Approximate stabilization of an infinite dimensional quantum stochastic system

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joint work with
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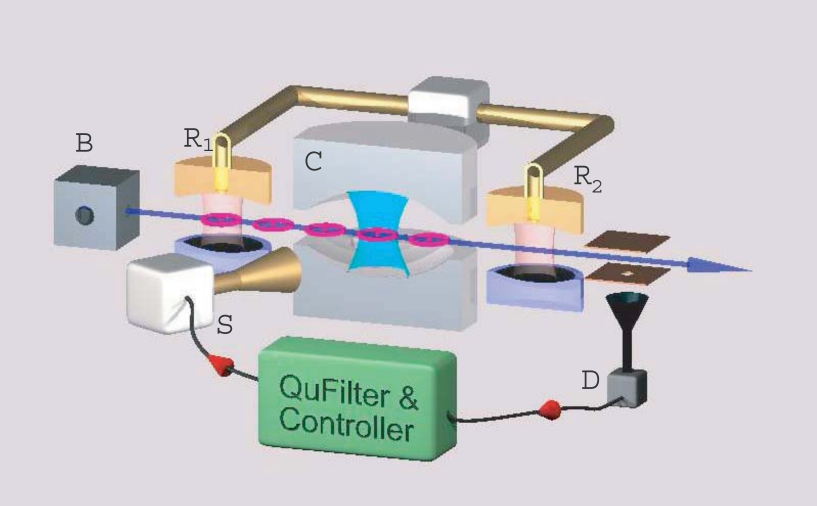
LKB photon-box

Experiment at Laboratoire Kastler-Brossel, École Normale Supérieure.



I. Dotsenko, M. Mirrahimi, M. Brune, S. Haroche, J.M. Raimond and P. Rouchon,
Phys. Rev. A., 2009.

LKB photon-box: measurement process (1)

Composite system

Field+atom: the Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_c$, where

$$\mathcal{H}_a = \text{span}(|g\rangle, |e\rangle)(\equiv \mathbb{C}^2), \qquad \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \;\middle|\; (c_n) \in l^2(\mathbb{C}) \right\}.$$

Initial state: $|g\rangle \otimes |\psi\rangle$

Joint unitary evolution

$|\Psi(t)\rangle \in \mathcal{H}_a \otimes \mathcal{H}_c$ being the state of the composite system,

$$i \frac{d}{dt} |\Psi\rangle = ((H_a \otimes \mathbf{1}) + (\mathbf{1} \otimes H_c) + H_{ac}) |\Psi\rangle .$$

The state after this unitary evolution is necessarily of the form

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle ,$$

where $\mathcal{M}_g$ and $\mathcal{M}_e$ are operators acting on $\mathcal{H}_c$. Furthermore, the unitarity condition implies

$$\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}.$$

LKB photon-box: measurement process (2)

Final state is inseparable: we can not write

$$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle = \left(\tilde{\alpha} |g\rangle + \tilde{\beta} |e\rangle\right) \otimes \left(\sum_n \tilde{c}_n |n\rangle\right).$$

We can not associate to the cavity (nor to the atom) a well-defined wavefunction just before the measurement.

However, we can still compute the probability of having the atom in $|g\rangle$ or in $|e\rangle$:

$$P_g = \left\| \mathcal{M}_g |\psi\rangle \right\|^2_{\mathcal{H}_c}, \qquad P_e = \left\| \mathcal{M}_e |\psi\rangle \right\|^2_{\mathcal{H}_c}.$$

Measurement result

Meas. in $|g\rangle$: $$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|g\rangle \otimes \mathcal{M}_g |\psi\rangle}{\left\| \mathcal{M}_g |\psi\rangle \right\|_{\mathcal{H}_c}},$$

Meas. in $|e\rangle$: $$|g\rangle \otimes \mathcal{M}_g |\psi\rangle + |e\rangle \otimes \mathcal{M}_e |\psi\rangle \longrightarrow \frac{|e\rangle \otimes \mathcal{M}_e |\psi\rangle}{\left\| \mathcal{M}_e |\psi\rangle \right\|_{\mathcal{H}_c}}.$$

LKB photon-box: quantum trajectories

State space

Hilbert space of the cavity field: $\mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n \left| n \right\rangle \;\middle|\; (c_n) \in l^2(\mathbb{C}) \right\}$ ..

