On the quadratic finite element approximation of $1-d$ waves: propagation, observation, control and numerical implementation

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The continuous model

The $1-d$ wave equation with non-homogeneous boundary conditions:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ y(0,t) = 0, \ y(1,t) = v(t), \ t > 0, \\ y(x,0) = y^0(x), \ y_t(x,0) = y^1(x), \ x \in (0,1). \end{cases} \tag{1}$$

Exact controllability: $\forall (y^1, y^0) \in \mathcal{V}' = H^{-1} \times L^2$, $\exists v \in L^2(0,T)$ s.t. $y(x,T) = y_t(x,T) = 0$.
The adjoint $1-d$ wave equation with homogeneous boundary conditions:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0, \\ u(x,T) = u^0(x), \ u_t(x,T) = u^1(x), \ x \in (0,1). \end{cases} \tag{2}$$

Well-posed in the energy space $\mathcal{V} := H_0^1 \times L^2$. The energy is conserved in time:

$$\mathcal{E}(u^0,u^1) = \frac{1}{2}(||u(\cdot,t)||^2_{H_0^1} + ||u_t(\cdot,t)||^2_{L^2}) = \frac{1}{2}(||u^0||^2_{H_0^1} + ||u^1||^2_{L^2}).$$

The **observability inequality** holds for all solutions of (2), provided $T \geq 2$:

$$\mathcal{E}(u^0,u^1) \leq C(T) \int_0^T |u_x(1,t)|^2 \, dt. \tag{3}$$

Hilbert Uniqueness Method (HUM): exact controllability $\Leftrightarrow$ observability inequality

Lions J.L., *Contrôlabilité exacte, perturbations et stabilisation des systèmes distribués*, Masson, 1988.

The HUM control

The HUM control v has the explicit form

$$v(t) = \tilde{v}(t) := \tilde{u}_x(1,t), \tag{4}$$

where $\tilde{u}(x,t)$ is the solution of (2) corresponding to the minimum $(\tilde{u}^0, \tilde{u}^1) \in \mathcal{V}$ of

$$\mathcal{J}(u^0,u^1) = \frac{1}{2}\int_0^T |u_x(1,t)|^2\,dt - \langle (y^1,-y^0),(u(\cdot,0),u_t(\cdot,0))\rangle_{\mathcal{V}',\mathcal{V}}. \tag{5}$$

Example: $y^1 \equiv 0$ and the initial position y^0 given by

$$y^0(x) = H(x) := \begin{cases} 1, & x \in [0,1/2) \\ -1, & x \in [1/2,1], \end{cases} \tag{6}$$

for which the optimal control is

$$\tilde{v}(t) = \tilde{v}_H(t) = \begin{cases} -1/2, & t \in (0,1/2] \cup (1,3/2], \\ 1/2, & t \in (1/2,1] \cup (3/2,2). \end{cases} \tag{7}$$

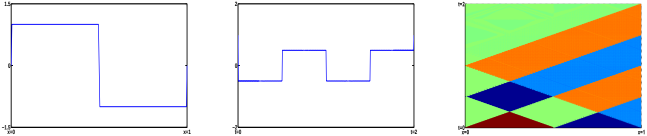

Figure: The initial position $H(x)$ (left) versus the HUM control $\tilde{v}_H$ (middle) versus the solution of the controlled problem (right) (red = 1, orange = 1/2, green = 0, cyan = -1/2 and blue = -1).

Linear and quadratic FEM spaces

$N \in \mathbb{N}$, $h = 1/(N+1)$. An **uniform grid** of $[0,1]$: nodes x_j and midpoints $x_{j+1/2}$.

$$\mathcal{U}_1^h := \{u \in H_0^1(0,1) \text{ s.t. } u|_{I_j} \in \mathcal{P}_1(I_j),\ 0 \le j \le N\} = \text{span}\{\phi_{1,j}^h\}$$

$$\mathcal{U}_2^h := \{u \in H_0^1(0,1) \text{ s.t. } u|_{I_j} \in \mathcal{P}_2(I_j),\ 0 \le j \le N\} = \text{span}\{\phi_{2,j}^h\} \oplus \text{span}\{\phi_{2,j+1/2}^h\}$$

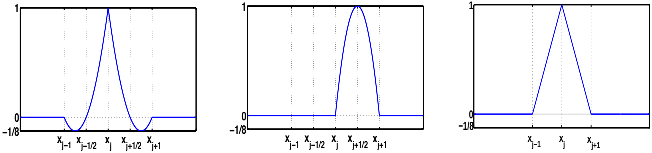

Figure: The basis functions: $\phi_{2,j}^h$ (left), $\phi_{2,j+1/2}^h$ (middle) and $\phi_{1,j}^h$ (right).

