Partitions and compositions: A tale of two symmetries

Sarah K Mason
Wake forest University

Algebraic Combinatorixx
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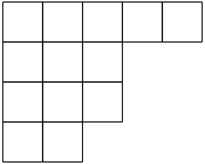
Partition

A **partition** of n is a weakly decreasing sequence of positive integers which sum to n.

Example: $13 = 5 + 3 + 3 + 2$

$$\lambda = (5, 3, 3, 2)$$

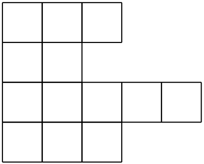


Composition

A **composition** of a positive integer n is a sequence of positive integers which sum to n.

Example: $13 = 3 + 2 + 5 + 3$

$$\alpha = (3, 2, 5, 3)$$



symmetric functions in n variables (Sym_n)

$\pi f(X) = f(X)$ for any permutation π.
(Indexed by partitions.)

Examples (Sym_3)

- $x_1^2 + x_2^2 + x_3^2$
- $x_1^3 x_2 + + x_1^3 x_3 + x_1 x_2^3 + x_2^3 x_3 + x_1 x_3^3 + x_2 x_3^3$

Non-example

- $x_1^2 x_2 + x_2^2 x_3^2$

quasisymmetric functions in n variables ($QSym_n$)

$\sigma f(X) = f(X)$ for any shift σ of the nonzero exponents.
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Examples ($QSym_3$)

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- $x_1^3 x_2 + x_1^3 x_3$ ~~$x_2^3 x_3$~~

Semi-standard Young tableau (SSYT)

rows: weakly decreasing
columns: strictly decreasing

$T =$

9	8	8	6	3
7	6	3		
6	4	1		
3	2			

$$x^T = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

Semi-standard composition tableau (CT)

rows: weakly decreasing
left column: strictly increasing
columns: $a \leq b \Rightarrow b > c$

c	a
	b

$F =$

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Schur functions

$$s_\lambda(x_1, \cdots, x_n) = \sum_{T \in SSYT(\lambda)} x^T$$

Quasisymmetric Schurs

$$\mathcal{QS}_\gamma(x_1, \cdots, x_n) = \sum_{F \in CT(\gamma)} x^F$$

$$s_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$x_1^2x_2 + x_1^2x_3 + x_1x_2^2 +$$
$$2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

$$\mathcal{QS}_{2,1,3}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & 3 \\ \hline\end{array}$$

$$x_1^2x_2^2x_3^2 + x_1^2x_2x_3^3$$

Schur functions

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$$x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$$
$$+x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2$$

Quasisymmetric Schurs

$$\mathcal{QS}_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3$$

$$\mathcal{QS}_{1,2}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 3 & 2 \\ \hline\end{array}\quad
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\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 3 & 3 \\ \hline\end{array}$$

$$x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2$$

Schur functions

$$s_{2,1}(x_1, x_2, x_3) =$$

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$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$
$$+ x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

Quasisymmetric Schurs

$$\mathcal{QS}_{2,1}(x_1, x_2, x_3) =$$

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$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

$$\mathcal{QS}_{1,2}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \hline 3 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 3 & 3 \\ \hline\end{array}$$

$$x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

$$\boxed{s_{2,1}(x_1, x_2, x_3) = QS_{2,1}(x_1, x_2, x_3) + QS_{1,2}(x_1, x_2, x_3)}$$

Proposition (Haglund, Luoto, M., van Willigenburg)

$$s_\lambda = \sum_{\alpha^+ = \lambda} \mathcal{QS}_\alpha$$

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Corollary

If a function is **symmetric** and **quasisymmetric Schur positive**, then it is **Schur positive**!

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Proof (example):

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Proof (example):

2	2	2	1	1
3				
5	4			
6	6	6		
9	8	5		

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$$\xleftarrow{\rho}$$

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9
6
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2

$\xleftarrow{\rho}$

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9	8
6	6
5	4
3	2
2	

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$$\xrightarrow{\rho^{-1}}$$

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5	4	2		
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2				

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6
9

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Proof (example):

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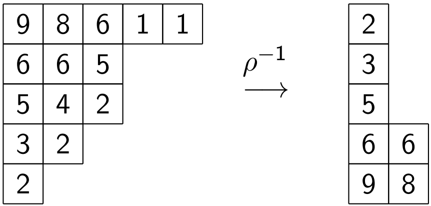

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Schur functions...

