Vanishing Results for Hall-Littlewood Polynomials

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Classical restriction rules in the representation theory of real Lie groups:

Theorem

For any integer $n \geq 0$ and partition λ with at most n parts, we have

$$\int_{O \in O(n)} s_\lambda(O)dO = \begin{cases} 0, & \text{if } \lambda \neq 2\mu \text{ for any } \mu \\ 1, & \text{if } \lambda = 2\mu \text{ for some } \mu \end{cases}$$

(where the integral is with respect to Haar measure on the orthogonal group). Similarly, for n even, we have

$$\int_{S \in Sp(n)} s_\lambda(S)dS = \begin{cases} 0, & \text{if } \lambda \neq \mu^2 \text{ for any } \mu \\ 1, & \text{if } \lambda = \mu^2 \text{ for some } \mu \end{cases}$$

(where the integral is with respect to Haar measure on the symplectic group).

Rephrase identities using eigenvalue densities for the orthogonal and symplectic groups. Symplectic integral can be rephrased as

$$\int_T s_\lambda(z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_n^{\pm 1}) \prod_{1\le i\le n} |z_i - z_i^{-1}|^2 \prod_{1\le i<j\le n} |z_i + z_i^{-1} - z_j - z_j^{-1}|^2 dT,$$

where

$$T = \{(z_1, \ldots, z_n) : |z_1| = \cdots = |z_n| = 1\}$$
$$dT = \prod_j \frac{dz_j}{2\pi\sqrt{-1}z_j}$$

are the n-torus and Haar measure, respectively. The orthogonal group case is similar, but more involved.

E. Rains, M. Vazirani: (q,t)-generalizations of these vanishing integrals:

Theorem (Rains-Vazirani '07)

For $n \geq 0$, $l(\lambda) \leq 2n$, $|q|, |t| < 1$, the integral

$$\int_T P_\lambda(\dots, z_i^{\pm 1}, \dots; q, t) \prod_{1 \leq i \leq n} \frac{(z_i^{\pm 2}; q)}{(tz_i^{\pm 2}; q)} \prod_{1 \leq i < j \leq n} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)}{(tz_i^{\pm 1} z_j^{\pm 1}; q)} dT$$

vanishes unless $\lambda = \mu^2$ for some μ.

Proof uses Hecke algebra techniques, fails for $q = 0$.

Definition

The Hall-Littlewood polynomial $P_\lambda(x_1, \ldots, x_n; t)$ indexed by λ is

$$\frac{1}{v_\lambda(t)} \sum_{\omega \in S_n} \omega\Big(x^\lambda \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\Big),$$

where we write x^λ for $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and ω acts on the subscripts of the x_i. The normalization $1/v_\lambda(t)$ has the effect of making the coefficient of x^λ equal to unity.

- Gives a $\mathbb{Z}[t]$-basis for $\mathbb{Z}[t][x_1, \ldots, x_n]^{S_n}$
- At $t = 0$ recover Schur functions
- Macdonald polynomial reduces to Hall-Littlewood polynomial at $q = 0$

Our results:

- develop a combinatorial technique for proving Rains,Vazirani's results (and some new identities) at Hall-Littlewood level
- allows us to compute nonzero values, rational functions in t
- orthogonal group cases: can introduce an extra parameter, obtain t-generalizations of identities studied by Forrester, Rains (involving Pfaffians)
- prove some conjectures of Rains in the case $q = 0$

Symplectic case:

Theorem (V. '10)

Let $l(\lambda) \leq 2n$, then

$$\frac{1}{\int \tilde{\Delta}_K^{(n)}(\pm\sqrt{t}, 0, 0)dT} \int P_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm\sqrt{t}, 0, 0)dT$$

$$= \begin{cases} \frac{[n]_{t^2}!}{v_\mu(t^2)}, & \text{if } \lambda = \mu^2 \text{ for some } \mu \\ 0, & \text{otherwise.} \end{cases}$$

$O^+(2n)$ case:

Theorem (V. '10)