Linear/quaratic semi-discretization of the adjoint problem (2):

$$\begin{cases} \text{Find } u_p^h(\cdot,t) \in \mathcal{U}_p^h \text{ s.t. } \frac{d^2}{dt^2}(u_p^h(\cdot,t),\varphi_p^h)_{L^2} + (u_p^h(\cdot,t),\varphi_p^h)_{H_0^1} = 0, \forall \varphi_p^h \in \mathcal{U}_p^h, \\ u_p^h(x,T) = u_p^{h,0}(x), u_{p,t}^h(x,T) = u_p^{h,1}(x),\ x \in (0,1), \end{cases} \tag{8}$$

which can be written as a system of second-order linear differential equations (ODEs):

$$M_p^h \mathbf{U}_{p,tt}^h(t) + S_p^h \mathbf{U}_p^h(t) = 0,\ \mathbf{U}_p^h(T) = \mathbf{U}_p^{h,0},\ \mathbf{U}_{p,t}^h(T) = \mathbf{U}_p^{h,1},\ p = 1,2, \tag{9}$$

where M_1^h, S_1^h - tri-diagonal and M_2^h, S_2^h - pentha-diagonal mass and stiffness matrices.

Discrete functional setting

Discrete analogues of $H_0^1(0,1)$, $L^2(0,1)$ or $H^{-1}(0,1)$:

$$\mathcal{H}_p^{h,i} := \{\mathbf{F}_p^h = (F_{p,j/p})_{1\leq j\leq pN+p-1} \in \mathbb{C}^{pN+p-1} \text{ s.t. } ||\mathbf{F}_p^h||_{h,i,p} < \infty\}, \; i = 1, 0, -1.$$

Inner products defining the discrete spaces $\mathcal{H}_p^{h,i}$, $i = 1, 0, -1$:

$$(\mathbf{E}_p^h, \mathbf{F}_p^h)_{h,i,p} := ((M_p^h (S_p^h)^{-1})^{1-i} S_p^h \mathbf{E}_p^h, \mathbf{F}_p^h)_{p,e}, i = 1, 0, -1.$$

Discrete energy space and its **dual**: $\mathcal{V}_p^h := \mathcal{H}_p^{h,1} \times \mathcal{H}_p^{h,0}$ and $\mathcal{V}_p^{h,'} := \mathcal{H}_p^{h,-1} \times \mathcal{H}_p^{h,0}$.

Problem (9) is well-posed in $\mathcal{V}_p^h$. The total energy is conserved in time:

$$\mathcal{E}_p^h(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = \frac{1}{2}(||\mathbf{U}_p^h(t)||_{h,1,p}^2 + ||\mathbf{U}_{p,t}^h(t)||_{h,0,p}^2) = \frac{1}{2}(||\mathbf{U}_p^{h,0}||_{h,1,p}^2 + ||\mathbf{U}_p^{h,1}||_{h,0,p}^2). \quad (10)$$

Discrete observability inequality

$$\mathcal{E}_p^h(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) \leq C_p^h(T) \int_0^T ||B_p^h \mathbf{U}_p^h(t)||_{p,e}^2 \, dt, \quad (11)$$

where

$$B_{p,ij} := \begin{cases} -\frac{1}{h}, & (i,j) = (pN+p-1, pN), \\ 0, & \text{otherwise}. \end{cases} \quad (12)$$

The solution of (9) admits the following Fourier representation

$$\mathbf{U}_p^h(t) = \sum_{\pm} \sum_{k=1}^{pN+p-1} \widehat{u}_{p,\pm}^k \exp(\pm it\lambda_p^{h,k})\varphi_p^{h,k}.$$

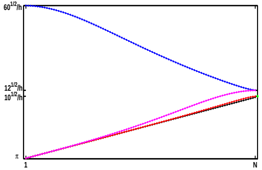

Figure: The square roots of the eigenvalues, λ_p^h: the continuous, acoustic, optic, resonant modes and $p = 1$.

$$\lambda_1(\eta) \sim \eta + \eta^3/24 + \eta^5/1920, \quad \lambda_2^a(\eta) \sim \eta + \eta^5/1440.$$

$C_p^h(T)$ in (11) blows-up. !!!

Infante J.-A., Zuazua E., *Boundary observability for the space semidiscretization of the 1-d wave equation*, M2AN, 1999.

Marica A., Zuazua E., *On the quadratic finite element approximation of $1-d$ waves: propagation, observation and control*, in preparation.

Marica A., Zuazua E., *Localized solutions and filtering mechanisms for the DG semi-discretizations of the 1-d wave equation*, C. R. Acad. Sci. Paris Ser. I, 2010.