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Quasisymmetric Schur functions...

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- have many nice combinatorial properties.
- generalize to non-symmetric Macdonald polynomials ($\hat{E}_\gamma(X; q, t)$).

Let $D(T)$ be the set of all i such that $i+1$ appears weakly to the right of i in T, let $\beta(S)$ be the composition obtained from the differences between consecutive elements of a set S, and let F_γ be the fundamental quasisymmetric function corresponding to the composition γ. Then:

Theorem (Gessel 84)

$$s_\lambda = \sum_T F_{\beta(D(T))}$$

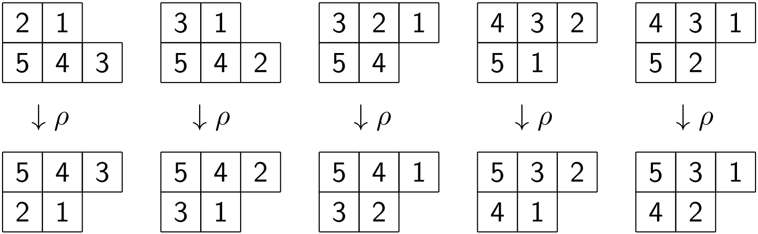
where the sum is over all standard contretableaux, T, of shape λ.

Theorem (HLMvW)

$$\mathcal{QS}_\alpha = \sum_T F_{\beta(D(T))}$$

where the sum is over all standard contretableaux, T, of shape $\lambda(\alpha)$ that map under ρ^{-1} to a CT of shape α.

Example



$D = \{2\}$ $\quad D = \{1, 3\}$ $\quad D = \{3\}$ $\quad D = \{1, 4\}$ $\quad D = \{2, 4\}$

$$s_{3,2} = F_{2,3} + F_{1,2,2} + F_{3,2} + F_{1,3,1} + F_{2,2,1}$$

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Example

$$
\begin{array}{ccc} 2 & 1 & \\ 5 & 4 & 3 \end{array}
\qquad
\begin{array}{ccc} 3 & 1 & \\ 5 & 4 & 2 \end{array}
\qquad
\begin{array}{ccc} 3 & 2 & 1 \\ 5 & 4 & \end{array}
\qquad
\begin{array}{ccc} 4 & 3 & 2 \\ 5 & 1 & \end{array}
\qquad
\begin{array}{ccc} 4 & 3 & 1 \\ 5 & 2 & \end{array}
$$

$$
\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho
$$

$$
\begin{array}{ccc} 5 & 4 & 3 \\ 2 & 1 & \end{array}
\qquad
\begin{array}{ccc} 5 & 4 & 2 \\ 3 & 1 & \end{array}
\qquad
\begin{array}{ccc} 5 & 4 & 1 \\ 3 & 2 & \end{array}
\qquad
\begin{array}{ccc} 5 & 3 & 2 \\ 4 & 1 & \end{array}
\qquad
\begin{array}{ccc} 5 & 3 & 1 \\ 4 & 2 & \end{array}
$$

$$
D = \{2\} \qquad D = \{1,3\} \qquad D = \{3\} \qquad D = \{1,4\} \qquad D = \{2,4\}
$$

$$
s_{3,2} = F_{2,3} + F_{1,2,2} + F_{3,2} + F_{1,3,1} + F_{2,2,1}
$$

$$
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$$

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$$

Example

$$
\begin{array}{ccccc}
\begin{array}{|c|c|c|}\hline 2 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & 3 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & 2 \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \\
\downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho \\
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \\
D = \{2\} & D = \{1,3\} & D = \{3\} & D = \{1,4\} & D = \{2,4\}
\end{array}
$$

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Example

$$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \color{red}{3} & \color{red}{1} \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 5 & 4 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 5 & 2 \\ \cline{1-2} \end{array}$$

$$\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho$$

$$\begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline 2 & 1 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 5 & 4 & 2 \\ \hline 3 & 1 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 5 & 4 & 1 \\ \hline 3 & 2 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 5 & 3 & 2 \\ \hline 4 & 1 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2} \end{array}$$

$$D = \{2\} \qquad \color{red}{D = \{1,3\}} \qquad D = \{3\} \qquad D = \{1,4\} \qquad D = \{2,4\}$$

