Braid arrangement: $A^{\mathbb{C}}_{n,2}$ hyperplanes $\in \mathbb{C}^n$ $z_i = z_j$ $\forall i \neq j$

Complement: $\mathbb{C}^n - A^{\mathbb{C}}_{n,2} = M(A^{\mathbb{C}}_{n,2})$

$\boxed{1962}$ Fadell, Neuwirth: $M(A^{\mathbb{C}}_{n,2})$ are $K(\pi,1)$ spaces

$\boxed{1963}$ Fox, Neuwirth: $\pi_1(M(A^{\mathbb{C}}_{n,2})) \simeq$ pure braid gr. $\simeq \mathrm{Ker}(\varphi)$

$\varphi$: braid group $\to S_n$

$\boxed{1972}$ Deligne: complement of the complexification of any simplicial arr. is a $K(\pi,1)$ space

$\Rightarrow$ compl. of complexification of hyp. arrg. of type $W$ are $K(\pi,1)$ spaces

$\boxed{1971}$ Brieskorn: $W_{n,2}$ real finite reflection group

$\pi_1(M(W^{\mathbb{C}}_{n,2})) \simeq$ pure Artin gr. of type $W$ $\simeq \mathrm{Ker}(\varphi)$

$\varphi$: Artin group $\to W$

$\boxed{1992}$ Arvola: gives a description of $\pi_1$ of any complex hyp. arrang. in terms of generators and relations.

∴ Sometimes one can "replace" a group defined in terms of the topology of a space with a group defined in terms of the combinatorics of the space

3-equal arrangement $A^{\mathbb{R}}_{n,3}$: co-dim 2 subsp. $x_{i_1} = x_{i_2} = x_{i_3}$

Complement: $\mathbb{R}^n - A^{\mathbb{R}}_{n,3} = M(A_{n,3})$

$\boxed{1996}$ Khovanov: $M(A^{\mathbb{R}}_{n,3})$ is a $K(\pi,1)$ space

$\pi_1(M^{\mathbb{R}}_{n,3}) \simeq$ pure triplet gr. $\simeq \mathrm{Ker}(\psi)$

$\psi$: triplet group $\longrightarrow S_n$

$\boxed{1998}$ Davis, Januszkiewicz, Scott

Compl. of any collection of co-dim 2 subsp. (embedded in a hyp. arr. of type $W$) invariant under $W$ is $K(\pi,1)$

$\boxed{2009}$ B-Severs, White

$\pi_1(M(\text{3-parabolic arr. of type } W)) \simeq$ discrete fund. gr $\simeq \mathrm{Ker}(\psi)$

$\psi: W' \longrightarrow W$ and more generally

$\pi_1$ (complement in $\mathbb{R}$ of co-dim 2 sub. invariant under $W$)

$\simeq \mathrm{Ker}\, \psi'$

$\psi': W'' \longrightarrow W$

B.S.W give a description of $\pi_1(M(\text{3-parabolic}))$ in terms of generators and relations

What makes things work so well? Beautiful interplay between geometry of arrangement (Coxeter complex and its dual, a zonotope) and the (algebraic) group structure (parabolic subgroups, cosets)

1. Real Reflection group
2. Braid group, Artin group, triplet group
3. Explicit correspondence between geometry and group
4. Definition of K-parabolic arrangements
5. Explicit statement of theorem in full generality
6. Sketch of proof

Note: all reflection groups are finite

① Finite real reflection group: finite group generated by reflections

- reflection $s_\alpha$: linear op. on $\mathbb{R}^n$ that sends a vector $\alpha$ to $-\alpha$ and fixes the hyperplane $\perp$ to $\alpha$

## Examples.

$S_n$ ($A_{n-1}, n\geqslant 2$): let $\sigma \in S_n$ act on $\mathbb{R}^n$ by permuting the standard basis

Transposition $(i\,j)(e_i - e_j) = -(e_i - e_j)$ note: Transp. are the only refl of $S_n$

$H_{ij} = \{(x_1,\dots,x_i,\dots,x_j,\dots,x_n) \mid x_i = x_j\}$ hyperplane arrangement of Type A. $\# \text{ " } = \binom{n}{2}$

$B_n$: "signed permutations" reflections of $S_n$, add refl. $e_i \to -e_i$

$H^B = H_i \cup H_{ij} \cup H_{i,-j}$ hyperplane arrangement of Type B: $n + 2\binom{n}{2} = n^2$

$D_n$: "even signed permutations"

$H^D = H_{ij} \cup H_{i,-j}$ hyperplane arrangement of type D: $2\binom{n}{2} = n(n-1)$