Micu S., *Uniform boundary controllability of a semi-discrete 1-D wave equation*, Numer. Math., 2002.

$$\mathcal{B}_1^h := \{\mathbf{F}_1^h = (F_{1,j})_{1\le j\le N} \text{ s.t. } F_{1,2j+1} = (F_{1,2j} + F_{1,2j+2})/2,\ \forall 0 \le j \le (N-1)/2\}$$

$$\mathcal{B}_2^h := \{\mathbf{F}_2^h = (F_{2,j/2})_{1\le j\le 2N+1} \text{ s.t. } F_{2,j+1/2} = (F_{2,j} + F_{2,j+1})/2,\ \forall 0 \le j \le N,$$
$$\text{and } F_{2,2j+1} = (F_{2,2j} + F_{2,2j+2})/2,\ \forall 0 \le j \le (N-1)/2\}.$$

Theorem

For all initial data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) \in (\mathcal{B}_p^h \times \mathcal{B}_p^h) \cap \mathcal{V}_p^h$ *in the adjoint problem (9) and for all* $T \ge 2$, *the observability inequality (11) holds uniformly as* $h \to 0$.

Uniform observability results for the fully discrete conservative scheme:

$$\mathbf{U}_p^{h,k+1} - 2\mathbf{U}_p^{h,k} + \mathbf{U}_p^{h,k-1} + (\delta t)^2 (M^h)^{-1} S^h (\mathbf{U}_p^{h,k+1} + \mathbf{U}_p^{h,k-1})/2 = 0.$$

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- Ignat L., Zuazua E., *Convergence of a two-grid algorithm for the control of the wave equation*, J.Eur.Math.Soc., 2009.
- Loreti P., Mehrenberger M., *An Ingham type proof for a two-grid observability theorem*, EAIM: COCV, 2008.
- Marica A., Zuazua E., *On the quadratic finite element approximation of $1-d$ waves: propagation, observation and control*, in preparation.
- Negreanu M., Zuazua E., *Convergence of a multigrid method for the controllability of a 1-d wave*

On the discrete control problem

For a particular solution $\tilde{\mathbf{U}}_p^h(t)$ of the adjoint problem (9), consider the non-homogeneous problem

$$M_p^h \mathbf{Y}_{p,tt}^h(t) + S_p^h \mathbf{Y}_p^h(t) = -(B_p^h)^* B_p^h \tilde{\mathbf{U}}_p^h(t), \quad \mathbf{Y}_p^h(0) = \mathbf{Y}_p^{h,0}, \quad \mathbf{Y}_{p,t}^h(0) = \mathbf{Y}_p^{h,1}. \tag{13}$$

The discrete quadratic functional:

$$\mathcal{J}_p^h(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = \frac{1}{2} \int_0^T ||B_p^h \mathbf{U}_p^h(t)||_{p,e}^2 \, dt - \langle (\mathbf{Y}_p^{h,1}, -\mathbf{Y}_p^{h,0}), (\mathbf{U}_p^h(0), \mathbf{U}_{p,t}^h(0)) \rangle_{\mathcal{V}_p^{h,\prime}, \mathcal{V}_p^h}, \tag{14}$$

$\mathbf{U}_p^h(t)$ being the solution of the adjoint problem (9) with initial data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1})$.

The uniform observability inequality (11) in the class of initial data $\mathcal{B}_p^h \times \mathcal{B}_h^p$ implies

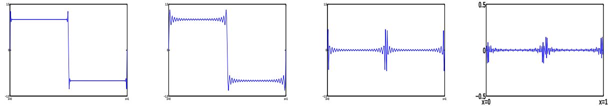
- the coercivity of $\mathcal{J}_p^h$
- the convergence of the last component of $B_p^h \tilde{\mathbf{U}}_p^h(t)$ to the continuous optimal control $\tilde{v}$.

$\tilde{\mathbf{H}}_p^h := (\tilde{H}_{p,j/p})_{1\le j\le pN+p-1}$, where $\tilde{H}_{p,j/p} = (H, \phi^h_{p,j/p})_{L^2}$, for all $1 \le j \le pN + p - 1$.

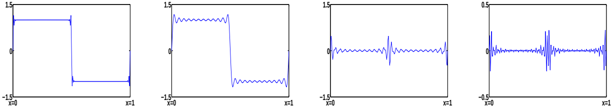
The numerical approximation of $H(x)$ we consider is $\mathbf{Y}_p^{h,0} = \mathbf{H}_p^h := (M_p^h)^{-1}\tilde{\mathbf{H}}_p^h$.
Some projections:

$$\mathbf{H}^h_{1,lo} = \sum_{k=1}^{(N-1)/2} (\mathbf{H}_1^h, \varphi_1^{h,k})_{h,0,1}\varphi_1^{h,k}, \quad \mathbf{H}^h_{1,hi} = \sum_{k=(N+1)/2}^{N} (\mathbf{H}_1^h, \varphi_1^{h,k})_{h,0,1}\varphi_1^{h,k},$$

$$\mathbf{H}^{h,\alpha}_{2,lo} = \sum_{k=1}^{(N-1)/2} (\mathbf{H}_2^h, \varphi_2^{h,\alpha,k})_{h,0,2}\varphi_2^{h,\alpha,k}, \quad \mathbf{H}^{h,\alpha}_{2,hi} = \sum_{k=(N+1)/2}^{N} (\mathbf{H}_2^h, \varphi_2^{h,\alpha,k})_{h,0,2}\varphi_2^{h,\alpha,k}, \ \alpha \in \{a, o\}.$$

