Diagrammatic Kazhdan-Lusztig theory for (walled) Brauer algebras

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$$\mathrm{GL}_n \quad \to \quad V^{\otimes r} \otimes (V^*)^{\otimes s} \quad \leftarrow \quad B_{r,s}(n).$$

More generally, for each $\delta \in \mathbb{C}$ and any positive integers r and s we can define the **Brauer algebra** $B_r(\delta)$ [Brauer '37] and the **walled Brauer algebra** $B_{r,s}(\delta)$ [Koike '89; Turaev '89; BCHLLS '94].

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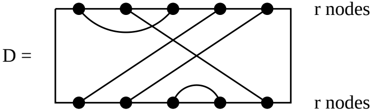

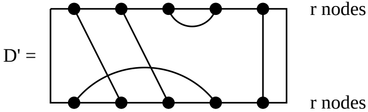

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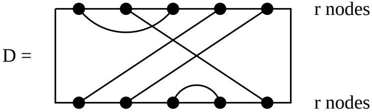
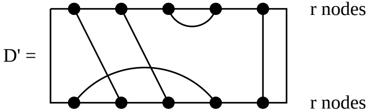
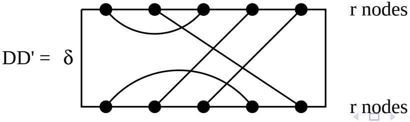




and multiplication given by concatenation of diagrams and scalar multiplication by δ^k where k is the number of loops in the concatenation.



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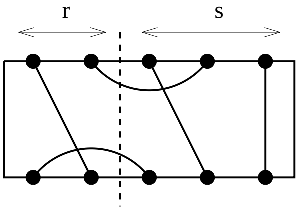

where horizontal edges must cross the wall and vertical edges cannot cross the wall.

3. Blocks and reflection groups

Simple $B_r(\delta)$-modules indexed by partitions of degree $r, r-2, r-4, \dots$
Simple $B_{r,s}(\delta)$-modules indexed by bipartitions of bidegree $(r,s), (r-1,s-1), (r-2,s-2), \dots$

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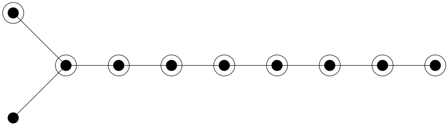
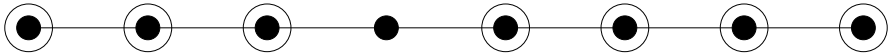
What are their block structures?

Consider the maximal parabolic subgroups

$$A_{r-1} \subset D_r$$



$$A_{r-1} \times A_{s-1} \subset A_{r+s-1}$$



Theorem 1. (i) [CDM] The simple $B_r(\delta)$-modules can be identified with A_{r-1}-dominant integral weights. Then two simple modules are in the same block if and only if they are in the same D_r-orbit.

(ii) [CDDM] The simple $B_{r,s}(\delta)$-modules can be identified with $A_{r-1} \times A_{s-1}$-dominant integral weights. Then two simple modules are in the same block if and only if they are in the same A_{r+s-1}-orbit.

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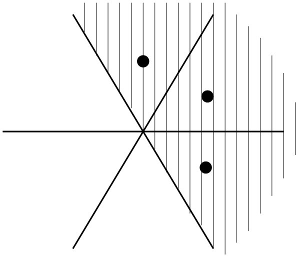

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Theorem 2. [CDM][M][CD] The **decomposition numbers** for the Brauer and walled Brauer algebras are given by parabolic KL-polynomials (in the sense of Soergel's algorithm) of type $A \subset D$ and $A \times A \subset A$ respectively.

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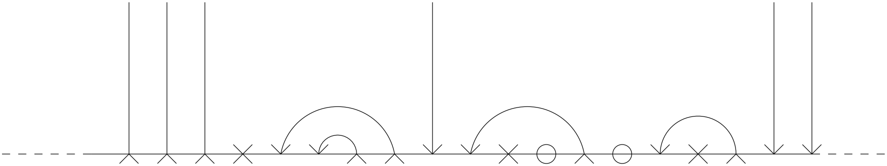
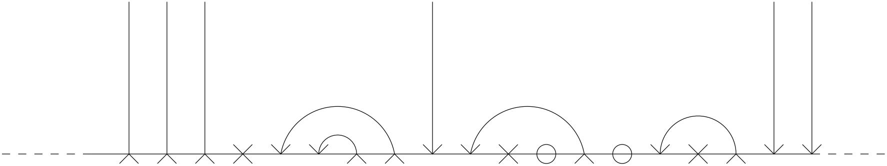
These parabolic KL-polynomials are monomials which can be described using the **cap/curl diagram** associated to a (bi)partition (see [Brundan, Stroppel] for type $A \times A \subset A$ and [CD] for type $A \subset D$).



cap diagram associated to a bipartition



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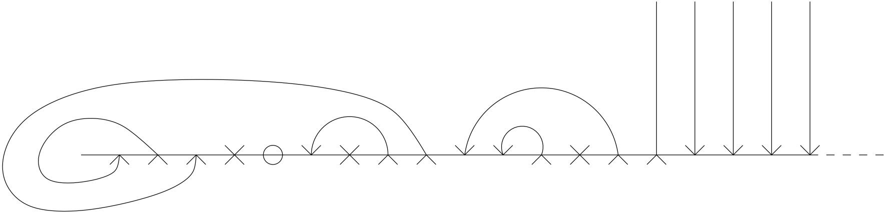

curl diagram associated to a partition

Theorem 3. [CD] (see also [BS]) The dimensions of $\mathrm{Ext}^i(\Delta(\lambda), L(\mu))$ are given by parabolic KL-polynomials in the sense of Lascoux-Schutzenberger ($A \times A \subset A$) and Boe (other types). These can also be described using the cap/curl diagrams.

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Thank you!