A Geometric Interpretation of the Characteristic Polynomial of a Hyperplane Arrangement

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joint work with Mathias Drton and Ed Swartz

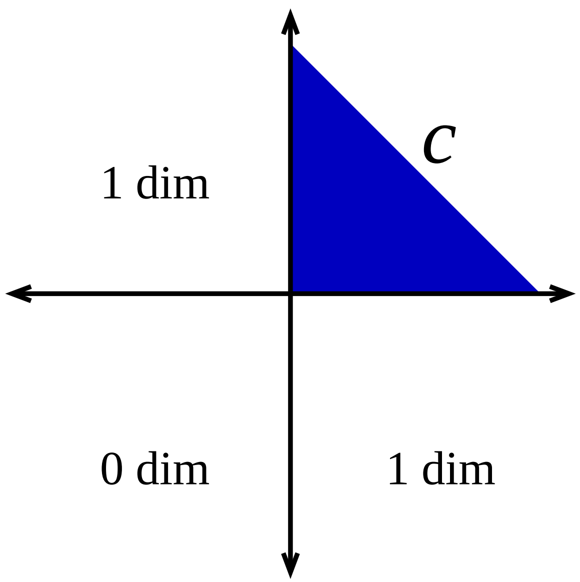
OPEN PROBLEM

Does there exist a real central hyperplane arrangement with all cones isometric that is not a reflection arrangement?

Projection Volumes

- $\mathcal{C}$ = Polyhedral cone in $\mathbb{R}^n$, $\pi_{\mathcal{C}}(x)$ = orthogonal projection onto $\mathcal{C}$
- $\pi_{\mathcal{C}}(x)$ is k-dim if it is in the relative interior of a k-dim face of $\mathcal{C}$
- ν_k = ratio of volume of $\mathbb{R}^n$ for which $\pi_{\mathcal{C}}(x)$ is k-dimensional

PROBLEM *Given a cone $\mathcal{C}$ compute the projection volumes ν_k*



(1/4, 1/2, 1/4)

STATISTICAL MOTIVATION: HYPOTHESIS TESTING

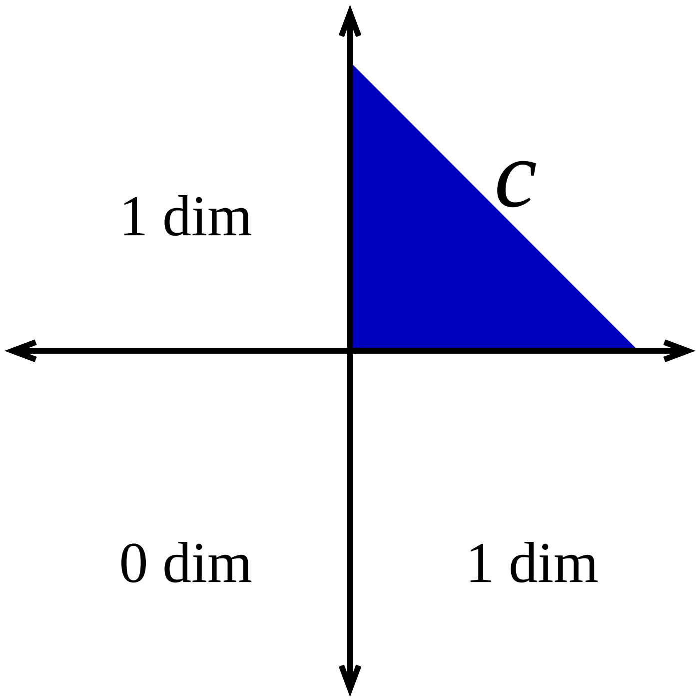
- Likelihood Ratio Testing
- Mixture of chi-square distributions
- Projection volumes = weights of the distribution

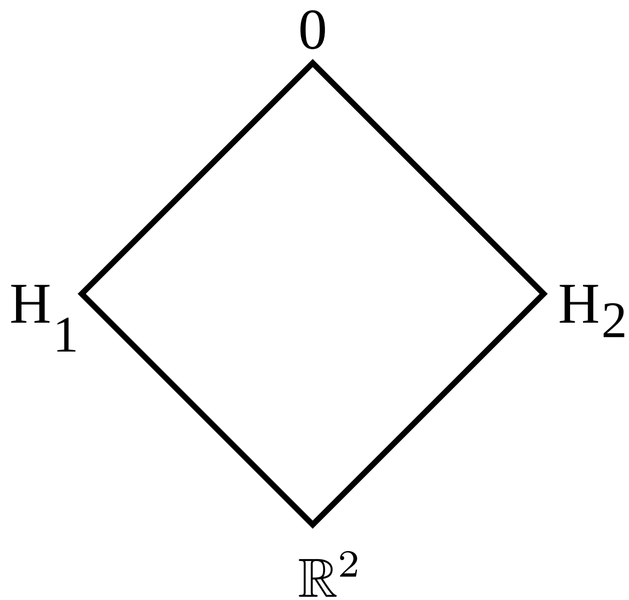
Hyperplane Arrangements

- $\mathcal{L}(\mathcal{H})$ = Set of all intersections of collections of hyperplanes of $\mathcal{H}$

