Littlewood-Richardson rule for Schur P-functions

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Partition

$\lambda = (\lambda_1, \ldots, \lambda_l) \vdash k$:

$$\lambda_1 \geq \cdots \geq \lambda_l \geq 0, \ \sum \lambda_i = k$$

Schur function [Jacobi 1841, Schur 1901]

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\left| x_i^{\lambda_j + n - j} \right|}{\left| x_i^{n-j} \right|}$$

$s_\lambda(x_1, x_2, \ldots, x_n)$ is a symmetric polynomial:

$s_\lambda(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) = s_\lambda(x_1, x_2, \ldots, x_n)$ for all $\pi \in \mathfrak{S}_n$

Ring of Symmetric functions

$$\Lambda = \mathbb{C}[m_\lambda] = \bigoplus_{n \geq 0} \Lambda^n$$

$m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$ is a monomial symmetric function

- powersum symmetric functions

  $p_{(3,2)} = (x_1^3 + x_2^3 + \cdots)(x_1^2 + x_2^2 + \cdots)$

- elementary symmetric functions

  $e_{(3,2)} = (x_1x_2x_3 + x_1x_2x_4 + \cdots)(x_1x_2 + x_1x_3 + x_2x_3 \cdots)$

- complete homogeneous symmetric functions

  $h_{(3,2)} = (x_1^3 + \cdots + x_1^2x_2 + \cdots + x_1x_2x_3 + \cdots)(x_1^2 + \cdots + x_1x_2 + \cdots)$

- Schur functions

Jacobi-Trudi determinants & combinatorial model

$$s_\lambda = |h_{\lambda_i - i + j}| \qquad s_{\lambda'} = |e_{\lambda_i - i + j}|$$

Semistandard tableau of shape $\lambda = (4,3,3)$: $\begin{array}{cccc} 1 & 2 & 2 & 3 \\ 2 & 5 & 6 & \\ 4 & 6 & 8 & \end{array}$

$$s_\lambda = \sum_T x^T, \quad T\text{: semistandard tableau}$$

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + 2 x_1 x_2 x_3 + \cdots$$

$$\begin{array}{cc} 1 & 1 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 1 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

Bender-Knuth involution: a proof of symmetry of s_λ

$$\sigma_2 \left(\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & \underline{2} & \underline{2} & \underline{2} & \underline{3} \\ 2 & \underline{2} & \underline{3} & \underline{3} & 3 & 3 & & & & \\ 3 & & & & & & & & & \end{array} \right)$$

$$\longrightarrow \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & \underline{2} & \underline{3} & \underline{3} & \underline{3} \\ 2 & \underline{2} & \underline{2} & \underline{3} & 3 & 3 & & & & \\ 3 & & & & & & & & & \end{array}$$

(isomorphic) graded algebras

- $\Lambda = \bigoplus_n \Lambda^n$: ring of symmetric functions

$$s_\lambda(x) s_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x)$$

- $R = \bigoplus_n R^n$: class functions (representations) on $\mathfrak{S}_n$'s

$$\chi^\lambda \cdot \chi^\mu = \sum_\nu c^\nu_{\lambda,\mu} \chi^\nu$$

- finite dimensional polynomial representations of $GL_m(\mathbb{C})$

$$V(\lambda) \otimes V(\mu) = \bigoplus_\nu c^\nu_{\lambda,\mu} V(\nu)$$

- $H^*(Gr(n,m))$: cohomology ring of a Grassmannian

$$\sigma_\lambda \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu$$

- $\mathbb{C}[\mathsf{S}_\lambda]$: subalgebra of the tableaux algebra generated by S_λ's where $\mathsf{S}_\lambda = \sum \mathsf{T}$

$$\begin{matrix} 1 & 1 \\ 3 & \end{matrix} \cdot \begin{matrix} 1 & 2 \\ 2 & \end{matrix} = \begin{matrix} * & * & 1 & 2 \\ * & * & 2 & \\ 1 & 1 & & \\ 3 & & & \end{matrix} \underset{(\mathrm{jdt})}{=} \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & & & \\ 3 & & & \end{matrix}$$