$I_2(m)$: dihedral group of order $2m$, $m$ rotations, $m$ reflections
regular $m$-sided gone

$E_6, E_7, E_8, F_4, H_3, H_4$

Root systems $\Phi$ : finite set of nonzero vectors $\alpha \in \mathbb{R}^n$ s.t:

a) $\Phi \cap \mathbb{R}\langle\alpha\rangle = \{\alpha, -\alpha\}$ $\forall \alpha \in \Phi$

b) $s_\alpha \Phi = \Phi$ $\forall \alpha \in \Phi$

W : group generated by all reflections $s_\alpha \in \Phi$ . can do better

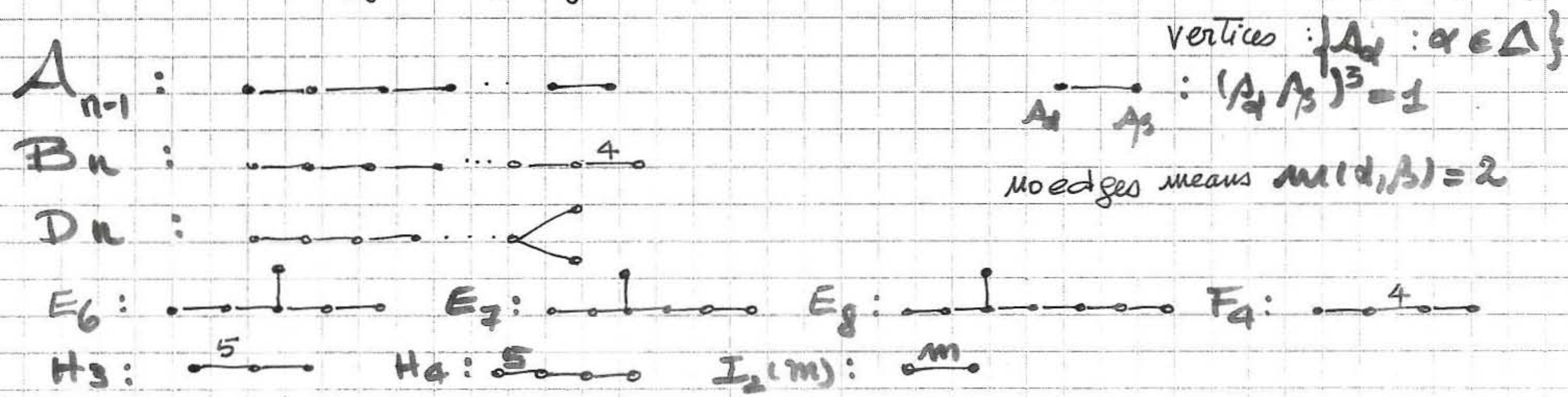
Simple root systems $\Delta$ : linearly ind. subset $\Delta \subset \Phi$ that spans $\mathbb{R}^n$ and such that all roots can be written as lin. comb. of $\alpha \in \Delta$ with coeff all same sign

$\therefore$ Simple root system exist (and infact are conjugate to one another)

$$m(\alpha,\beta) = \text{order of } s_\alpha s_\beta \in W$$

Theorem : fix a simple system $\Delta$ in $\Phi$ $\Rightarrow$ W is generated by $S = \{s_\alpha \mid \alpha \in \Delta\}$ subject only to relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ $(\alpha, \beta \in \Delta)$

$\Rightarrow$ one can classify all f.r.g (over $\mathbb{R}$) using Coxeter graphs independently of simple system



Example : $S_n = A_{n-1} \to S = \{ (i\ i+1) : 1 \leq i \leq n-1 \}$

1) $s_i^2 = 1 \quad \forall i$ (involutions)

2) $s_i s_j = s_j s_i$ if $|i-j| \geq 2$ (commutation)

3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (braid relations)

(2)

$\therefore$ Every reflection group $W$ has a hyperplane arrangement associated to it

$$\mathcal{W}_{n,2} = \{ H_\alpha : \alpha \in \Phi \}$$

Restate results of 60's and 70's : $M(\mathcal{W}_{n,2}^{\mathbb{C}})$ are $K(\pi, 1)$ spaces (Deligne)

$\pi_1(M(\mathcal{W}_{n,2}^{\mathbb{C}})) \simeq$ pure Artin group of type $W \simeq \operatorname{Ker}(\varphi)$

Artin group : $W' =$ "$W - 1$" i.e: $W'$ is a group generated by $S$ a simple system for $W$ but subject only to