$s_{3,2} = F_{2,3} + \color{red}{F_{1,2,2}} + F_{3,2} + F_{1,3,1} + F_{2,2,1}$

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Example

$$\begin{array}{ccccc}
\begin{array}{|c|c|c}\hline 2 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array} &
\begin{array}{|c|c|c}\hline \color{red}{3} & \color{red}{1} & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \\
\downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho \\
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline \color{red}{3} & \color{red}{1} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} &
\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \\
D = \{2\} & \color{red}{D = \{1,3\}} & D = \{3\} & D = \{1,4\} & D = \{2,4\}
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Example

$$
\begin{array}{ccccc}
\begin{array}{|c|c|c}\hline 2 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array}
&
\begin{array}{|c|c|c}\hline 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array}
&
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
&
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
&
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\\
\downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho & \downarrow \rho
\\
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
&
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&
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&
\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
&
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Example

2 1 5 4 3	3 1 5 4 2	3 2 1 5 4	4 3 2 5 1	4 3 1 5 2
$\downarrow \rho$	$\downarrow \rho$	$\downarrow \rho$	$\downarrow \rho$	$\downarrow \rho$
5 4 3 2 1	5 4 2 3 1	5 4 1 3 2	5 3 2 4 1	5 3 1 4 2
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Example

$$\begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c|c}\hline 3 & 1 & \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline \color{red}{4} & 3 & 2 \\ \hline 5 & \color{red}{1} & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}$$

$$\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho$$

$$\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}$$

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\qquad
\begin{array}{|c|c|c}\hline 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

$$
\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho
$$

$$
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 5 & 4 & 1 \\ \hline 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
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$$

$$
D = \{2\} \qquad D = \{1, 3\} \qquad D = \{3\} \qquad D = \{1, 4\} \qquad D = \{2, 4\}
$$

$$
s_{3,2} = F_{2,3} + F_{1,2,2} + F_{3,2} + F_{1,3,1} + F_{2,2,1}
$$

$$
QS_{2,3} = F_{2,3} + F_{1,2,2}
$$

$$
QS_{3,2} = F_{3,2} + F_{1,3,1} + F_{2,2,1}
$$

Example

$$
\begin{array}{|c|c|c}\hline 2 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{3} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c}\hline 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 4 & \multicolumn{1}{c|}{2} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 5 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
$$

$$\downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho \qquad \downarrow \rho$$

$$
\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}
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Example

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$$
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$$

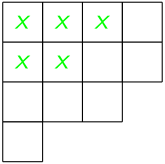
skew shape

diagram for partition λ/μ

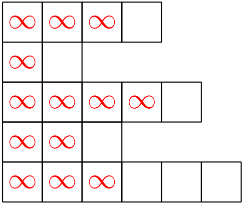
skew diagram

composition diagram with extended basement

$$(4, 4, 3, 1)/(3, 2)$$



$$(4, 2, 5, 3, 6)/(3, 1, 4, 2, 3)$$



Yamanouchi

A **Yamanouchi** word is a word in which any prefix contains at least as many occurences of j as of $j+1$, for each $j \geq 1$.

Yamanouchi: 11212321

not Yamanouchi: 1123131

contre-Yamanouchi

A **contre-Yamanouchi** word is a word in which any prefix contains at least as many occurences of j as $j-1$.

contre-Yamanouchi: 33232123

not contre-Yamanouchi: 3321313

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Littlewood-Richardson SSYT

A **Littlewood-Richardson SSYT** is a skew tableau whose reading word (L-R, bottom-top) is a Yamanouchi word.

x	x	x	x	1
x	x	1	1	2
1	2	3		

1 2 3 1 1 2 1

Littlewood-Richardson CT

A **Littlewood-Richardson CT** is a skew composition whose reading word (L-R, bottom-top) is a Yamanouchi word.

∞	∞	∞	1		
∞	1				
∞	∞	∞	∞	2	
∞	∞	2			
∞	∞	∞	3	3	3

3 2 3 1 3 2 1

Littlewood-Richardson Rule

In the expansion

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu,$$

the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of LRST of shape ν/λ, weight μ.