- $\mathsf{P}[\mathsf{S}_\lambda]$: subalgebra of the plactic algebra $\mathsf{P}[\mathsf{X}]$ generated by $\mathcal{S}_\lambda$'s where $\mathcal{S}_\lambda = \sum [w]$

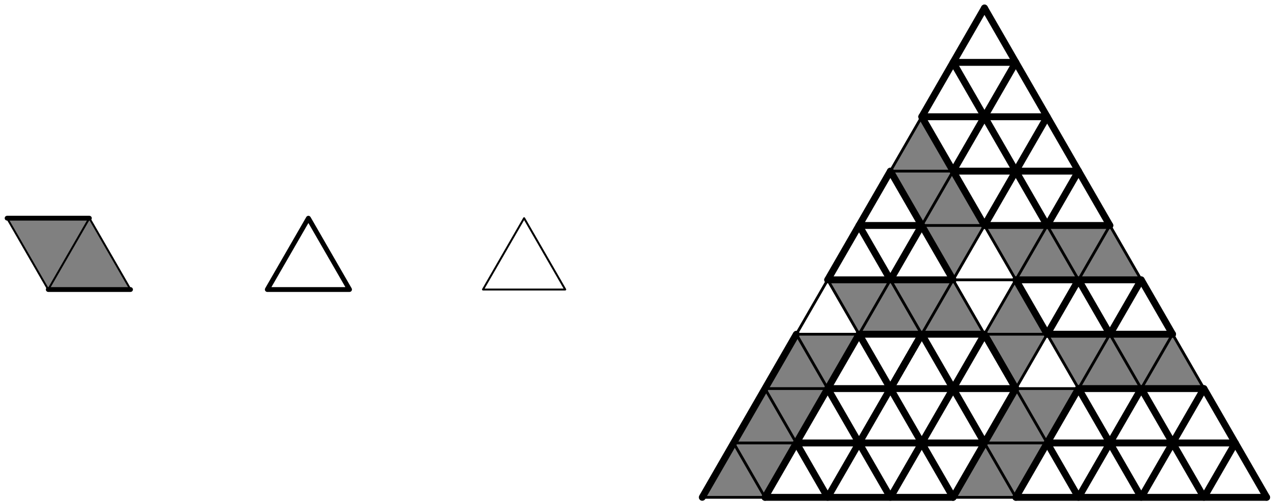
$$3\ 1\ 1 * 2\ 1\ 2 = 3\ 1\ 1\ 2\ 1\ 2 \equiv_{\mathsf{K}} 3\ 2\ 1\ 1\ 1\ 2$$

Knuth relation:

1. $yzx \equiv_K yxz$ if $x < y \leq z$
2. $xzy \equiv_K zxy$ if $x \leq y < z$

Littlewood-Richardson coefficients

$$
\begin{aligned}
c^{\nu}_{\lambda,\mu} \quad &= \quad \text{number of LR-tableaux} \\
&= \quad \text{number of integral honeycombs (hives)} \\
&= \quad \text{number of puzzles} \\
&= \quad \cdots
\end{aligned}
$$



$\lambda = (4, 2), \mu = (3, 2), \nu = (7, 4) \subseteq (7, 7)$

[Stembridge 02] $s_\lambda s_\mu = \sum s_{\lambda+wt(T)}$, where T: semistandard tableau of shape μ such that $\lambda + wt(T_{\geq j})$ is a partition for all $j \geq 1$.

