Alternating Sign Matrices and Schur Functions

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Three Objects and a Formula

Object 1

Alternating Sign Matrices

Alternating Sign Matrix

- Square matrices with entries from 0, 1, or -1
- Each row and column contains at least one 1; first and last nonzero elements of each row and column are 1
- Nonzero entries in each row and column alternate in sign

Alternating Sign Matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Alternating sign matrices (ASM) generalize permutation matrices

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Alternating Sign Matrix

The number A(*m*) of *mxm* ASM is:

$$A(m) = \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+j)!}$$

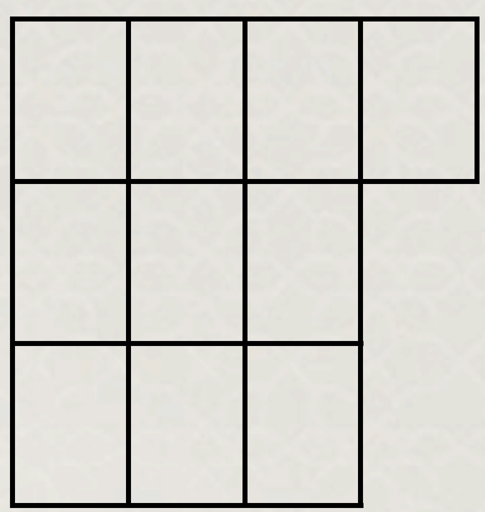
- This was the Alternating Sign Matrix Conjecture
- See D.M. Bressoud, Proof and Confirmations: The Story of the Alternating Sign Matrix Conjecture, Cambridge UP: 1999

Object 2

Tableaux

Partitions

- Given a partition, λ, with parts $\lambda_1, \lambda_2, \ldots, \lambda_k$, can be represented graphically by a diagram:



$$\lambda = (4, 3, 3)$$

Tableaux

- Fill diagram with entries according to the following rules:
  - entries weakly increase across rows
  - entries strictly increase down columns

1	1	2	4
2	3	3	
4	4	5	

Weighting Tableaux

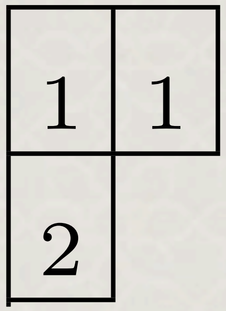
- Weight each entry *i* in the tableau by x_i
- Then each tableau has weight $x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$
  - For example, the weight of this tableau is

1	1	2	4
2	3	3	
4	4	5	

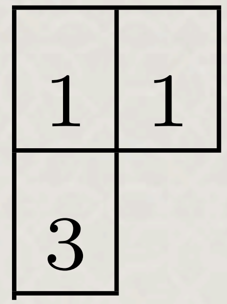
$$x_1^2 x_2^2 x_3^2 x_4^3 x_5$$

Schur Functions

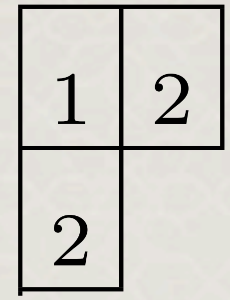
$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(T)}$$