Theorem (HLMvW 08)

$$\mathcal{QS}_\alpha \cdot s_\lambda = \sum_\beta c^\beta_{\alpha,\lambda} \mathcal{QS}_\beta,$$

where $c^\beta_{\alpha,\lambda}$ is the number of LRCT of shape β/α and with content λ^*.

Example

$$s_{2,1}s_{2,1} = s_{4,1,1} + s_{4,2} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,2} + s_{2,2,1,1}$$

x	x	1	1
x			
2			

x	x	1	1
x	2		

x	x	1
x	1	2

x	x	1
x	1	
2		

x	x	1
x	2	
1		

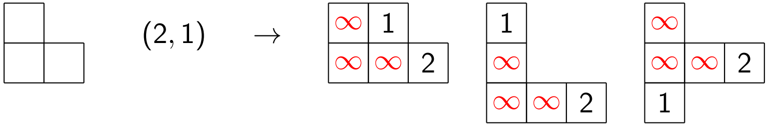
x	x	1
x		
1		
2		

x	x
x	1
1	2

x	x
x	1
1	
2	

Example

$$\mathcal{QS}_{1,2} \cdot s_{1,1} = \mathcal{QS}_{2,3} + \mathcal{QS}_{1,1,3} + \mathcal{QS}_{1,3,1}$$



$Sym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[\mathbf{x}]$ $Sym_n(\mathbb{Q}) \hookrightarrow QSym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[\mathbf{x}]$

A classical result

The following are equivalent:

1. Sym_n is a polynomial ring, generated by the elementary symmetric polynomials $\mathcal{E}_n = \{e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\}$;
2. the ring $\mathbb{Q}[\mathbf{x}]$ is a free Sym_n-module;
3. the coinvariant space $\mathbb{Q}[\mathbf{x}]_{\mathfrak{S}_n} = \mathbb{Q}[\mathbf{x}]/(\mathcal{E}_n)$ has dimension $n!$.

Bergeron-Reutenauer conjectures

- $QSym_n(\mathbb{Q})$ is a free module over $Sym_n(\mathbb{Q})$;
- dim of coinvariant space $QSym_n(\mathbb{Q})/(\mathcal{E}_n)$ is $n!$;
- Pure, inverting comps index a stable basis for $QSym_n(\mathbb{Q})/(\mathcal{E}_n)$.

$Sym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[\mathbf{x}]$ $\quad$ $Sym_n(\mathbb{Q}) \hookrightarrow QSym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[\mathbf{x}]$

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$Sym_n(\mathbb{Q}) \hookrightarrow \mathbb{Q}[x_1, \ldots, x_n] = \mathbb{Q}[\mathbf{x}]$ $\quad$ $Sym_n(\mathbb{Z}) \hookrightarrow QSym_n(\mathbb{Z}) \hookrightarrow \mathbb{Z}[\mathbf{x}]$

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Bergeron-Reutenauer conjectures

- $QSym_n(\mathbb{Z})$ is a free module over $Sym_n(\mathbb{Z})$; Lauve-M (2009)
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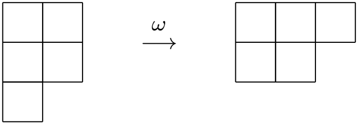
$\omega : Sym_n \to Sym_n$

- endomorphism
- $\omega(e_\lambda) = h_\lambda$
- $\omega(s_\lambda) = s_\lambda^T$

Example:

$\omega(s_{2,2,1}) = s_{3,2}$



$\omega : QSym_n \to QSym_n$

- endomorphism
- $\omega(F_\beta(x_1, x_2, \ldots, x_n)) = F_{\hat{\beta}}(x_n, \ldots, x_2, x_1)$

Theorem (M.-Remmel)

$\omega(QS_\alpha^{col}(x_1, \ldots, x_n)) = QS_\alpha^{row}(x_n, \ldots, x_1)$

Moreover:

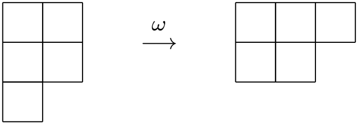
$$s_\lambda^{row} = \sum_{\tilde{\alpha}=\lambda} QS_\alpha^{row}$$

$\omega : Sym_n \to Sym_n$

- endomorphism
- $\omega(e_\lambda) = h_\lambda$
- $\omega(s_\lambda) = s_\lambda^T$