(Proof) $s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}$, where $a_\lambda = |x_i^{\lambda_j}|$, $\rho = (n-1, \ldots, 1, 0)$.

$$\begin{aligned} a_{\lambda+\rho} s_\mu &= \sum_{w \in S_n} \varepsilon(w) x^{w(\lambda+\rho)} \sum_{T \in S(\mu)} x^{wt(T)} \\ &= \sum_{T \in S(\mu)} \sum_{w \in S_n} \varepsilon(w) x^{w(\lambda+\rho+wt(T))} = \sum_{T \in S(\mu)} a_{\lambda+wt(T)+\rho} \end{aligned}$$

For $n = 4$, $\lambda = (3, 3, 1)$, $\mu = (4, 2, 1)$,

$$T = \begin{array}{cccc} 1 & 3 & \underline{4} & 4 \\ 2 & 4 & & \\ 3 & & & \end{array} \quad \longleftrightarrow \quad T^* = \begin{array}{cccc} 1 & 3 & \underline{4} & 4 \\ 2 & 4 & & \\ 4 & & & \end{array}$$

$$\lambda + wt(T) + \rho = (7, 6, 4, 3) \quad \longleftrightarrow \quad \lambda + wt(T^*) + \rho = (7, 6, 3, 4)$$

$$a_{\lambda+wt(T)+\rho} = -a_{\lambda+wt(T^*)+\rho}$$

Important Theorems

Horn's inequalities Let λ, μ and ν be partitions of lengths at most n. Then $c^{\nu}_{\lambda\mu} > 0$ if and only if $|\nu| = |\lambda| + |\mu|$ and for all $r \leq n$, $\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$ holds for all $(I, J, K) \in R^n_r$.

Saturation For a positive integer N,

$$c^{\nu}_{\lambda\mu} > 0 \quad \text{if and only if} \quad c^{N\nu}_{N\lambda N\mu} > 0$$

Fulton's conjecture For a positive integer N,

$$c^{\nu}_{\lambda\mu} = 1 \quad \text{if and only if} \quad c^{N\nu}_{N\lambda N\mu} = 1$$

Interior If λ, μ, ν are partitions with n distinct parts, and each of the Horn inequalities holds strictly, then $c^{\nu}_{\lambda\mu}$ is at least 2.

Identities of LR-coefficients

Symmetry $c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda}$

Conjugation $c^{\nu}_{\lambda,\mu} = c^{\widetilde{\nu}}_{\widetilde{\lambda},\widetilde{\mu}}$

Reduction I For any three indices $0 \le i, j, k \le n$ with $i + j = k + n$, if $\lambda_i + \mu_j = \nu_k$ then $c^{\nu}_{\lambda\mu} = c^{\nu - \nu_k}_{\lambda - \lambda_i, \mu - \mu_j}$.

Reduction II If there are λ_i, μ_j, ν_k with $i + j = k - 1$ such that $\lambda_{i+1} < \lambda_i$, $\mu_{j+1} < \mu_j$, $\nu_k < \nu_{k-1}$ and $\lambda_i + \mu_j \ge \nu_1 + \nu_k + 1$, then $c^{\nu}_{\lambda,\mu} = c^{\nu - (1^{k-1})}_{\lambda - (1^i),\, \mu - (1^j)}$.

Factorization If $c^{\nu}_{\lambda\mu} > 0$ and there exists $(I, J, K) \in R^n_r$ for some $r < n$ such that $\sum_{k \in K} \nu_k = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$, then $c^{\nu}_{\lambda\mu} = c^{\nu_K}_{\lambda_I \mu_J} c^{\nu_{K^c}}_{\lambda_{I^c} \mu_{J^c}}$.

Combinatorial proofs

Symmetry Benkart, Sottile and Stroomer, "Tableau switching: algorithms and applications," JCTA 1996

Conjugation Hanlon and Sundaram, "On a bijection between Littlewood-Richardson fillings of conjugate shape," JCTA 1992

Reductions Cho, Jung and Moon,

"A combinatorial proof of the reduction formula for Littlewood-Richardson coefficients," JCTA 2007

"A bijective proof of the second reduction formula for Littlewood-Richardson coefficients," BKMS 2008

Factorization King, Tollu and Toumazet "Factorisation of Littlewood-Richardson coefficients," JCTA 2009