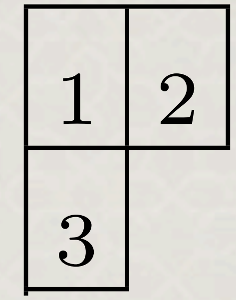
$x_1^2 x_2$



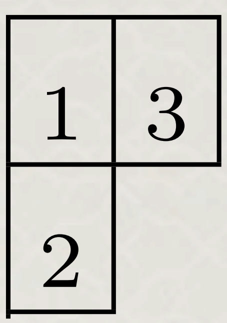
$x_1^2 x_3$



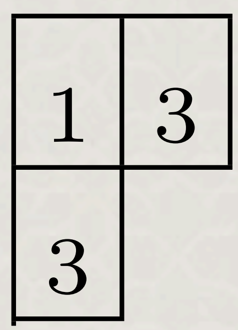
$x_1 x_2^2$



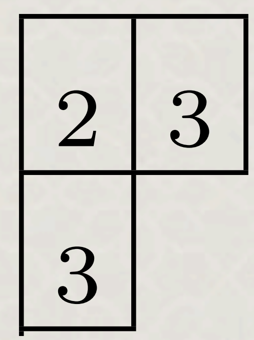
$x_1 x_2 x_3$



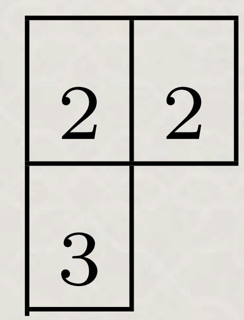
$x_1 x_2 x_3$



$x_1 x_3^2$



$x_2 x_3^2$



$x_2^2 x_3$

A formula

Tokuyama's Formula

Tokuyama's Formula

- Proved by Tokuyama in 1988 using representation theory of general linear groups
- Proved by Okada in 1990 using algebraic manipulations on monotone triangles (equivalent to alternating sign matrices)

Playing with Formulas

- Tokuyama's formula:

$$\prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} (x_i + t x_j)\, s_\lambda(\mathbf{x}) = \sum_{ST \in \mathcal{ST}^\mu(n)} t^{\mathrm{hgt}(ST)} (1+t)^{\mathrm{str}(ST)-n}\, \mathbf{x}^{\mathrm{wgt}(ST)}$$

t-deformation of a Weyl denominator formula

Shifted Tableaux

- weakly increasing in rows
- weakly increasing down columns
- strictly increasing down left-to-right diagonals

$ST =$

1	1	1	2	2	2	3	3	5
	2	2	3	3	4	5	5	6
		3	3	4	4	5	6	
			4	5	5	5		
				5	6	6		
					6			

Shifted Tableaux

$$ST = \begin{array}{|c|c|c|c|c|c|c|c|c|} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ & & 3 & 3 & 4 & 4 & 5 & 6 & \\ & & & 4 & 5 & 5 & 5 & & \\ & & & & 5 & 6 & 6 & & \\ & & & & & 6 & & & \end{array} \in \mathcal{ST}^{986431}(6) \text{ with } \begin{array}{l} \mathrm{wgt}(ST) = (3, 5, 6, 4, 8, 5) \\ \mathrm{str}(ST) = 12, \quad \mathrm{hgt}(ST) = 6. \end{array}$$

- wgt(ST)=weight of the shifted tableau
- str(ST)=disjoint connected components of ribbon strips
- hgt(ST)=height of the tableau

Back to ASM: μ-ASM

- $\mu=\mu_1, \mu_2, \ldots, \mu_k$ is a partition
- Rectangular matrices with entries from 0, 1, or -1
- Nonzero entries in each row and column alternate in sign
- Each row and column contains at least one 1; first and last nonzero elements of each row are 1
- First nonzero element in each column is 1
- Last nonzero element is 1 in column q if $q=\mu_i$ for some i, and 0 otherwise

ASM statistics

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- Four kinds of zeros:
  - NE, SW, NW, SE
- Two kinds of ones:
  - WE (+1s), NS (-1s)

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} NE & NE & WE & NW & NW & NW & NW & NW & NW \\ NE & NE & SE & WE & NW & NW & NW & NW & NW \\ WE & NW & NS & SE & NE & WE & NW & NW & NW \\ SE & NE & NE & SE & WE & NS & NE & NE & WE \\ SE & NE & WE & NS & SE & NE & NE & WE & SW \\ SE & NE & SE & WE & NS & WE & NW & SW & SW \end{bmatrix}$$

Tokuyama for ASM

- H. and King, 2007:

$$\prod_{1\le i<j\le n}(x_i+y_j)\, s_\lambda(\mathbf{x}) = \sum_{A\in\mathcal{A}^\mu(n)}\prod_{k=1}^{n} x_k^{NE_k(A)} y_k^{SE_k(A)} (x_k+y_k)^{NS_k(A)}$$

Or, if you like *t*'s....

$$\prod_{1\le i<j\le n}(x_i+tx_j)\, s_\lambda(\mathbf{x}) = \sum_{A\in\mathcal{A}^\mu(n)}\prod_{k=1}^{n} t^{SE_k(A)}(1+t)^{NS_k(A)} x_k^{NE_k(A)+SE_k(A)+NS_k(A)}$$

Primed Shifted Tableaux

- weak increase across each row
- weak increase down each column
- no two identical unprimed entries in any column
- no two identical primed entries in any row
- no primed element on the main diagonal

1	1	1	$2'$	2	2	3	3	5
	2	2	$3'$	3	$4'$	$5'$	5	$6'$
		3	3	$4'$	4	$5'$	6	
			4	$5'$	5	5		
				5	$6'$	6		
					6			

Proof idea...