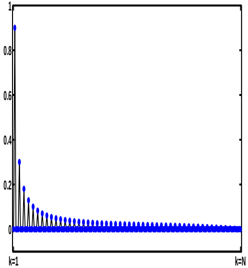


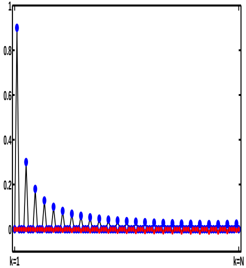
(a) $\mathbf{H}_1^h$ (b) $\mathbf{H}_{1,lo}^h$ (c) $\mathbf{H}_{1,hi}^h$ (d) $\mathbf{H}_{2,lo}^{h,o}$



(e) $\mathbf{H}_2^h$ (f) $\mathbf{H}_{2,lo}^{h,a}$ (g) $\mathbf{H}_{2,hi}^{h,a}$ (h) $\mathbf{H}_{2,hi}^{h,o}$

Figure: The discrete Heaviside functions $\mathbf{H}_p^h$ and their projections.





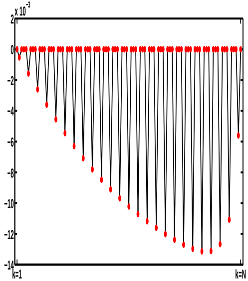


Figure: The Fourier coefficients of $\mathbf{H}_p^h$ for $p = 1$ (left), $p = 2$ (center, blue=acoustic, red=optic), $p = 2$ - the optic branch (right).

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Conjugate gradient algorithm without filtering

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Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then
$\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction
$(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Step 4.b. Solve (13) with trivial initial data and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{D}_{p,n}^h(t)$ and denote the solution by $\mathbf{Y}_{p,n+1}^h(t)$. Take $\mathbf{Z}_{p,n}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,n+1,t}^h(T)$ and $\mathbf{Z}_{p,n}^{h,1} = \mathbf{Y}_{p,n+1}^h(T)$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Step 4.b. Solve (13) with trivial initial data and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{D}_{p,n}^h(t)$ and denote the solution by $\mathbf{Y}_{p,n+1}^h(t)$. Take $\mathbf{Z}_{p,n}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,n+1,t}^h(T)$ and $\mathbf{Z}_{p,n}^{h,1} = \mathbf{Y}_{p,n+1}^h(T)$.

Step 4.c. Set $\rho_{p,n} := -\frac{||\mathbf{G}_{p,n}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,n}^{h,1}||_{h,0,p}^2}{(\mathbf{Z}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,0})_{h,1,p} + (\mathbf{Z}_{p,n}^{h,1}, \mathbf{D}_{p,n}^{h,1})_{h,0,p}}$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Step 4.b. Solve (13) with trivial initial data and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{D}_{p,n}^h(t)$ and denote the solution by $\mathbf{Y}_{p,n+1}^h(t)$. Take $\mathbf{Z}_{p,n}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,n+1,t}^h(T)$ and $\mathbf{Z}_{p,n}^{h,1} = \mathbf{Y}_{p,n+1}^h(T)$.

Step 4.c. Set $\rho_{p,n} := -\frac{||\mathbf{G}_{p,n}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,n}^{h,1}||_{h,0,p}^2}{(\mathbf{Z}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,0})_{h,1,p} + (\mathbf{Z}_{p,n}^{h,1}, \mathbf{D}_{p,n}^{h,1})_{h,0,p}}$.

Step 4.d. $(\mathbf{U}_{p,n+1}^{h,0}, \mathbf{U}_{p,n+1}^{h,1}) := (\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1}) + \rho_{p,n}(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1}) \in \mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1}) := \nabla \mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0}, \mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0}, \mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then
$\mathbf{G}_{p,0}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1} = \mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2} \geq \epsilon$, compute the first descent direction
$(\mathbf{D}_{p,0}^{h,0}, \mathbf{D}_{p,0}^{h,1}) = -(\mathbf{G}_{p,0}^{h,0}, \mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0}, \mathbf{U}_p^{h,1}) = (\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Step 4.b. Solve (13) with trivial initial data and $\tilde{\mathbf{U}}_p^h(t) = \mathbf{D}_{p,n}^h(t)$ and denote the solution by $\mathbf{Y}_{p,n+1}^h(t)$. Take $\mathbf{Z}_{p,n}^{h,0} = -(S_p^h)^{-1} M_p^h \mathbf{Y}_{p,n+1,t}^h(T)$ and $\mathbf{Z}_{p,n}^{h,1} = \mathbf{Y}_{p,n+1}^h(T)$.