Let $l(\lambda) \leq 2n$, then

$$\frac{1}{\int \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})dT} \int P_\lambda^{(2n)}(x_i^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{[2n]!}{v_\lambda(t)}\Big[(-\alpha)^{\#\text{odd parts of } \lambda} + (-\alpha)^{\#\text{even parts of } \lambda}\Big].$$

In both cases, $\tilde{\Delta}_K^{(n)}$ is the symmetric Koornwinder density:

$$\tilde{\Delta}_K^{(n)}(x; a, b, c, d; t)$$
$$= \frac{1}{2^n n!} \prod_{1 \leq i \leq n} \frac{1 - x_i^{\pm 2}}{(1 - a x_i^{\pm 1})(1 - b x_i^{\pm 1})(1 - c x_i^{\pm 1})(1 - d x_i^{\pm 1})} \prod_{1 \leq i < j \leq n} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}}.$$

Definition

Let $H_m(z;t)$ denote the Rogers–Szegő polynomial

$$H_m(z;t) = \sum_{i=0}^{m} z^i \begin{bmatrix} m \\ i \end{bmatrix}_t,$$

where

$$\begin{bmatrix} m \\ i \end{bmatrix}_t = \begin{cases} \frac{[m]!}{[m-i]![i]!}, & \text{if } m \geq i > 0 \\ 1, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

is the t-binomial coefficient. Rogers–Szegő polynomials satisfy the following second-order recurrence:

$$H_m(z;t) = (1+z)H_{m-1}(z;t) - (1-t^{m-1})zH_{m-2}(z;t).$$

Recent Littlewood summation identity of Warnaar:

Theorem (Warnaar '07)

We have the following formal identity:

$$\sum_{\lambda} P_\lambda(x;t)\Big[\Big(\prod_{i>0} H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}\Big]$$
$$= \prod_{j<k}\frac{1-tx_jx_k}{1-x_jx_k}\prod_{j}\frac{(1-\alpha x_j)(1-\beta x_j)}{(1-x_j)(1+x_j)}.$$

Unifies some well-known summation identities of Macdonald, Kawanaka.

Theorem (V. '10)

Let $l(\lambda) \le 2n$, then we have the following integral identity

$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$=\frac{[2n]!}{v_\lambda(t)}\Big[\Big(\prod_{i\ge 0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\ge 0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$+\Big(\prod_{i\ge 0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\ge 0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big].$$

Using a method of Rains that produces summation results from integral identities, we can obtain Warnaar's result in $\lim n \to \infty$. Thus, this can be viewed as a finite-dimensional analog of that result.

Some of the results in Rains, Vazirani involve Koornwinder polynomials (instead of Macdonald polynomials). These polynomials are a 6-parameter BC_n-symmetric family of Laurent polynomials that contain the Macdonald polynomials as suitable limits of the parameters. Unlike in the Macdonald case, there is no known closed formula for these polynomials at $q = 0$. We use the defining properties to obtain an explicit formula at $q = 0$:

- BC_n-symmetric (invariant under permutations, reciprocals) Laurent polynomial
- triangular with respect to dominance ordering, i.e.,

$$K_\lambda^{(n)}(z; t_0, \ldots, t_3; t) = m_\lambda + \sum_{\mu < \lambda} c_\mu^\lambda m_\mu$$

- satisfies an orthogonality relation wrt symmetric Koornwinder density

Theorem (V. -)

Let λ be a partition with $l(\lambda) \leq n$ and $|t|, |t_0|, \ldots, |t_3| < 1$. Then the Koornwinder $q = 0$ polynomial $K_\lambda(z_1, \ldots, z_n; t_0, \ldots, t_3; t)$ indexed by λ is defined by

$$\frac{1}{v_\lambda(t_0, \ldots, t_3; t)} \sum_{\omega \in B_n} \omega \left(\prod_{1 \leq i \leq n} u_\lambda(z_i) \prod_{1 \leq i < j \leq n} \frac{1 - t z_i^{-1} z_j}{1 - z_i^{-1} z_j} \frac{1 - t z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \right),$$

where

$$u_\lambda(z_i) = \begin{cases} 1 & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i} \frac{(1-t_0 z_i^{-1})(1-t_1 z_i^{-1})(1-t_2 z_i^{-1})(1-t_3 z_i^{-1})}{1-z_i^{-2}} & \text{if } \lambda_i > 0. \end{cases}$$