- Use an association between ASM and primed shifted tableaux...

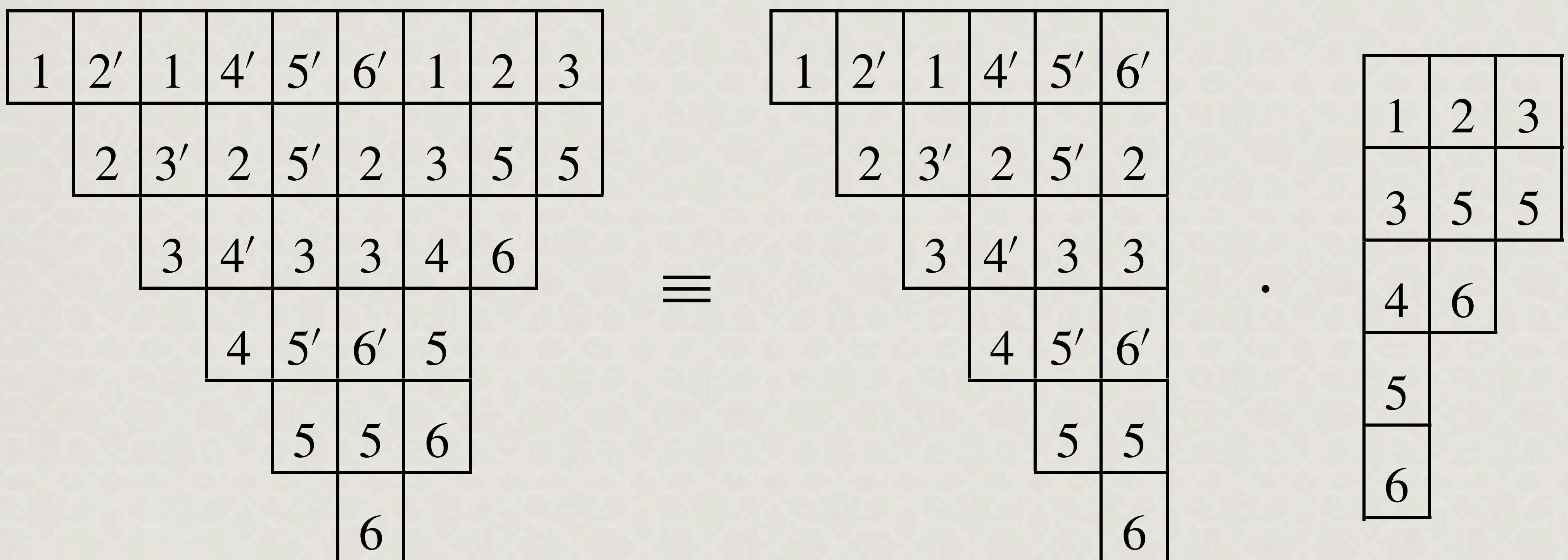
$$
PST = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 2' & 3' & 3 & 4 & 4 & 4 \\
\hline
\multicolumn{1}{c|}{} & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\
\cline{2-9}
\multicolumn{2}{c|}{} & 3 & 4' & 4 & 4 & 5 & 6 & \multicolumn{1}{c}{} \\
\cline{3-8}
\multicolumn{3}{c|}{} & 4 & 5' & 5 & 6' & \multicolumn{2}{c}{} \\
\cline{4-7}
\multicolumn{4}{c|}{} & 5 & 6' & 6 & \multicolumn{2}{c}{} \\
\cline{5-7}
\multicolumn{5}{c|}{} & 6 & \multicolumn{3}{c}{} \\
\cline{6-6}
\end{array}
\implies M(PST) = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2' & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 3' & 3' & 3 & 0 & 0 & 0 \\
4 & 4' & 4 & 4 & 4' & 0 & 4 & 4 & 4 \\
5 & 5' & 5 & 0 & 5 & 5' & 5 & 5 & 0 \\
6 & 6' & 6 & 6' & 0 & 6 & 0 & 0 & 0
\end{bmatrix}
$$

$$
PST = \begin{array}{ccccccccc}
1 & 1 & 1 & 2' & 3' & 3 & 4 & 4 & 4 \\
 & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\
 & & 3 & 4' & 4 & 4 & 5 & 6 & \\
 & & & 4 & 5' & 5 & 6' & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\implies M(PST) = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2' & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 3' & 3' & 3 & 0 & 0 & 0 \\
4 & 4' & 4 & 4 & 4' & 0 & 4 & 4 & 4 \\
5 & 5' & 5 & 0 & 5 & 5' & 5 & 5 & 0 \\
6 & 6' & 6 & 6' & 0 & 6 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\implies A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

...and use jeu de taquin on the primed shifted tableau...