Stochastic evolution: ψ_k the wave function after the measurement of atom number $k-1$.

$$\left| \psi_{k+1} \right\rangle = \begin{cases} \dfrac{D_{\alpha_k} \mathcal{M}_g \left| \psi_k \right\rangle}{\left\| \mathcal{M}_g \left| \psi_k \right\rangle \right\|_{\mathcal{H}}} & \text{Detect. in } \left| g \right\rangle \left(\text{proba. } \left\| \mathcal{M}_g \left| \psi_k \right\rangle \right\|_{\mathcal{H}}^2 \right) \\ \dfrac{D_{\alpha_k} \mathcal{M}_e \left| \psi_k \right\rangle}{\left\| \mathcal{M}_e \left| \psi_k \right\rangle \right\|_{\mathcal{H}}} & \text{Detect. in } \left| e \right\rangle \left(\text{proba. } \left\| \mathcal{M}_e \left| \psi_k \right\rangle \right\|_{\mathcal{H}}^2 \right) \end{cases}$$

Here

- $\mathcal{M}_g$ and $\mathcal{M}_e$ are measurement operators (bounded operators on the Hilbert space $\mathcal{H}_c$) satisfying $\mathcal{M}_g^\dagger \mathcal{M}_g + \mathcal{M}_e^\dagger \mathcal{M}_e = \mathbf{1}$.
- D_α is unitary operator of the form $\exp(\alpha a^\dagger - \alpha^* a)$ with operator a defined on $\mathcal{H}_c$ with its domain a subset of $\mathcal{H}_c$ (annihilation operator), and where α is a complex control.

Physical operators

Annihilation and creation operators

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots \\ \sqrt{1} & 0 & 0 & \cdots & \cdots \\ 0 & \sqrt{2} & 0 & \cdots & \cdots \\ 0 & 0 & \sqrt{3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \sqrt{n+1} & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Photon counting operator: $\mathbf{N} = a^\dagger a = \text{diag}(0, 1, 2, 3, \cdots)$.
Domains:

$$\mathcal{D}(a) = \mathcal{D}(a^\dagger) = \{\sum_{n=0}^{\infty} c_n |n\rangle \mid (c_n)_{n=0}^{\infty} \in h^1(\mathbb{C})\}, \quad \mathcal{D}(\mathbf{N}) = \{\sum_{n=0}^{\infty} c_n |n\rangle \mid (c_n)_{n=0}^{\infty} \in h^2(\mathbb{C})\}$$

where $\quad h^k(\mathbb{C}) = \{(c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \mid \sum_{n=0}^{\infty} n^k |c_n|^2 < \infty\}$.

Dispersive measurement operators and displacement operator

- $\mathcal{M}_g = \cos(\phi_0 + \mathbf{N}\vartheta)$ and $\mathcal{M}_e = \sin(\phi_0 + \mathbf{N}\vartheta)$.
- $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$: operator $\alpha a^\dagger - \alpha^* a$ being anti-Hermitian and densely defined in $\mathcal{H}$, it defines a strongly continuous group of isometries on $\mathcal{H}$.

Control problem: finite-dimensional approximation

Hilbert space after a Galerkin approximation:

$$\mathcal{H}_c = \left\{ \sum_{n=0}^{n^{\max}} c_n |n\rangle \mid (c_n)_{n=0}^{n^{\max}} \in \mathbb{C} \right\}$$

Control goal: to stabilize the Fock state $|\bar{n}\rangle$.

