On the factorization of Sobolev inequalities through classes of functions

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(joint work with J. Bastero and J. Bernués)

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Sobolev inequality

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported C^1 function.

Sobolev inequality

$$\|\nabla f\|_p \geq \mathbf{C_{p,n}} \|f\|_q \ , \qquad p \in [1, n), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$$

where $\|\nabla f\|_p^p = \int_{\mathbb{R}^n} |\nabla f(x)|^p dx$

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- **Improvements and extensions from Analysis (right hand side)** Moser-Trudinger, Hanson, Brezis-Wainger, Maly-Pick, Tartar, Bastero-Milman-Ruiz, Martin...
- **Improvements and extensions from Geometry (left hand side)**. Lutwak, Yang, Zhang, Cianchi, Haberl, Schuster, Xiao...
- **The results**

Two remarks on Sobolev inequality

- The case $p = 1$ is equivalent to the isoperimetric inequality,

$$\|\nabla f\|_1 \geq n\, \omega_n^{\frac{1}{n}} \|f\|_{\frac{n}{n-1}} \iff S(\partial K) \geq n\, \omega_n^{\frac{1}{n}} |K|_n^{\frac{n-1}{n}}$$

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- Sobolev inequality follows from Polya-Szegö rearrangement inequality

$$\|\nabla f\|_p \geq \|\nabla f^\circ\|_p \qquad\qquad p \geq 1$$

where $f^\circ(x) := f^*(\omega_n |x|^n)$, is the radial extension to $\mathbb{R}^n$ of the nonincreasing rearrangement f^*

$$f^*(t) = \inf\{\lambda > 0 \;\; |\{|f| > \lambda\}| \leq t\}.$$

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Extensions and improvements (Analysis)

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The case p=n

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- Moser and Trudinger, 1969-71, introduced an Orlicz space $\mathcal{MT}$ and showed

$$\|\nabla f\|_n \geq c_n \|f\|_{\mathcal{MT}}$$

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Dependence on the support of f

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Tartar, Maly-Pick, and Bastero-Milman-Ruiz, 1998-2003 introduced *classes of functions.* For $1 \leq p < \infty$ denote

$$\mathcal{A}_{\infty,p}(\mathbb{R}^n) = \{f; \|f\|_{\infty,p} = \left(\int_0^\infty (f^{**}(t) - f^*(t))^p \frac{dt}{t^{p/n}}\right)^{1/p} < \infty\}$$

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Bastero-Milman-Ruiz, 2003

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t), \quad a.e. \quad t \geq 0$$

Extensions and improvements (Analysis)

As a Corollary,

Bastero-Milman-Ruiz, 2003

$$\|\nabla f\|_n \geq (n-1)\,\omega_n^{\frac{1}{n}}\,\|f\|_{\infty,n} \geq c_n\|f\|_{H_n}$$

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Once classes of functions are allowed,

Martin-Milman, 2010

$$\|\nabla f\|_p \geq \mathbf{c_{n,p}}\|f\|_{\infty,p} \geq \mathbf{c'_{n,p}}\|f\|_q \qquad 1 \leq p < n$$

No dependence on the support of f.

Extensions and improvements (Geometry)

Consider the class of functions

$$\mathcal{E}_p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}; \mathcal{E}_p(f) := \frac{1}{I_p} \left(\int_{S^{n-1}} \|D_u f\|_p^{-n} du \right)^{-\frac{1}{n}} < \infty \right\}$$

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where $D_u f(x) = \langle \nabla f(x), u \rangle$

and $I_p^p = \int_{S^{n-1}} |u_1|^p du$ is a normalization constant so that

$$\mathcal{E}_p(f^\circ) = \|\nabla f^\circ\|_p$$

Extensions and improvements (Geometry)

Zhang ($p = 1$, 1999), Lutwak-Yang-Zhang (general case, 2002)

$$\mathcal{E}_p(f) \geq \mathcal{E}_p(f^\circ) \ , \qquad 1 \leq p < \infty$$

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By Jensen's inequality

$$\|\nabla f\|_p = \frac{1}{I_p}\left(\int_{S^{n-1}} \|D_u f(x)\|_p^p\right)^{\frac{1}{p}} \geq \frac{1}{I_p}\left(\int_{S^{n-1}} \|D_u f(x)\|_p^{-n}\right)^{-\frac{1}{n}} = \mathcal{E}_p(f)$$

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Corollary

Let $p \geq 1$

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Let $p \in [1, n)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$

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Remark and Observation

- The case $p = 1$ (Zhang) is equivalent to the Petty projection inequality.
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Using Zhang's original ideas ($p = 1$) and techniques from the usual proof of the Polya-Szegö inequality, one has (penalty on the constants)

Proposition (A., Bastero, Bernués.)

Let $1 \leq p < \infty$ then

$$\mathcal{E}_p(f^\circ) \leq \frac{I_p}{I_1}\mathcal{E}_p(f) \sim \sqrt{p}\mathcal{E}_p(f)$$

Asymmetric case

Haberl, Schuster, Xiao, $\geq$ 2009, stated the asymmetric case $\mathcal{E}_p^+(\mathbb{R}^n)$

$$\mathcal{E}_p^+(f) := \frac{2^{1/p}}{I_p} \left(\int_{S^{n-1}} \|D_u^+ f\|_p^{-n} du \right)^{-\frac{1}{n}}$$

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Theorem

Let $p \geq 1$

$$\mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \mathcal{E}_p^+(f^\circ) = \mathcal{E}_p(f^\circ)$$

Improvements and extensions (Geometry)

Cianchi, Lutwak, Yang, Zhang,(symmetric case) and Haberl, Schuster, Xiao, (asymmetric case) in $\geq$ 2009:

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The case p>n (and therefore negative q!)

$$\mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \left(\frac{p'}{|q|}\right)^{\frac{1}{p'}} n\omega_n^{1/n} |\mathrm{supp}\, f|_n^{1/q} \|f\|_\infty \qquad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

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and the constants *depending on the size of the support of* f are sharp.

The two approaches

$\boxed{1 \leq p < n}$

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\|\nabla f\|_p &\geq \qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq \mathbf{c_{n,p}}\|f\|_{\infty,p} \geq \mathbf{c'_{n,p}}\|f\|_q \\
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Results

Theorem

Let $1 \le p < \infty$ and $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{n}$. Then

$$\mathcal{E}_p^+(f^\circ) \ge \left(1 - \frac{1}{q}\right) n\, \omega_n^{1/n} \|f\|_{\infty,p}$$

The constant is sharp

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Proposition

Let $p > n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and f a compactly supported C^1 function. Then,

$$\alpha_{\mathbf{n,p}} \|f\|_{\infty,p} \geq \sup_{t>0}\{(\|f\|_\infty - f^*(t))\, t^{1/q}\} \geq \|f\|_\infty |\text{supp}\, f|_n^{1/q}$$

for some $\alpha_{\mathbf{n,p}} > 0$ (independent of the support of f)

The proof gives $\alpha_{\mathbf{n,p}} = \left(\left(p(1-\frac{1}{q})\right)^{p'/p} + \frac{|q|}{p'}\right)^{\frac{1}{p'}}$

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$$\begin{aligned} \|\nabla f\|_p &\geq \mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \qquad\qquad \geq c_{n,p}\|f\|_\infty \\ \|\nabla f\|_p &\geq \mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \mathbf{c_{n,p}}\|f\|_{\infty,p} \end{aligned}$$

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