Schur P-functions [Schur 1911]

For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell) \vdash k$,

$$Q_\lambda(x_1, x_2, \ldots, x_n) = \mathrm{Pf}(Q_{(\lambda_i, \lambda_j)}),$$

where *Pfaffian* of a $2m \times 2m$ skew symmetric matrix $A = (a_{ij})$ is

$$\mathrm{Pf}(A) = \sum_{w \in S_{2m}} \varepsilon(w) \prod_{i=1}^{m} a_{w(2i-1)\,w(2i)},$$

for $w(2i-1) < w(2i)$ and $w(1) < w(3) < \cdots < w(2m-3) < w(2m-1)$

$$Q_{(r,s)} = q_r q_s + 2\sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i},$$

$$q_r(x_1, \ldots, x_n) = 2\sum_{i=1}^{n} x_i^r \prod_{j \neq i} \frac{x_i + x_j}{x_i - x_j}$$

$$Q_\lambda(x_1, x_2, \ldots, x_n) = 2^\ell P_\lambda(x_1, x_2, \ldots, x_n)$$

A specialized Hall-Littlewood function ($t = -1$)

$$P_\lambda(x_1, x_2, \ldots, x_n) = \frac{1}{(n-\ell)!} \sum_{w \in S_n} \prod_{i=1}^{n} x_{w(i)}^{\lambda_i} \prod_{i \leq \ell, i<j} \frac{x_{w(i)} + x_{w(j)}}{x_{w(i)} - x_{w(j)}}$$

$$P_\lambda(x_1, x_2, \ldots, x_n) = \frac{\mathrm{Pf}\begin{pmatrix} \frac{x_i - x_j}{x_i + x_j} & x_i^{\lambda_j} \\ -x_j^{\lambda_i} & 0 \end{pmatrix}}{\mathrm{Pf}\left(\frac{x_i - x_j}{x_i + x_j}\right)}$$

$P_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial and $\{P_\lambda\}$ forms a basis of $\Gamma = \mathbb{C}[q_1, q_2, q_3, \ldots] \subset \Lambda$

Combinatorial model for P_λ

Marked shifted semistandard tableaux of shape $\lambda = (5,3,2)$ on letters $\{1' < 1 < 2' < 2 < \cdots\}$:

$$\begin{array}{ccccc} 1 & 1 & 1 & 2' & 2 \\ & 2 & 2 & 5' & \\ & & 4 & 5' & \end{array}$$

$$P_\lambda = \sum_T x^T, \ \text{T: marked shifted semistandard tableau}$$

$P_{(r)} = \frac{1}{2} q_r$

$P_{(3,1)}(x_1, x_2) = x_1^3 x_2 \ + \ x_1^2 x_2^2 \ + \ x_1^2 x_2^2 \ + \ x_1 x_2^3$

$$\begin{array}{ccc} 1 & 1 & 1 \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 1 & 2' \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 1 & 2 \\ & 2 & \end{array} \quad \begin{array}{ccc} 1 & 2' & 2 \\ & 2 & \end{array}$$

Combinatorial model for P_λ

A word $w = w_1 w_2 \cdots w_m$ on the set of alphabets $\{1, 2, \ldots, n\}$ is a *hook word* if there is $1 \le m' \le m$ such that

$$w_1 > w_2 > \cdots > w_{m'} \le w_{m'+1} \le \cdots \le w_m .$$

A semistandard decomposition tableau (SSDT) R of shape λ is a filling of $\mathcal{S}(\lambda)$ such that

1. the word R_i obtained by reading the ith row of R from the left is a hook word of length λ_i, and
2. R_i is a hook word of maximum length in $R_\ell R_{\ell-1} \cdots R_i$ for all $i = 1, \ldots, \ell - 1$.