Step 4.c. Set $\rho_{p,n} := -\frac{||\mathbf{G}_{p,n}^{h,0}||_{h,1,p}^2 + ||\mathbf{G}_{p,n}^{h,1}||_{h,0,p}^2}{(\mathbf{Z}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,0})_{h,1,p} + (\mathbf{Z}_{p,n}^{h,1}, \mathbf{D}_{p,n}^{h,1})_{h,0,p}}$.

Step 4.d. $(\mathbf{U}_{p,n+1}^{h,0}, \mathbf{U}_{p,n+1}^{h,1}) := (\mathbf{U}_{p,n}^{h,0}, \mathbf{U}_{p,n}^{h,1}) + \rho_{p,n}(\mathbf{D}_{p,n}^{h,0}, \mathbf{D}_{p,n}^{h,1})$.

Step 4.e. $(\mathbf{G}_{p,n+1}^{h,0}, \mathbf{G}_{p,n+1}^{h,1}) := (\mathbf{G}_{p,n}^{h,0}, \mathbf{G}_{p,n}^{h,1}) + \rho_{p,n}(\mathbf{Z}_{p,n}^{h,0}, \mathbf{Z}_{p,n}^{h,1})$.

Conjugate gradient algorithm without filtering

Step 1. Solve the adjoint problem (9) with arbitrary data $(\mathbf{U}_p^{h,0},\mathbf{U}_p^{h,1})=(\mathbf{U}_{p,0}^{h,0},\mathbf{U}_{p,0}^{h,1})\in\mathcal{V}_p^h$, for example the trivial one. This step yields the solution $\mathbf{U}_{p,0}^h(t)$.

Step 2. Compute the first gradient $(\mathbf{G}_{p,0}^{h,0},\mathbf{G}_{p,0}^{h,1}):=\nabla\mathcal{J}_p^h(\mathbf{U}_{p,0}^{h,0},\mathbf{U}_{p,0}^{h,1})$ by solving (13) with initial data $(\mathbf{Y}_{p,0}^{h,0},\mathbf{Y}_{p,0}^{h,1})$ and $\tilde{\mathbf{U}}_p^h(t)=\mathbf{U}_{p,0}^h(t)$. This produces the solution $\mathbf{Y}_{p,0}^h(t)$. Then $\mathbf{G}_{p,0}^{h,0}=-(S_p^h)^{-1}M_p^h\mathbf{Y}_{p,0,t}^h(T)$ and $\mathbf{G}_{p,0}^{h,1}=\mathbf{Y}_{p,0}^h(T)$.

Step 3. If $(||\mathbf{G}_{p,0}^{h,0}||_{h,1,p}^2+||\mathbf{G}_{p,0}^{h,1}||_{h,0,p}^2)^{1/2}\geq\epsilon$, compute the first descent direction $(\mathbf{D}_{p,0}^{h,0},\mathbf{D}_{p,0}^{h,1})=-(\mathbf{G}_{p,0}^{h,0},\mathbf{G}_{p,0}^{h,1})$.

Step 4. Given $(\mathbf{U}_{p,n}^{h,0},\mathbf{U}_{p,n}^{h,1})$, $(\mathbf{G}_{p,n}^{h,0},\mathbf{G}_{p,n}^{h,1})$ and $(\mathbf{D}_{p,n}^{h,0},\mathbf{D}_{p,n}^{h,1})$ in $\mathcal{V}_p^h$, compute them $n+1$:

Step 4.a. Solve (9) with data $(\mathbf{U}_p^{h,0},\mathbf{U}_p^{h,1})=(\mathbf{D}_{p,n}^{h,0},\mathbf{D}_{p,n}^{h,1})$ and denote the solution by $\mathbf{D}_{p,n}^h(t)$.

Step 4.b. Solve (13) with trivial initial data and $\tilde{\mathbf{U}}_p^h(t)=\mathbf{D}_{p,n}^h(t)$ and denote the solution by $\mathbf{Y}_{p,n+1}^h(t)$. Take $\mathbf{Z}_{p,n}^{h,0}=-(S_p^h)^{-1}M_p^h\mathbf{Y}_{p,n+1,t}^h(T)$ and $\mathbf{Z}_{p,n}^{h,1}=\mathbf{Y}_{p,n+1}^h(T)$.

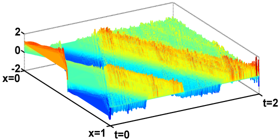
Step 4.c. Set $\rho_{p,n}:=-\frac{||\mathbf{G}_{p,n}^{h,0}||_{h,1,p}^2+||\mathbf{G}_{p,n}^{h,1}||_{h,0,p}^2}{(\mathbf{Z}_{p,n}^{h,0},\mathbf{D}_{p,n}^{h,0})_{h,1,p}+(\mathbf{Z}_{p,n}^{h,1},\mathbf{D}_{p,n}^{h,1})_{h,0,p}}$.