...to create a pair of tableaux



One corresponding to

$$\prod_{1 \le i \le j \le n} (x_i + y_j)$$

...and the other corresponding to $s_\lambda(\mathbf{x})$

Another perspective

$$Z(\mathfrak{S}^{\Gamma}_{\lambda}) = \prod_{i<j}(t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n)$$

$$Z(\mathfrak{S}^{\Delta}_{\lambda}) = \prod_{i<j}(t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n)$$

where *Z* is the partition function.....

(Brubaker, Bump, Friedberg, 2009)

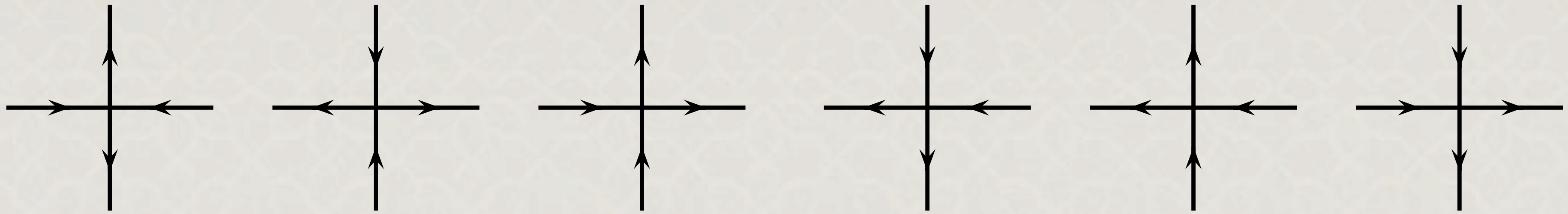
Object 3

Square Ice

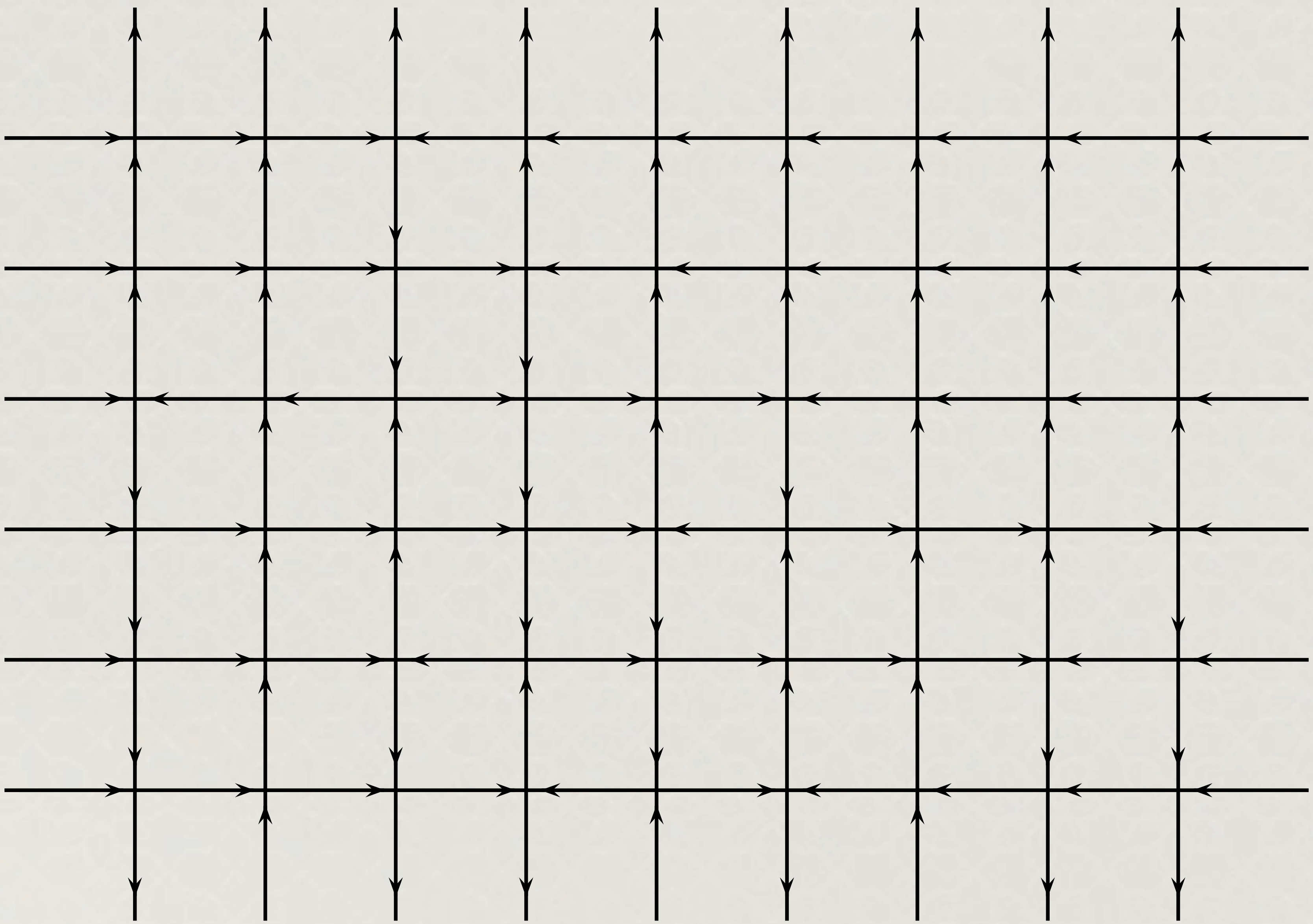
Square Ice