$$P_\lambda = \sum_T x^T, \ T\text{: SSDT}$$

$$P_{(3,1)}(x_1,x_2) = x_1^3x_2 \;+\; x_1^2x_2^2 \;+\; x_1^2x_2^2 \;+\; x_1x_2^3$$

$$\begin{array}{ccc} 2 & 1 & 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 1 \\ & 2 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 2 \\ & 1 & \end{array} \quad \begin{array}{ccc} 2 & 1 & 2 \\ & 2 & \end{array}$$

[Serrano 2010] There is a weight preserving bijection between the set of shifted semistandard tableaux and the set of SSDT's

$$\begin{array}{cccc} 1 & 2' & 3' & 3 \\ & 2 & 3' & 4 \\ & & 3 & \end{array} \quad \overset{\text{mixed insertion}}{\longleftarrow} \quad 3\,3\,2\,3\,4\,2\,1\,3 \quad \overset{\text{SK insertion}}{\longrightarrow} \quad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ & 3 & 2 & 3 \\ & & 3 & \end{array}$$

Lowest weight SSDT and highest weight SSDT

$$\lambda = (9, 8, 6, 4, 3)$$

5	4	3	2	1	|	1	1	3	4
	5	4	3	2	|	2	2	4	5
		5	4	3	|	3	3	5	
			5	4	|	4	4		
				5	|	5	5		

5	4	3	2	|	1	1	1	1	1
	5	4	3	|	1	2	2	2	2
		5	4	|	1	2	3	3	
			4	|	1	2	3		
				|	1	2	3		

Bender-Knuth type involutions: proof of symmetry of P_λ

[Stembridge 1990] For each $1 \le k \le n-1$, an involution is defined on the set of shifted semistandard tableaux

[C] Lascoux-Schützenberger involutions defined on the set of words are Bender-Knuth type involution on SSDT's

$$\sigma_1 \begin{pmatrix} 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ & 6 & 5 & 2 & 1 & 4 & \\ & & 5 & 1 & 2 & 3 & \end{pmatrix} = \begin{matrix} 6 & 5 & 4 & 2 & 1 & 1 & 3 \\ & 6 & 5 & 2 & 1 & 4 & \\ & & 5 & 2 & 2 & 3 & \end{matrix}$$

$$5\,\underline{1}\,\boxed{2}\,3\,6\,5\,\boxed{2}\,\boxed{1}\,4\,6\,5\,4\,\boxed{2}\,\boxed{1}\,\boxed{1}\,3 \rightarrow 5\,\underline{2}\,\boxed{2}\,3\,6\,5\,\boxed{2}\,\boxed{1}\,4\,6\,5\,4\,\boxed{2}\,\boxed{1}\,\boxed{1}\,3$$

Related algebras

- Γ: subring of symmetric functions

$$P_\lambda(x)P_\mu(x) = \sum_\nu f_{\lambda,\mu}^{\nu} P_\nu(x)$$

- $H^*(OG(n+1,2n+2))$: cohomology ring of orthogonal maximal isotropic Grassmannian

$$\tau_\lambda\tau_\mu = \sum_\nu f_{\lambda,\mu}^{\nu} \tau_\nu$$

- projective representations of $\mathfrak{S}_n$
- shifted plactic monoid (L. Serrano 2010)

Shifted Littlewood-Richardson coefficients

[Stembridge, 1989]

$$f^{\nu}_{\lambda,\mu} = \text{ number of LRS-tableaux}$$

When $\nu = (5,4,2,1)$, $\lambda = (3,1)$, $\mu = (4,3,1)$,

$$\mathsf{T} = \begin{array}{ccccc} \cdot & \cdot & \cdot & 1' & 1 \\ & \cdot & 1 & 1 & 2' \\ & & 2 & 2 & \\ & & & 3 & \end{array}$$ is an LRS tableau:

- T is a marked shifted semistandard tableau of shape ν/λ and content μ

- Let $w\hat{w} = a_1 a_2 \cdots a_{2m}$ for $w = 1\,1'\,2'\,1\,1\,2\,2\,3$, $\hat{w} = 4'\,3'\,3'\,2'\,2'\,2\,1\,2'$. For each $a_i = k+1$ or $a_i = (k+1)'$, there are more k's than $(k+1)$'s in $a_1 \cdots a_{i-1}$.