Step 4.d. $(\mathbf{U}_{p,n+1}^{h,0},\mathbf{U}_{p,n+1}^{h,1}):=(\mathbf{U}_{p,n}^{h,0},\mathbf{U}_{p,n}^{h,1})+\rho_{p,n}(\mathbf{D}_{p,n}^{h,0},\mathbf{D}_{p,n}^{h,1})$.

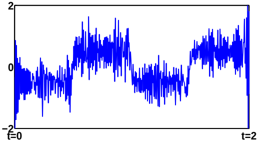
Step 4.e. $(\mathbf{G}_{p,n+1}^{h,0},\mathbf{G}_{p,n+1}^{h,1}):=(\mathbf{G}_{p,n}^{h,0},\mathbf{G}_{p,n}^{h,1})+\rho_{p,n}(\mathbf{Z}_{p,n}^{h,0},\mathbf{Z}_{p,n}^{h,1})$.

Step 4.f. $(\mathbf{D}_{p,n+1}^{h,0},\mathbf{D}_{p,n+1}^{h,1}):=-(\mathbf{G}_{p,n+1}^{h,0},\mathbf{G}_{p,n+1}^{h,1})+\frac{||\mathbf{G}_{p,n+1}^{h,0}||_{h,1,p}^2+||\mathbf{G}_{p,n+1}^{h,1}||_{h,0,p}^2}{||\mathbf{G}_{p,n}^{h,0}||_{h,1,p}^2+||\mathbf{G}_{p,n}^{h,1}||_{h,0,p}^2}(\mathbf{D}_{p,n}^{h,0},\mathbf{D}_{p,n}^{h,1})$.

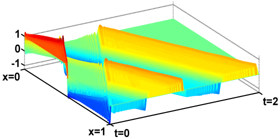
Numerical results (I)



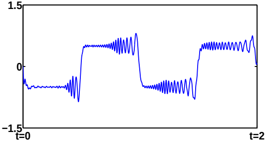
(a) Solution for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_1$, $\mathbf{Y}^{h,1}_1 = 0$



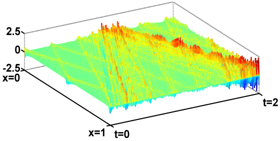
(b) Control for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_1$, $\mathbf{Y}^{h,1}_1 = 0$



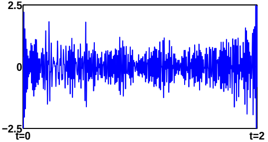
(c) Solution for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_{1,lo}$, $\mathbf{Y}^{h,1}_1 = 0$



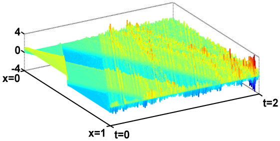
(d) Control for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_{1,lo}$, $\mathbf{Y}^{h,1}_1 = 0$



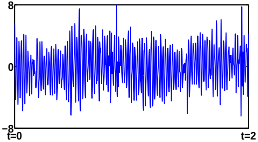
(e) Solution for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_{1,hi}$, $\mathbf{Y}^{h,1}_1 = 0$



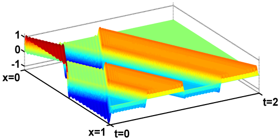
(f) Control for $\mathbf{Y}^{h,0}_1 = \mathbf{H}^h_{1,hi}$, $\mathbf{Y}^{h,1}_1 = 0$



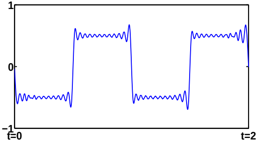
(a) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_2^h$, $\mathbf{Y}_2^{h,1} = 0$



(b) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_2^h$, $\mathbf{Y}_2^{h,1} = 0$

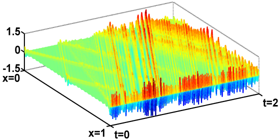


(c) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$

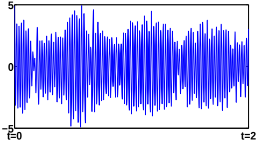


(d) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$

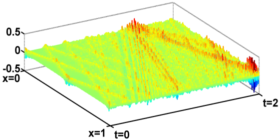
Figure: Solutions of the controlled problem (13) and the corresponding numerical controls for $p = 2$ arising by minimizing $\mathcal{J}_2^h$ over the whole space $\mathcal{V}_2^h$.