Example:

$$\omega(s^{col}_{2,2,1}) = s^{col}_{3,2} = s^{row}_{2,2,1}$$



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- $\omega(F_\beta(x_1, x_2, \ldots, x_n)) = F_{\hat{\beta}}(x_n, \ldots, x_2, x_1)$

Theorem (M.-Remmel)

$$\omega(QS^{col}_\alpha(x_1, \ldots, x_n)) = QS^{row}_\alpha(x_n, \ldots, x_1)$$

Moreover:

$$s^{row}_\lambda = \sum_{\tilde{\alpha}=\lambda} QS^{row}_\alpha$$

column-strict SSYT

rows: weakly increasing
columns: strictly increasing

$T =$

9	8	8	6	3
7	6	3		
6	4	1		
3	2			

$$x^T = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

column-strict CT

rows: weakly decreasing
left column: strictly increasing
columns: $a \leq b \Rightarrow b > c$

c	a
	b

$F =$

3	2	1		
6	6	3		
7	4			
9	8	8	6	3

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

row-strict SSYT

rows: strictly increasing
columns: weakly increasing

$T =$

9	8	7	3	1
7	6	3		
7	4	1		
5	4			

$$x^T = x_1^2 x_3^2 x_4^2 x_5 x_6 x_7^3 x_8 x_9$$

row-strict CT

rows: strictly decreasing
left column: weakly increasing
columns: $a < b \Rightarrow b \geq c$

c	a
	b

$F =$

5	4	3		
7	6	3		
7	4			
9	8	7	3	1

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

row-strict SSYT

rows: strictly increasing
columns: weakly increasing

$$T = \begin{array}{|c|c|c|c|c|} \hline 9 & 8 & 7 & 3 & 1 \\ \hline 7 & 6 & 3 \\ \cline{1-3} 7 & 4 & 1 \\ \cline{1-3} 5 & 4 \\ \cline{1-2} \end{array}$$

$$x^T = x_1^2 x_3^2 x_4^2 x_5 x_6 x_7^3 x_8 x_9$$

row-strict CT

rows: strictly decreasing
left column: weakly increasing
columns: $a < b \Rightarrow b \geq c$

$$\begin{array}{|c|c|} \hline c & a \\ \hline \end{array}$$

$$\begin{array}{cc} & \fbox{b} \end{array}$$

$$F = \begin{array}{|c|c|c|c|c|} \hline 5 & 4 & 3 \\ \cline{1-3} 7 & 6 & 3 \\ \cline{1-3} 7 & 4 \\ \hline 9 & 8 & 7 & 3 & 1 \\ \hline \end{array}$$

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

row-strict SSYT

rows: strictly increasing
columns: weakly increasing

$T =$

9	8	7	3	1
7	6	3		
7	4	1		
5	4			

$$x^T = x_1^2 x_3^2 x_4^2 x_5 x_6 x_7^3 x_8 x_9$$

row-strict CT

rows: strictly decreasing
left column: weakly increasing
columns: $a < b \Rightarrow b \geq c$

c	a
	b

$F =$

5	4	3		
7	6	3		
7	4			
9	8	7	3	1

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9$$

$$s_{2,1,1}(x_1, x_2, x_3) \quad =$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

$$s_{2,1,1}(x_1, x_2, x_3) \quad = \quad QS_{2,1,1} \quad + \quad QS_{1,2,1} \quad + \quad QS_{1,1,2}$$

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 3 & 3 \\ \hline\end{array}$$

$$s_{2,1,1}(x_1, x_2, x_3) \quad = \quad QS_{2,1,1} \quad + \quad QS_{1,2,1} \quad + \quad QS_{1,1,2}$$