- So-called because it models in a two dimensional grid the orientation of molecules in frozen water.
- Also called the six-vertex model.

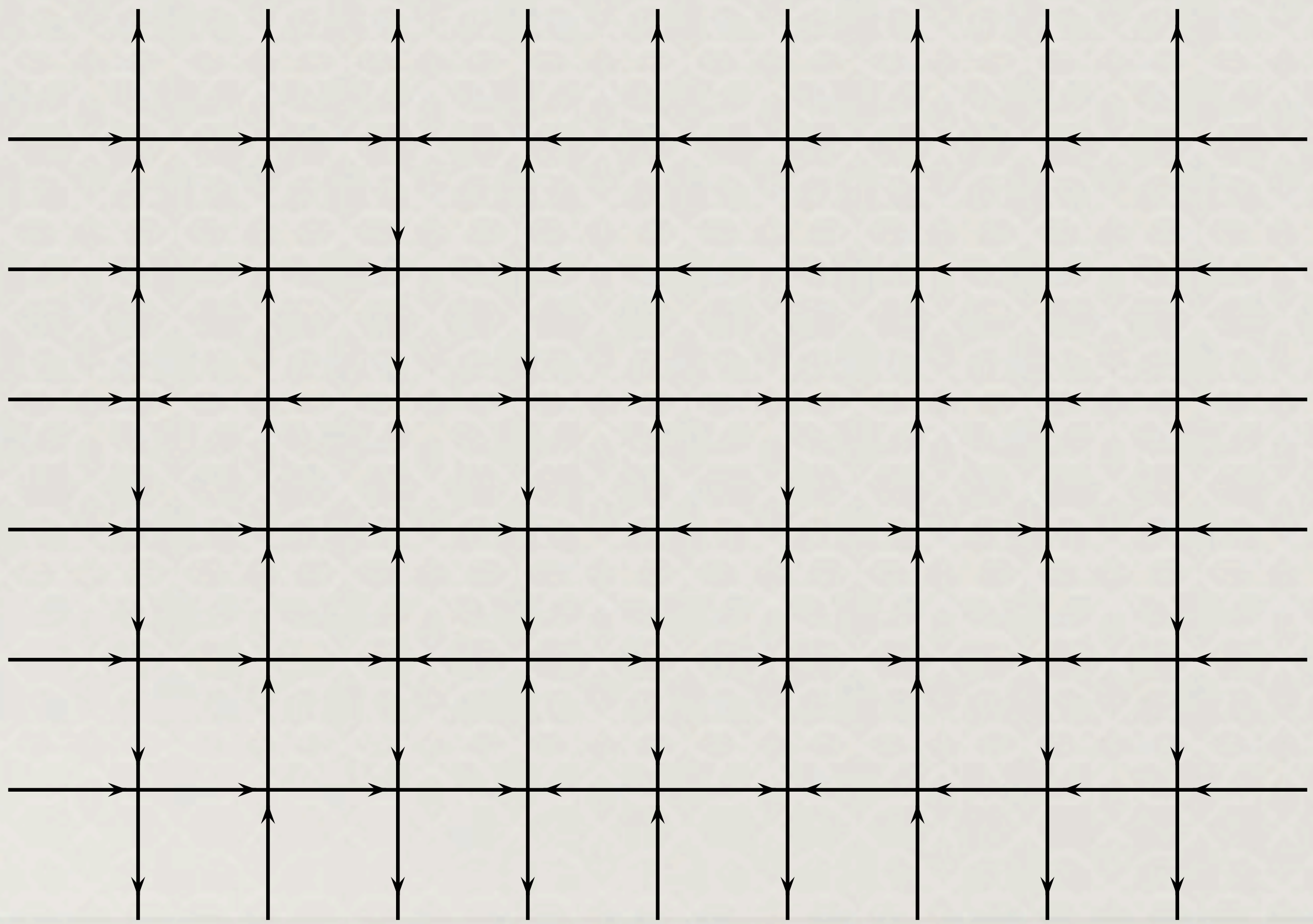
$$\begin{array}{c}\uparrow\\ \rightarrow\cdot\leftarrow\\ \downarrow\end{array}\qquad\begin{array}{c}\downarrow\\ \leftarrow\cdot\rightarrow\\ \uparrow\end{array}\qquad\begin{array}{c}\uparrow\\ \rightarrow\cdot\rightarrow\\ \uparrow\end{array}\qquad\begin{array}{c}\downarrow\\ \leftarrow\cdot\leftarrow\\ \downarrow\end{array}\qquad\begin{array}{c}\uparrow\\ \leftarrow\cdot\leftarrow\\ \uparrow\end{array}\qquad\begin{array}{c}\downarrow\\ \rightarrow\cdot\rightarrow\\ \downarrow\end{array}$$



