuation methods

'Holy grail' [FWI with focusing]

Convexification by *extensions*

$$\tilde{\mathbf{X}} = \arg\min_{\mathbf{X}\in\boldsymbol{\mathcal{X}}} \|\mathbf{X}\|_{\boldsymbol{\mathcal{A}}} \quad \text{subject to} \quad \|\mathbf{D} - \boldsymbol{\mathcal{F}}[\mathbf{X};\mathbf{Q}]\|_{2,2} \le \sigma$$

$$\tilde{\mathbf{m}} = \mathrm{diag}\{\mathbf{S}^H\mathbf{X}\} \quad \text{with} \quad \mathbf{X} \quad \text{the extension}$$

$\boldsymbol{\mathcal{F}}[\mathbf{X};\mathbf{Q}] := \mathbf{P}\bar{\mathbf{H}}^{-1}[\mathbf{S}^H\mathbf{X}]\mathbf{Q}$, $\mathbf{S}^H\mathbf{X}$ *positive-definite matrix*

1. $\|\mathbf{X}\|_{\boldsymbol{\mathcal{A}}} = \|\overbrace{A_h}^{\text{annihilator}}\mathbf{X}\|_{1,2}$ [Symes, '09]

2. $\|\mathbf{X}\|_{\boldsymbol{\mathcal{A}}} = \|\mathbf{X}\|_{*}$ **?**

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Further reading

Compressive sensing

- *Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information by Candes, 06*.
- *Compressed Sensing* by D. Donoho, '06

Simultaneous simulations, imaging, and full-wave inversion:

- *Faster shot-record depth migrations using phase encoding* by Morton & Ober, '98.
- *Phase encoding of shot records in prestack migration* by Romero et. al., '00.
- *High-resolution wave-equation amplitude-variation-with-ray-parameter (AVP) imaging with sparseness constraints* by Wang & Sacchi, '07
- Efficient Seismic Forward Modeling using Simultaneous Random Sources and Sparsity by N. Neelamani et. al., '08.
- *Compressive simultaneous full-waveform simulation* by FJH et. al., '09.
- *Fast full-wavefield seismic inversion using encoded sources* by Krebs et. al., '09
- *Randomized dimensionality reduction for full-waveform inversion* by FJH & X. Li, '10

Stochastic optimization and machine learning:

- *A Stochastic Approximation Method* by Robbins and Monro, 1951
- *Neuro-Dynamic Programming* by Bertsekas, '96
- *Robust stochastic approximation approach to stochastic programming* by Nemirovski et. al., '09
- *Stochastic Approximation approach to Stochastic Programming* by Nemirovski
- *An effective method for parameter estimation with PDE constraints with multiple right hand sides. by* Eldad Haber, Matthias Chung, and Felix J. Herrmann. '10
- Seismic waveform inversion by stochastic optimization. Tristan van Leeuwen, Aleksandr Aravkin and FJH, 2010.

Full-waveform inversion with extensions

- *Migration velocity analysis and waveform inversion* by Symes Geophysical Prospecting, 56: 765–790, 2008.
- *The seismic reflection inverse problem* by Symes, Inverse Problems 25, 2009.

SLIM

Thank you

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