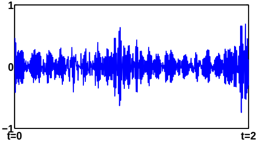
(a) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$



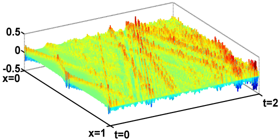
(b) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$



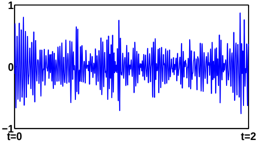
(c) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(d) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(e) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(f) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$

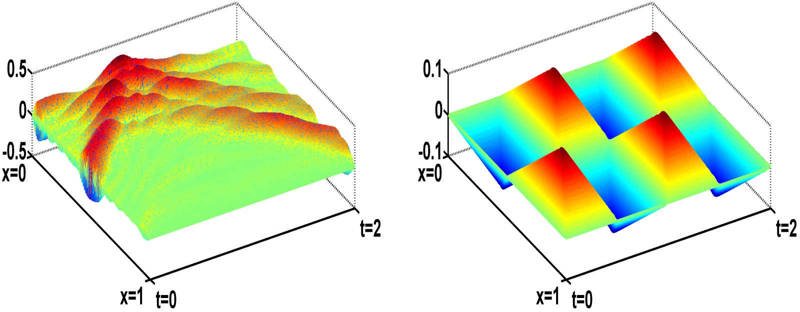


Figure: Typical solution of the adjoint problem (9) corresponding to the minimizer $(\tilde{\mathbf{U}}_p^{h,0}, \tilde{\mathbf{U}}_p^{h,1})$ of $\mathcal{J}_p^h$ over $\mathcal{V}_p^h$ (left) or over $(\mathcal{B}_p^h \times \mathcal{B}_p^h) \times \mathcal{V}_p^h$ (right).

Implementation of the conjugate gradient with filtering

Modifications on **Step 2**: $\mathbf{F}_2^{h,0} = -M_2^h \mathbf{Y}_{2,0,t}^h(T)$ and $\mathbf{F}_2^{h,1} = M_2^h \mathbf{Y}_{2,0}^h(T)$.

The Gateaux derivative of $\mathcal{J}_2^h$ at $(\mathbf{U}_{2,0}^{h,0}, \mathbf{U}_{2,0}^{h,1})$ is:

$$\mathcal{J}_2^{h,'}(\mathbf{U}_{2,0}^{h,0}, \mathbf{U}_{2,0}^{h,1})(\mathbf{U}_2^{h,0}, \mathbf{U}_2^{h,1}) = (\mathbf{F}_2^{h,0}, \mathbf{U}_2^{h,0})_{2,e} + (\mathbf{F}_2^{h,1}, \mathbf{U}_2^{h,1})_{2,e} = (\mathbf{G}_{2,0}^{h,0}, \mathbf{U}_2^{h,0})_{h,1,2} + (\mathbf{G}_{2,0}^{h,1}, \mathbf{U}_2^{h,1})_{h,0,2}.$$

First restriction operator Π: $(\Pi \mathbf{E}_2^h)_j = E_{2,2j}$, for all $1 \leq j \leq (N-1)/2$.

When both $(\mathbf{U}_2^{h,0}, \mathbf{U}_2^{h,1})$ and $(\mathbf{G}_{2,0}^{h,0}, \mathbf{G}_{2,0}^{h,1})$ belong to $\mathcal{B}_2^h \times \mathcal{B}_2^h$

$$(\mathbf{G}_{2,0}^{h,0}, \mathbf{U}_2^{h,0})_{h,1,2} + (\mathbf{G}_2^{h,1}, \mathbf{U}_2^{h,1})_{h,0,2} = (\Pi \mathbf{G}_{2,0}^{h,0}, \Pi \mathbf{U}_2^{h,0})_{2h,1,1} + (\Pi \mathbf{G}_{2,0}^{h,1}, \Pi \mathbf{U}_2^{h,1})_{2h,0,1}.$$

The second restriction operator Γ
$(\Gamma \mathbf{E}_2^h)_j = E_{2,2j} + 3(E_{2,2j+1/2} + E_{2,2j-1/2})/4 + (E_{2,2j+1} + E_{2,2j-1})/2 + (E_{2,2j+3/2} + E_{2,2j-3/2})/4.$

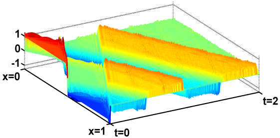
$$(\mathbf{F}_2^{h,0}, \mathbf{U}_2^{h,0})_{2,e} + (\mathbf{F}_2^{h,1}, \mathbf{U}_2^{h,1})_{2,e} = ((S_1^{2h})^{-1} \Gamma \mathbf{F}_2^{h,0}, \Pi \mathbf{U}_2^{h,0})_{2h,1,1} + ((M_1^{2h})^{-1} \Gamma \mathbf{F}_2^{h,1}, \Pi \mathbf{U}_2^{h,1})_{2h,0,1}$$

The two components of the gradient are explicitly given by

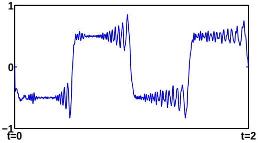
$$\mathbf{G}_{2,0}^{h,0} = \Pi^{-1}(S_1^{2h})^{-1} \Gamma \mathbf{F}_2^{h,0} \text{ and } \mathbf{G}_{2,0}^{h,1} = \Pi^{-1}(M_1^{2h})^{-1} \Gamma \mathbf{F}_2^{h,1},$$

where Π^{-1} is the inverse of the restriction operator Π defined as the linear interpolation on a grid of size $h/2$ of a function defined on a grid of size $2h$.