$WE(+1)$ $NS(-1)$ $NE(0)$ $SW(0)$ $NW(0)$ $SE(0)$



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} NE & NE & WE & NW & NW & NW & NW & NW & NW \\ NE & NE & SE & WE & NW & NW & NW & NW & NW \\ WE & NW & NS & SE & NE & WE & NW & NW & NW \\ SE & NE & NE & SE & WE & NS & NE & NE & WE \\ SE & NE & WE & NS & SE & NE & NE & WE & SW \\ SE & NE & SE & WE & NS & WE & NW & SW & SW \end{bmatrix}$$

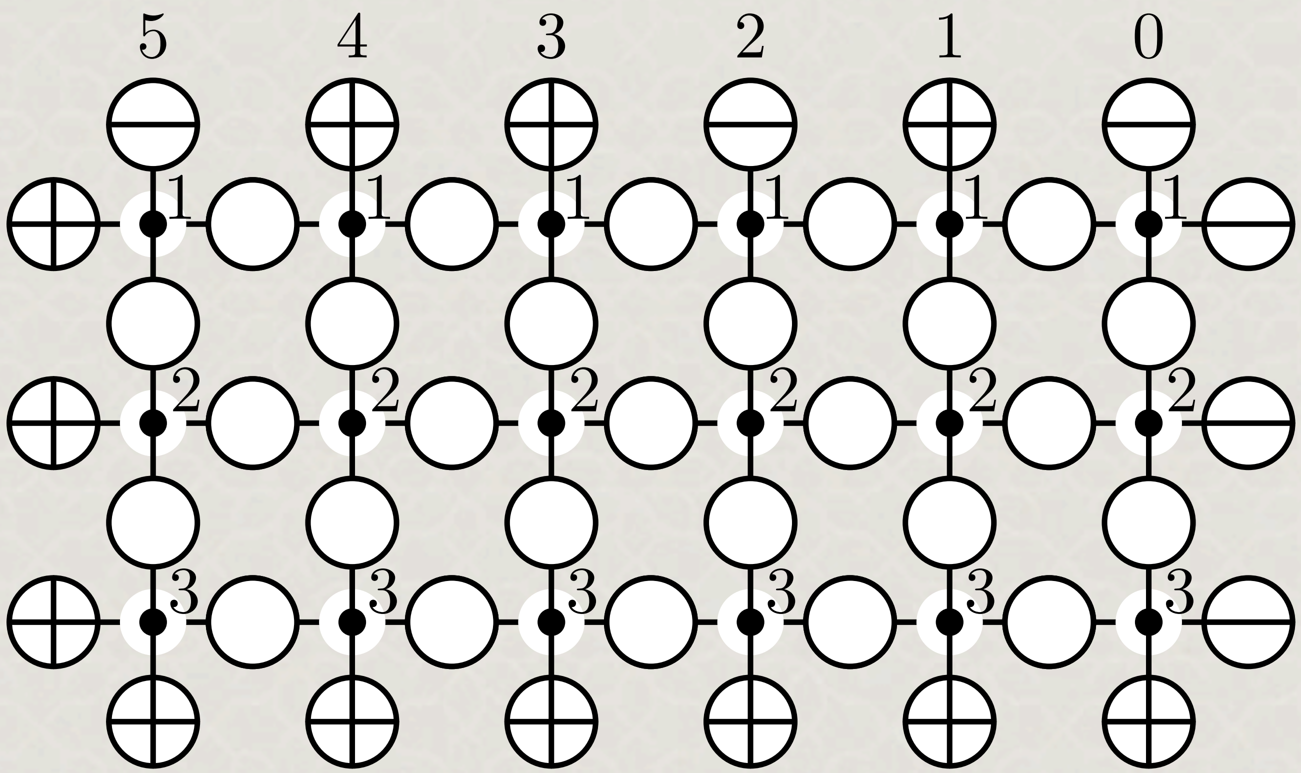
Boltzmann Weights

- Each vertex is assigned a weight called a **Boltzmann weight**. The value of this weight depends on the orientation of the adjacent edges.
- A **partition function** is the sum of the weights over all possible states.