Lyapunov approach: Lyapunov function $\mathcal{V}(\psi) = 1 - |\langle \bar{n} \mid \psi \rangle|^2$

Nous choisissons α_k tel que:

$$\mathbb{E}\left(\mathcal{V}(\psi_{k+1}) \mid \psi_k\right) \leq \mathcal{V}(\psi_k).$$

Proof of convergence is based on **stochastic versions of Lyapunov techniques (Doob's inequality and Kushner's invariance principle)**:

- I. Dotsenko, M. Mirrahimi, M. Brune, S. Haroche, J.M. Raimond, P. Rouchon, Phys. Rev. A, 2009.
- H. Amini, M. Mirrahimi and P. Rouchon, Submitted.

Reminder: stability and convergence of stochastic

Doob's Inequality

Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then

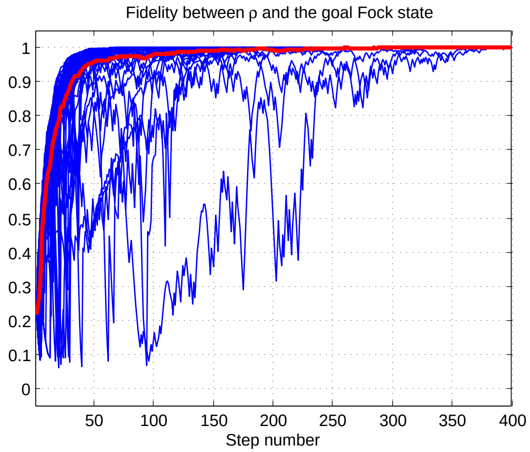
$$\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Kushner's invariance Theorem

Consider the same assumptions as that of the Doob's inequality. Let $\mu_0 = \sigma$ be concentrated on a state $x_0 \in Q_\lambda$, i.e. $\sigma(x_0) = 1$. Assume that $0 \leq k(X_n) \to 0$ in Q_λ implies that $X_n \to \{x \mid k(x) = 0\} \cap Q_\lambda \equiv K_\lambda$. For the trajectories never leaving Q_λ, X_n converges to K_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_n$ tend to the largest invariant set of measures $M_\infty \subset M$ whose support set is in K_λ. Finally, for the trajectories never leaving Q_λ, X_n converges, in probability, to the support set of M_∞.

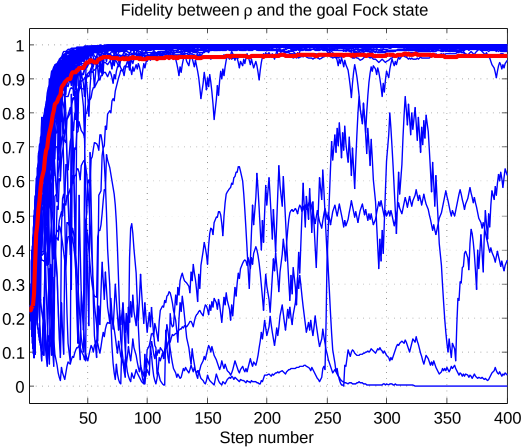
Finite-dimensional control problem: simulations

100 Random trajectories for a finite-dimensional approximation with a maximum photon number of 10 and where the target is the Fock state $|3\rangle$.



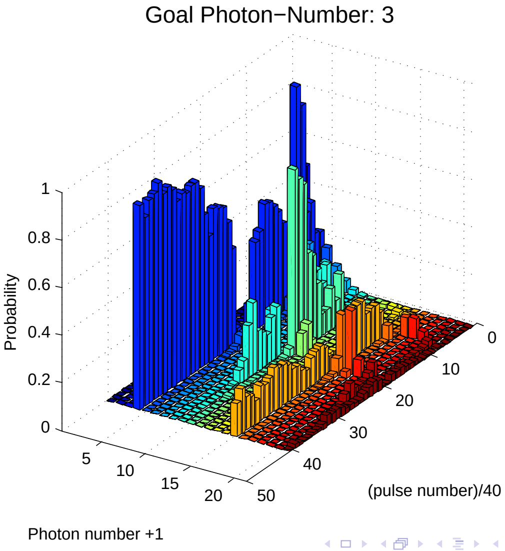
Control problems: infinite dimensions

100 Random trajectories with the same feedback law, where the controller is simulated based on a Galerkin approximation with a maximum photon number of 10 and the real system is simulated based on a Galerkin approximation with a maximum photon number of 20.