$$\downarrow \omega$$

$$s_{3,1}(x_1, x_2, x_3) = s^{row}_{2,1,1} \quad =$$

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

$$s_{2,1,1}(x_1, x_2, x_3) = QS_{2,1,1} + QS_{1,2,1} + QS_{1,1,2}$$

$$\downarrow \omega \qquad\qquad \downarrow \omega$$

$$s_{3,1}(x_1, x_2, x_3) = s^{row}_{2,1,1} = QS^{row}_{2,1,1}$$

2	2	2
1		

3	2	2
1		

3	3	2
1		

3	3	3
1		

3	3	3
2		

2	1
2	
2	

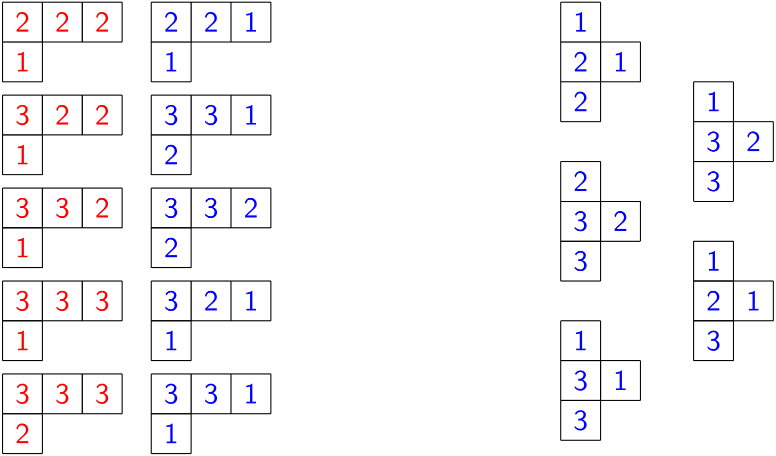
2	1
3	
3	

3	2
3	
3	

2	1
2	
3	

3	1
3	
3	

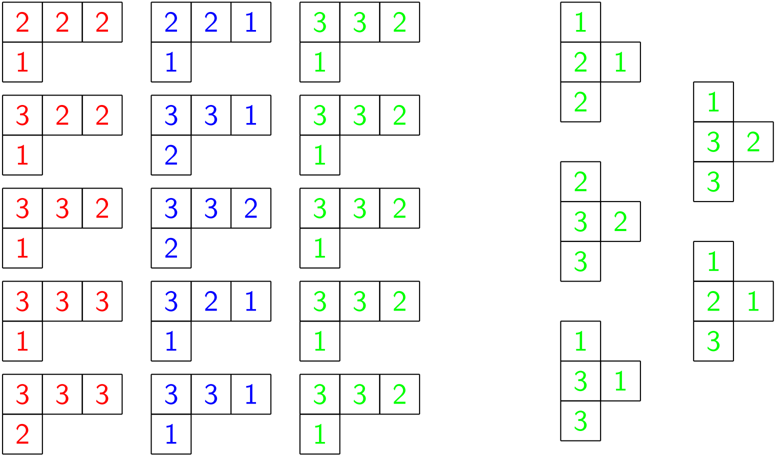
$$
\begin{array}{ccccccc}
s_{2,1,1}(x_1,x_2,x_3) & = & QS_{2,1,1} & + & QS_{1,2,1} & + & QS_{1,1,2} \\
\downarrow \omega & & \downarrow \omega & & \downarrow \omega & & \\
s_{3,1}(x_1,x_2,x_3) = s^{row}_{2,1,1} & = & {\color{red} QS^{row}_{2,1,1}} & {\color{blue} +} & {\color{blue} QS^{row}_{1,2,1}} & &
\end{array}
$$



$$s_{2,1,1}(x_1,x_2,x_3) = QS_{2,1,1} + QS_{1,2,1} + QS_{1,1,2}$$

$$\downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega \qquad \downarrow \omega$$

$$s_{3,1}(x_1,x_2,x_3) = s^{row}_{2,1,1} = QS^{row}_{2,1,1} + QS^{row}_{1,2,1} + QS^{row}_{1,1,2}$$



Further directions

- Quasisymmetric Hall-Littlewood polynomials
- Quasisymmetric Macdonald polynomials
- Quasisymmetric Schur P-functions (in progress here!)
- Representation theoretic interpretation (Steph van Willigenburg and Christine Bessenrodt)
- Multiplication rules (Jeff Ferreira)
- Basis for invariant space $QSym_n^r/Sym_n$ (in progress here!)

THANK YOU!!

- Haglund, Luoto, Mason & van Willigenburg, *Refinements of the Littlewood-Richardson rule*, Trans. Amer. Math. Soc. (2011).
- Lauve & Mason, *QSym over Sym has a stable basis*, J. Combin. Theory Ser. A (to appear).
- Mason & Remmel, *Row-strict quasisymmetric Schur functions* (in preparation).