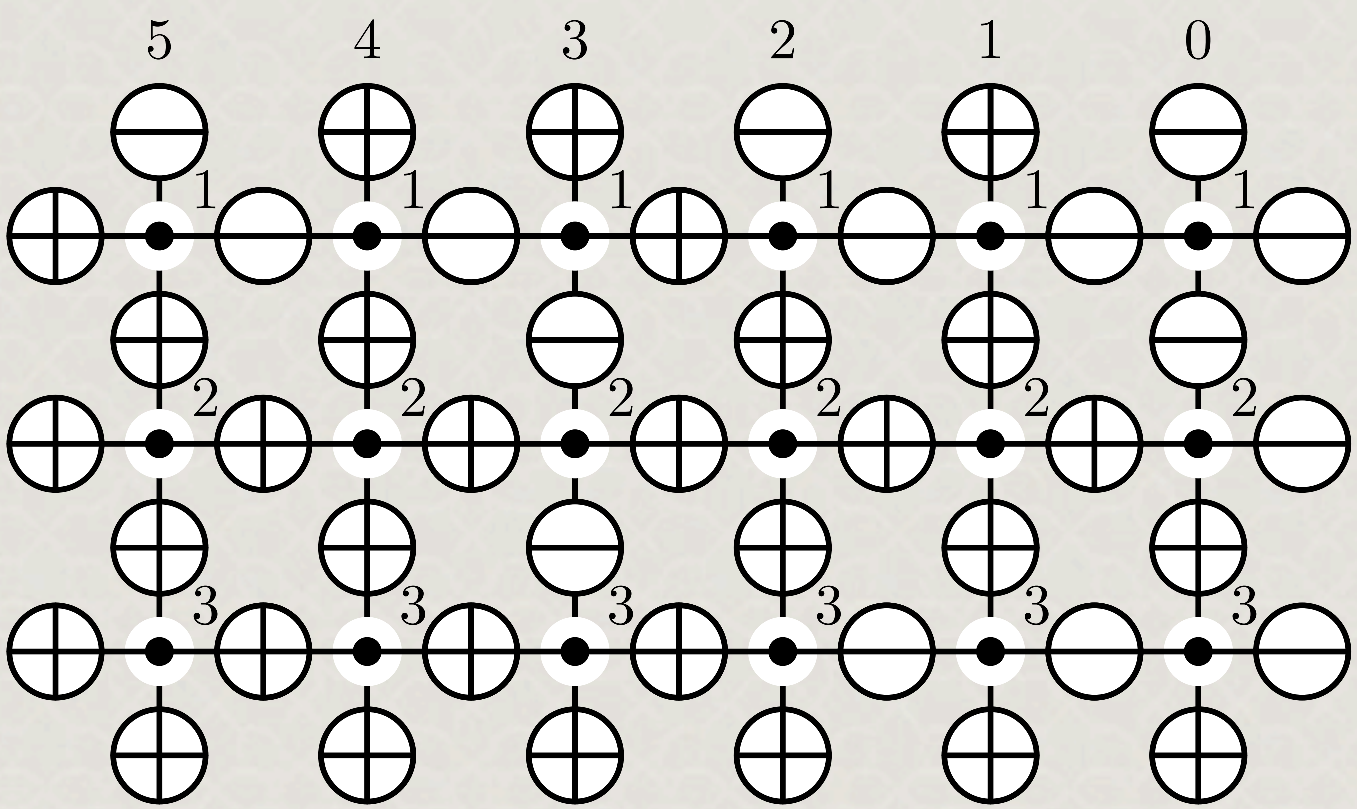
Boltzmann Weights

$$
\begin{array}{cccccc}
\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} &
\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} &
\begin{array}{ccc} & \uparrow & \\ \rightarrow & \cdot & \rightarrow \\ & \uparrow & \end{array} &
\begin{array}{ccc} & \downarrow & \\ \leftarrow & \cdot & \leftarrow \\ & \downarrow & \end{array} &
\begin{array}{ccc} & \uparrow & \\ \leftarrow & \cdot & \leftarrow \\ & \uparrow & \end{array} &
\begin{array}{ccc} & \downarrow & \\ \rightarrow & \cdot & \rightarrow \\ & \downarrow & \end{array} \\
1 & z_i(t_i+1) & t_i & z_i & z_i & 1
\end{array}
$$

- Set the arrows at the boundary either in or out (some restrictions apply)



- Look at all possible valid orientations for the arrows on the inside. Each set of valid orientations is a configuration.



- The weight of the configuration is the product of the Boltzmann weights of its vertices.
- In this case, $z_i^7 t_i(t_i + 1)$.
- The partition function is the sum over all configurations of the weight of the configuration, i.e. $\sum_{x \in \mathfrak{S}} w(x)$.

Proof idea...

Brubaker, Bump, Friedberg show that

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}$$

is the Schur function by showing it is symmetric in *z*, and independent of *t*.

- Then set *t* =-1 and show it is equivalent to the Weyl denominator formula.

Factorial Schur Functions

$$s_\lambda(x|a) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)})$$