  (include $\mathbb{R}^n$ for the empty intersection)

  $\mathcal{L}(\mathcal{H})$ forms a lattice under reverse inclusion of intersections.





Characteristic Polynomial

- The Characteristic polynomial:

$$\chi_{\mathcal{H}}(t) = \sum_{x \in L(\mathcal{H})} \mu(x) t^{\dim(x)}$$

Möbius function $\mu : L(\mathcal{H}) \to \mathbb{Z}$

$$\mu(\mathbb{R}^n) = 1 \text{ and } \sum_{z \leq y} \mu(z) = 0$$

- The Poincaré polynomial $\pi(\mathcal{H}, t)$ is related by:

$$\chi_{\mathcal{H}}(t) = t^{\mathsf{Rk}(\mathcal{H})} \pi(\mathcal{H}, -t^{-1})$$

REFLECTION GROUPS

- $\mathcal{W} \subset GL(\mathbb{R}^n)$: Finite real reflection group
- Reflection in $\mathbb{R}^n$ is an isometry fixing the points of a hyperplane (mirror of reflection)
- Reflection arrangement or Coxeter arrangement $\mathcal{H}$ is the collection of all mirrors of a finite reflection group.
- A Fundamental chamber is the closure $\mathcal{C}$ of a region of $\mathbb{R}^n \backslash \mathcal{H}$ (All chambers are isometric)

Coefficients

THEOREM *Let $\mathcal{W}$ be a finite reflection group, and $\chi_{\mathcal{W}}(t)$ the associated characteristic polynomial. The projection volumes ν_k are given by the coefficients of $\chi(t)$:*

$$\nu_k = \frac{|\alpha_k|}{|\alpha_n| + \cdots + |\alpha_0|} = \frac{|\alpha_k|}{\#\mathcal{W}}$$

- Connection to the group:

  Let x be a generic point in the fundamental chamber $\mathcal{C}$. Then the coefficient $|\alpha_k|$ is equal to the number of group elements $g \in \mathcal{W}$ for which the projection $\pi_{\mathcal{C}}(gx)$ is k-dimensional.

Coxeter Arrangements

- $|\alpha_k|$ is also known to be the number of group elements in $\mathcal{W}$ that leave fixed all points of some linear space of dimension $n-k$
- Top Coefficient: Action of $\mathcal{W}$ is simply transitive

$$\nu_n = 1/\#\mathcal{W}$$

- If $e_1, e_2, \ldots, e_n$ are the exponents of the group $\mathcal{W}$

$$\chi_{\mathcal{W}}(t) = (t-e_1)(t-e_2)\ldots(t-e_n)$$

- Bottom Coefficient: De Concini, Procesi, Stembridge, Denham

$$\nu_0 = \frac{|e_1 \cdots e_n|}{\#\mathcal{W}}$$

Averages over Arrangements

- Consider cones $\mathcal{C}$ given by the closure of a region of a linear hyperplane arrangement.
- Consider the *average* projection volumes over all regions.

EXAMPLE: Any two lines in $\mathbb{R}^2$

Average volumes will always be $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$

EXAMPLE: Any n lines in $\mathbb{R}^2$

Average volumes will always be $\left(\frac{1}{2n}, \frac{n}{2n}, \frac{n-1}{2n}\right)$

COEFFICIENTS

THEOREM *The average projection volumes are given by the absolute values of the coefficients of* $\chi(t)$*:*

$$\frac{\sum_{\mathcal{C}} \nu_k}{\#\mathcal{C}} = \frac{|\alpha_k|}{|\alpha_n| + \cdots + |\alpha_0|} = \frac{|\alpha_k|}{\#\mathcal{C}}$$

COROLLARY *If all regions of* $\mathcal{H}$ *are isometric, then the projection volumes of any region* $\mathcal{C}$ *are given by the coefficents of* $\chi(t)$:

$$\nu_k = \frac{|\alpha_k|}{\#\mathcal{C}}$$

ANGLE SUMS OF POLYTOPES

- Zonotopes
- Angle sums of polytopes
- Equiprojective polytope
- Angles to f-vectors to lattice