- The last occurrence of k' precedes the last occurrence of k in w.

Shifted Littlewood-Richardson coefficients

$$f^{\nu}_{\lambda,\mu} = \text{number of LRS-tableaux}$$
$$= \text{number of ??? SSDT?}$$

$$P_\lambda(x_n, x_{n-1}, \ldots, x_1) = \frac{\mathrm{Pf}_\lambda(x_n, \ldots, x_1)}{\mathrm{Pf}_0(x_n, \ldots, x_1)}$$

$$= \frac{1}{(n-\ell)!} \sum_{\pi \in S_n} \pi \left(x_n^{\lambda_1} x_{n-1}^{\lambda_2} \cdots x_{n-\ell+1}^{\lambda_\ell} \prod_{n-\ell+1 \le j,\ i<j} \frac{x_j + x_i}{x_j - x_i} \right)$$

Let $D_n = \mathrm{Pf}_0(x_n, \ldots, x_1) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i}$, then

$$D_n \cdot P_\lambda(x_n,\ldots,x_1)P_\mu(x_n,\ldots,x_1) \times (n-\ell)!$$

$$= \sum_{T\in\mathcal{Y}_n(\mu)} \left(\sum_{\pi\in S_n} \varepsilon(\pi)\ \pi\left(x_n^{\lambda_1}\cdots x_{n-\ell+1}^{\lambda_\ell}\ x^{\omega(T)} \prod_{1\le i<j\le n-\ell} \frac{x_j-x_i}{x_j+x_i} \right)\right)$$

$$= \sum_{\substack{R\in\mathcal{D}_n(\mu)\\ R \text{ is } \ell\text{-essential}}} \sum_{\pi\in S_n} \varepsilon(\pi)\pi\left(x^{r(\lambda)+\omega(R|_{n-\ell+1}^{n})} P_{sh(R|_1^{n-\ell})}(x_{n-\ell},\ldots,x_1) \prod_{1\le i<j\le n-\ell} \frac{x_j-x_i}{x_j+x_i} \right)$$

$$= \sum_R \frac{(n-\ell)!}{(n-\ell-k_R)!} \sum_{\pi\in S_n} \varepsilon(\pi)\pi\left(x^{r(\lambda)+\omega(R|_{n-\ell+1}^{n})} x_{n-\ell}^{\alpha_1^R}\cdots x_{n-\ell-k_R+1}^{\alpha_{k_R}^R} \prod_{1\le i<j\le n-\ell-k_R} \frac{x_j-x_i}{x_j+x_i} \right).$$

where $\alpha^R = (\alpha_1^R,\ldots,\alpha_{k_R}^R) = sh(R|_1^{n-\ell})$ and

an SSDT R is ℓ-essential if $\omega(R|_1^{n-\ell}) = (\rho_{n-\ell},\ldots,\rho_1)$ where $\rho = sh(R|_1^{n-\ell})$.

When $n = 3$,

$R_2 = \begin{array}{ccc} 3 & 1 & 2 \\ & 2 & \end{array}$ is not 1-essential: $P_{mix}(2\,1\,2) = \begin{array}{ccc} 1 & 2' & 2 \end{array}$ is of shape $(3, 0)$ and is not the lowest weight tableau of shape $(3, 0)$.

$$D_n \cdot P_\lambda(x_n, \ldots, x_1) P_\mu(x_n, \ldots, x_1) = \sum_{R:\ \ell\text{-essential}} D_n \cdot P_{\lambda + r(\omega(R))}(x_n, \ldots, x_1)$$

There are only 7 1-essential SSDT's among 24 SSDT's of shape $\mu = (3, 1)$, when $n = 3$:

SSDT R	2 1 2 2	3 2 2 2	3 2 1 2	2 1 3 2
$r(\omega(R))$	$(0, 3, 1)$	$(1, 3, 0)$	$(1, 2, 1)$	$(1, 2, 1)$
$\lambda + r(\omega(R))$	$(2, 3, 1)$	$(3, 3, 0)$	$(3, 2, 1)$	$(3, 2, 1)$

SSDT R	3 2 3 2	3 2 2 3	3 2 3 3
$r(\omega(R))$	$(2, 2, 0)$	$(2, 2, 0)$	$(3, 1, 0)$
$\lambda + r(\omega(R))$	$(4, 2, 0)$	$(4, 2, 0)$	$(5, 1, 0)$

$$
\begin{aligned}
& D_3 \cdot P_{(2,0,0)}(x_3,x_2,x_1)P_{(3,1,0)}(x_3,x_2,x_1) \\
&= D_3 \cdot P_{(2,3,1)}(x_3,x_2,x_1) + D_3 \cdot P_{(3,3,0)}(x_3,x_2,x_1) + D_3 \cdot P_{(3,2,1)}(x_3,x_2,x_1) \\
&\quad + D_3 \cdot P_{(3,2,1)}(x_3,x_2,x_1) + D_3 \cdot P_{(4,2,0)}(x_3,x_2,x_1) \\
&\quad + D_3 \cdot P_{(4,2,0)}(x_3,x_2,x_1) + D_3 \cdot P_{(5,1,0)}(x_3,x_2,x_1)
\end{aligned}
$$

An ℓ-essential SSDT R is λ-bad if $\mathrm{read}(R) = u_1u_2\cdots u_m$ is λ-bad: There is i such that $\lambda + r(\omega(u_1\ldots u_i))$ is not a strict partition.

Among 7 1-essential SSDT's of shape $\mu = (3,1)$,

$R_1 = \begin{matrix} 2 & 1 & 2 \\ & 2 & \end{matrix}$, $R_2 = \begin{matrix} 3 & 2 & 2 \\ & 2 & \end{matrix}$ and $R_3 = \begin{matrix} 2 & 1 & 3 \\ & 2 & \end{matrix}$ are λ-bad,

where $\lambda = (2)$:

$\mathrm{read}(R_1) = 2\,2\,1\,2$, $r(\omega(2\,2)) = (0,2,0)$ and $(2,0,0) + (0,2,0) \notin \mathcal{DP}$, Define a sign reversing involution on the set of λ-bad ℓ-essential SSDT's:

For $u = u_1 u_2 \dots u_m = \mathrm{read}(R)$ let i_0 be the first i such that $\lambda + r(\omega(u_1 \dots u_i)) \notin \mathcal{DP}$, and $u_{i_0} = k$ then

$$R^\diamond = \sigma_k^{i_0} \sigma_{k-1} \cdots \sigma_{k-d_R} \cdots \sigma_{k-1} \sigma_k^{i_0} (R)$$

Theorem [C]

$$P_\lambda(x_n, \dots, x_1) P_\mu(x_n, \dots, x_1) = \sum_R P_{\lambda + r(\omega(R))}(x_n, \dots, x_1),$$

where the sum runs over the λ-good ℓ-essential SSDT's of shape μ.

Remarks

1. S.-J. Kang et.al proved that the set of SSDT*'s forms a crystal of the quantum queer superalgebra $U_q(\mathfrak{q}(n))$ and obtain a similar result to the theorem on the decomposition of the product of two Schur P-functions using crystal basis theory.

2. We have constructed counter examples of the conjecture on plactic skew Schur P-functions made by Serrano.

   L. Serrano, *The shifted plactic monoid*, Math. Z. 266 (2010), no. 2, 363–392

Generalizations

- quasi-
- affine-
- - of Lie type (algebraic and geometric)
- $\mathsf{G/P} \rightsquigarrow \mathsf{G/B}$
- equivariant-
- quantum-
- K-theoretic

Thank you !!