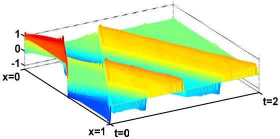
Numerical results (II)



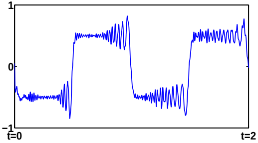
(a) Solution for $\mathbf{Y}_1^{h,0} = \mathbf{H}_1^h$, $\mathbf{Y}_1^{h,1} = 0$



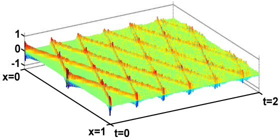
(b) Control for $\mathbf{Y}_1^{h,0} = \mathbf{H}_1^h$, $\mathbf{Y}_1^{h,1} = 0$



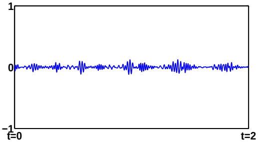
(c) Solution for $\mathbf{Y}_1^{h,0} = \mathbf{H}_{1,lo}^h$, $\mathbf{Y}_1^{h,1} = 0$



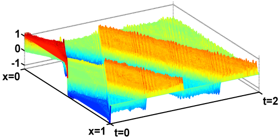
(d) Control for $\mathbf{Y}_1^{h,0} = \mathbf{H}_{1,lo}^h$, $\mathbf{Y}_1^{h,1} = 0$



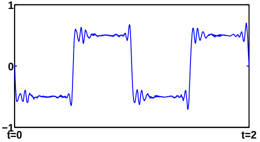
(e) Solution for $\mathbf{Y}_1^{h,0} = \mathbf{H}_{1,hi}^h$, $\mathbf{Y}_1^{h,1} = 0$



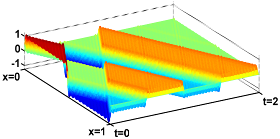
(f) Control for $\mathbf{Y}_1^{h,0} = \mathbf{H}_{1,hi}^h$, $\mathbf{Y}_1^{h,1} = 0$



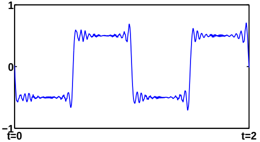
(a) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_2^h$, $\mathbf{Y}_2^{h,1} = 0$



(b) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_2^h$, $\mathbf{Y}_2^{h,1} = 0$

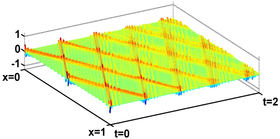


(c) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$

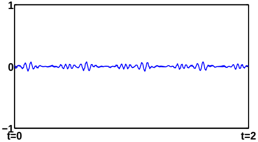


(d) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$

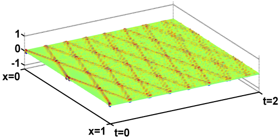
Figure: Solutions of the controlled problem (13) and the corresponding numerical controls for $p = 2$ arising by minimizing $\mathcal{J}_2^h$ over the whole space $\mathcal{V}_2^h$.



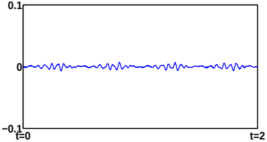
(a) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$



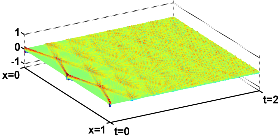
(b) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,a}$, $\mathbf{Y}_2^{h,1} = 0$



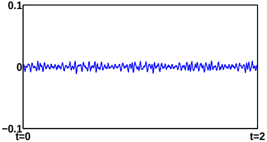
(c) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(d) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,lo}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(e) Solution for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$



(f) Control for $\mathbf{Y}_2^{h,0} = \mathbf{H}_{2,hi}^{h,o}$, $\mathbf{Y}_2^{h,1} = 0$

Conclusions and open problems

In this talk:

- we show numerically the high frequency pathological effects of the P_2 approximation of the boundary controllability problem for the $1-d$ wave equation we discovered from a theoretical point of view in [4].
- we also illustrate the efficiency of our bi-grid filtering algorithm in recovering the convergence of the numerical controls and compare our numerical results with the ones for the P_1 approximation.
- our conclusion is that after restricting the space over which we minimize the discrete functionals to the bi-grid one, we obtain more accurate controls for the quadratic approximation than for the linear one.
- the same analysis can be done for the DG method in [3].
- the filtering technique can be generalized to higher order finite elements approximation of waves ($p \geq 3$) on uniform meshes, a higher and higher accuracy of the numerical controls being expected.

Open problems:

- the high frequency effects of the numerical approximations on irregular meshes is a completely unknown open problem.
- higher-order FEM and DG approximations for other models like the Schrödinger equation.

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Thank you very much for your attention!