Connection to p-adic representation theory

The case $G = Gl_n(\mathbb{Q}_p)$, maximal compact subgroup $K = Gl_n(\mathbb{Z}_p)$ (due to Macdonald)

- $\mathcal{H}(G, K)$ is the Hecke algebra of G with respect to K: convolution algebra of compactly supported, K-bi-invariant, complex valued fcns on G

$$(f \star g)(x) = \int_G f(xy^{-1})g(y)dy$$

- Cartan decomposition

$$G = \bigsqcup_{\lambda \in \Lambda_n} Kp^\lambda K$$

  c_λ characteristic function of double coset $Kp^\lambda K$ - basis

Definition

A zonal spherical function (zsf) ω is a K-bi-invariant, eigenfunction of all the convolution operators, $\omega(1) = 1$.

- zsfs are parametrized by n tuples of non-zero complex numbers
- Fourier transform of characteristic function

$$\widehat{c_\lambda}(\omega_s) = (\omega_s \star c_\lambda)(1) = \int_G c_\lambda(x)\omega_s(x^{-1})dx$$
$$= \omega_s(p^{-\lambda})\times \text{ measure of } Kp^\lambda K = p^{<\lambda,\rho>}P_\lambda(p^{-s_1}, \ldots, p^{-s_n}; p^{-1})$$

- Plancherel-Godement theorem: there exists a measure μ such that
$$\int_G f_1(g)\overline{f_2(g)}dg = \int_T \hat{f}_1(\omega_s)\overline{\hat{f}_2(\omega_s)}d\mu(\omega_s)$$
- Fourier transformation extends to isomorphism of $\mathcal{H}(G, K)$ onto $\mathbb{C}[p^{\pm s_1}, \ldots, p^{\pm s_n}]^{S_n}$

Hall-Littlewood orthogonality for $t = p^{-1}$ can be transformed (via P-G theorem) into a statement about the Hecke algebra and intersection of double cosets:

$$\frac{1}{Z}\int_T P_\lambda^{(n)}(x;p^{-1})P_\mu^{(n)}(x^{-1};p^{-1})\tilde{\Delta}_S^{(n)}dT$$
$$= \frac{1}{p^{\langle\lambda+\rho\rangle+\langle\mu+\rho\rangle}}\int_{Gl_n(\mathbb{Q}_p)} c_\lambda(g)c_\mu(g)dg$$

Question: Given this, is there a p-adic interpretation of the Hall-Littlewood integration results?

Recall that the symplectic group is the set of fixed points of an order two endomorphism of GL_{2n}. There is an analogous order two endomorphism of $GL_{2n}(\mathbb{Q}_p)$, whose fixed point subgroup is isomorphic to $GL_n(\mathbb{Q}_p(\sqrt{a}))$, where $\mathbb{Q}_p(\sqrt{a})$ is an unramified quadratic extension of $\mathbb{Q}_p$. We prove the following result, which is a p-adic analogue of the classical formula describing the restriction of Schur polynomials on GL_{2n} to the symplectic group.

Theorem (V. -)

Let $l(\lambda) \leq 2n$, and $(G, H) = (Gl_{2n}(\mathbb{Q}_p), Gl_n(\mathbb{Q}_p(\sqrt{a})))$ and $K = Gl_{2n}(\mathbb{Z}_p)$. Let c_λ be the characteristic function in $\mathcal{H}(G, K)$. Then

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1}; p^{-1})\tilde{\Delta}_K^{(n)}(x; \pm p^{-1/2}, 0, 0; p^{-1})dT = \frac{1}{p^{<\lambda,\rho'>}}\int_H c_\lambda(h)dh,$$

where $\rho' = (n - 1/2, n - 3/2, \ldots, 1/2 - n) \in \mathbb{C}^{2n}$.