Control problems: infinite dimensions

A single trajectory showing the mass-loss through high-energy levels.



Control problem: infinite dimensions

Lyapunov approach

Lyapunov function:

$$\mathcal{V}(\psi) = \sum_{n=0}^{\infty} \sigma_n \left|\langle n \mid \psi\rangle\right|^2$$

$$+\delta\left(1 - f(|\langle n \mid \psi\rangle|^2)\right) + \delta\left(\cos^4(\phi_{\bar{n}}) + \sin^4(\phi_{\bar{n}}) - \left\|\mathcal{M}_g \left|\psi\right\rangle\right\|^2 - \left\|\mathcal{M}_e \left|\psi\right\rangle\right\|^2\right).$$

where $f(x) = (x + x^2)/2$, $\phi_{\bar{n}} = \phi_0 + \bar{n}\vartheta$ and

$$\sigma_n = \sum_{k=n+1}^{\bar{n}} \left(\frac{1}{k} - \frac{1}{k^2}\right), \quad n < \bar{n}, \quad \sigma_{\bar{n}} = 0, \quad \sigma_n = \sum_{k=\bar{n}+1}^{n} \left(\frac{1}{k} + \frac{1}{k^2}\right), \quad n > \bar{n}.$$

Feedback law: $\alpha_k = \alpha(\psi_k) := \text{argmin}_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathcal{V}(D_\alpha \left|\psi_k\right\rangle)$.

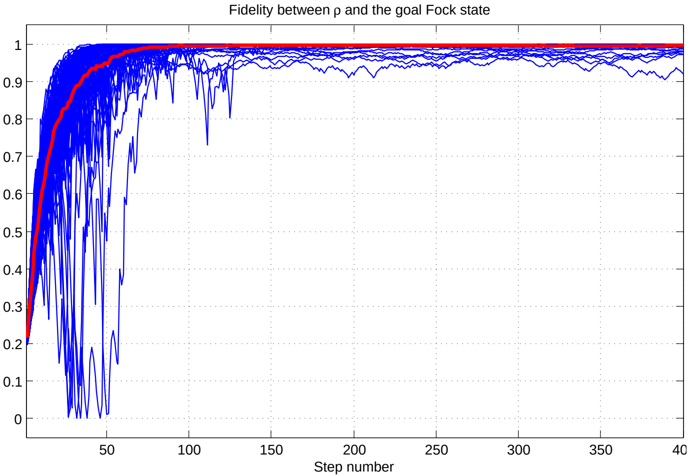
convergence result (recent joint work with Ram Somaraju and Pierre Rouchon)

For each $\epsilon > 0$, we can choose $\delta > 0$ small enough so that

$$\mathbb{P}(\left|\psi_k\right\rangle \to \left|\bar{n}\right\rangle) > 1 - \epsilon.$$

Control problem: infinite dimensions (Simulations)

100 Random trajectories with the above feedback law, where the controller is simulated based on a Galerkin approximation with a maximum photon number of 10 and the real system is simulated based on a Galerkin approximation with a maximum photon number of 20.



Proof's scheme (1)

We consider the Hilbert spaces l^2 and h^σ, together with the norms

$$\| |\psi\rangle \|_{l^2} = \sum_{n=0}^{\infty} |\langle n \mid \psi\rangle|^2 \qquad \text{and} \qquad \| |\psi\rangle \|_{h^\sigma} = \sum_{n=0}^{\infty} \sigma_n |\langle n \mid \psi\rangle|^2.$$

We consider the sequence of probability measures μ_k defined on the space of the wavefunctions and induced by the system's Markov chain.