- sum is over all tableaux of shape λ, and c(α) is the **content** of the square (c(α)=j-i for square α).

Weighted Tableaux

- Weight each entry *k* in position *i, j* by $x_k - a_{k+j-i}$

1	1	2	4
2	3	3	
4	4	5	

$$
\begin{array}{llll}
(x_1 - a_1) & (x_1 - a_2) & (x_2 - a_4) & (x_4 - a_7) \\
(x_2 - a_1) & (x_3 - a_3) & (x_3 - a_4) & \\
(x_4 - a_2) & (x_4 - a_3) & (x_5 - a_5) &
\end{array}
$$

Who are they?

- Factorial Schur functions....what are they good for?
  - Originally due to Biedenharn and Louck (1989) in a different form: x_k - k +1 +j - i.
  - Related to supersymmetric Schur functions (Macdonald, 1992 &1995 p54; Goulden and Greene, 1994)
  - Is there a connection to ASM?

Other Boltzmann weights

$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
$\rightarrow \cdot \leftarrow$	$\leftarrow \cdot \rightarrow$	$\rightarrow \cdot \rightarrow$	$\leftarrow \cdot \leftarrow$	$\leftarrow \cdot \leftarrow$	$\rightarrow \cdot \rightarrow$
$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
1	x_i/a_j	1	1	1	$x_i/a_j -1$

- McNamara 2009

Partition function and Factorial Schur Function

$$Z_\lambda(x|a) = \frac{x^\delta}{a^{(\lambda+\rho)'}} s_\lambda(x|a)$$

- McNamara 2009; Lascoux 2007 (in different language).

Proof idea...

- Show the symmetry of the partition function Z
- Use the “vanishing” properties of the factorial Schur function
- Show the partition function and the factorial Schur function are one and the same

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