Lemma 1

Applying the Doob's inequality, the set of measures $\{\mu_k\}$ is tight for the strong topology of l^2: indeed, taking $\mathcal{V}_0 := \mathbb{E}_{\mu_0}(\mathcal{V})$ and defining $\mathcal{B}_\epsilon = \{|\psi\rangle \mid \mathcal{V}(\psi) < \frac{\mathcal{V}_0}{\epsilon}\}$, we know by Doob's inequality that $\mu_k(\mathcal{B}_\epsilon) \geq 1 - \epsilon$. Furthermore, $\mathcal{B}_\epsilon$ is compact with respect to the l^2-strong topology.

Lemma 2

Applying Prokhorov's theorem, there exists a weakly converging subsequence μ_{k_n} such that, for each function $g(\psi)$ which is continuous with respect to the l^2-strong topology,

$$\mathbb{E}_{\mu_{k_n}}(g) \to \mathbb{E}_{\mu_\infty}(g).$$

Proof's scheme (2)

Bt the choice of the feedback law

$$\mathbb{E}\left(\mathcal{V}(\psi_{k+1}) \mid \psi_k\right) - \mathcal{V}(\psi_k) = -K_1(\psi_k) - K_2(\psi_k),$$

where the functions K_1 and K_2 are positive and

- K_1 is given by the difference of two lower semi-continuous functions with respect to l^2-strong topology;
- K_2 is continuous with respect to l^2-strong topology.

In particular

$$\mathbb{E}_{\mu_{k_{n+1}}}(\mathcal{V}) - \mathbb{E}_{\mu_{k_n}}(\mathcal{V}) = -\mathbb{E}_{\mu_{k_n}}(K_1) - \mathbb{E}_{\mu_{k_n}}(K_2).$$

Noting that $\mathbb{E}_{\mu_k}(\mathcal{V})$ is decreasing and bounded from below

$$\mathbb{E}_{\mu_{k_{n+1}}}(\mathcal{V}) - \mathbb{E}_{\mu_{k_n}}(\mathcal{V}) \to 0 \qquad \text{as } n \to \infty.$$

Thus

$$\mathbb{E}_{\mu_{k_n}}(K_2) \to 0 \qquad \text{as } n \to \infty.$$

This implies

$$\mathbb{E}_{\mu_\infty}(K_2) = 0.$$

Proof's scheme (3)

ω-limit set

There exists M_δ tending to $+\infty$ when δ tends to zero, such that,

$$\mathbb{E}_{\mu_\infty}(K_2) = 0$$
$$\Rightarrow$$
$$\text{supp}(\mu_\infty) \subset \{|\bar{n}\rangle\} \cup \{|m\rangle \ | \ m > M_\delta\}.$$

Final proposition

Noting that for $|m\rangle$ such that $m > M_\delta$,

$$\mathcal{V}(|m\rangle) \geq \sigma_m > \sigma_{M_\delta}$$

and applying the Doobs's inequality:

$$\mu_\infty(\{|\bar{n}\rangle\}) > 1 - \frac{\mathcal{V}_0}{\sigma_{M_\delta}}.$$

Summary

Objective: proving approximate stabilization whenever the pre-compactness is not ensured because of a mass-loss type phenomena .

$$dX = f(X)dt + \sigma(X)d\nu_t,$$

and $\mathcal{V}$ such that

$$\frac{d\mathbb{E}(\mathcal{V})}{dt} \leq -\mathbb{E}(K(X)).$$

A strict Lyapunov approach

The elements of $\chi = \{X \mid K(X) = 0\}$ are restricted to $X = \bar{X}$ or X such that $\mathcal{V}(X) > \mathcal{V}_0$:

1. $K(X)$ continuous for a weak-topology;
2. Decrease of $\mathcal{V}(X)$ prevents a mass-loss phenomena.