2) and 3) $m(\alpha, \beta) \ \forall \alpha \neq \beta$ and $m(\alpha, \alpha) = \infty$ no involutions

Pure Artin group : $\operatorname{Ker}(\varphi) \to \varphi : W' \longrightarrow W \quad \varphi(s_\alpha) = s_\alpha$

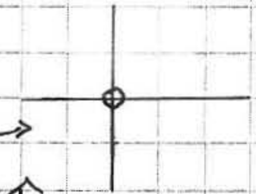
Example : $W = A_{n-1} \to$ braid group $= A_{n-1} - 1$

Infinite group where each strands that begin at $i$ ends at $i$ — pure braid group $= \operatorname{Ker}(\varphi)$

What happens if we look at $\mathbb{R}^n - \{ H_\alpha \mid \alpha \in \Phi \}$ "nothing interesting"

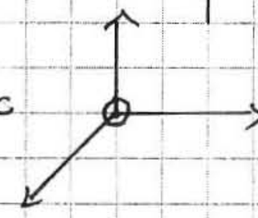
ex: $\mathbb{R}^2 - \ell$ 

disconnet space and each portion has $\pi_1 \sim 1$



but if we remove co-dimension 2 subspace $\to$

$\pi_1 \simeq \mathbb{Z}$



what if we remove codimension 3, 4, ... etc

$\pi_1$ is trivial again.

(3) $\therefore$ Study removal of co-dim 2 subsp. of $\mathbb{R}^n$

Definitions: - $G \subseteq W$ is a parabolic subgroup of $W \iff \exists\, T \subseteq S$ and $\omega \in W$ s.t. $G = \langle \omega T \omega^{-1} \rangle$

- $\mathcal{L}(H_W)$ : lattice of all intersections of hyp. $H_\alpha$, $\alpha \in \Phi_W$ ordered by reverse inclusion.

- $\mathcal{L}(W)$ : lattice of all parabolic subgroups of $W$

**example:** $\mathcal{L}(H_{A_{n-1}}) \sim$ partition lattice $(12/3578/46) = H_{12} \cap H_{46} \cap *$

$* = H_{35} \cap H_{57} \cap H_{78}$

$\mathcal{L}(W) \to \langle s_{35}, s_{57}, s_{78} \rangle = \omega \langle s_{12}, s_{23}, s_{34} \rangle \omega^{-1}$

$\boxed{1999}$ B- Ihrig : $\mathcal{L}(H_W) \simeq \mathcal{L}(W)$ via map let $X = \cap H_\alpha$ for some $\alpha \in \Phi$

$f^{-1}(G) =$

$f(X) = Gal(X) = \{ w \in W \,/\, wx = x \ \forall x \in X \} \longrightarrow Fix(G) = \{ x \in \mathbb{R}^n \,/\, gx = x \ \forall g \in G \}$

parabolic sub. of $W$ — is an intersection of hyp.

Crucial: for any intersection of hyperplanes, $\exists$ a parabolic subgroup

ex: $X = H_{12} \cap H_{23} = \{ (x, x, x, y, z, \dots) \}$

4 $Gal(X) = \langle s_{12}, s_{23} \rangle$

Definition :- let $W$ be a f.r.g. of rank $n$. $\mathcal{P}_{n,k}(W)$: collection of all irreducible parab. sub. of $W$ of rank $\underline{k-1}$

ex: $k=3$ $W = A_{n-1}$

↳ all irreducible parab. of rank 2

- $\mathcal{W}_{n,k} = \{ Fix(G) \,/\, G \in \mathcal{P}_{n,k}(W) \}$ = k-parabolic subspace arran.

ex: $k=3$ $W = A_{n-1}$ 3-equal arrangement

5

Theorem (B, Servers, White) (Davis, Jan. Scott)

Let $\mathbf{P}$ be a collection of rank 2 parabolic subgroups of $\mathbf{W}$ a f. refl. gr with simple system $\Delta$, that is closed under conjugation and let $\mathcal{W} = \{ Fix(G) \mid G \in \mathbf{P} \}$.

Let $M(\mathcal{W}) = \mathbb{R}^n - \mathcal{W}$. Define a Coxeter group $\mathbf{W}'$ as follows

$\mathbf{W}'$ is generated by a set $S$ $(|S| = |\Delta|)$ subject only to relation $m'(s,t) = \begin{cases} \infty & \text{if } \langle s,t \rangle \in \mathbf{P} \\ m(s,t) & \text{otherwise} \end{cases}$

Let $\varphi' : \mathbf{W}' \longrightarrow \mathbf{W}$ $\quad \varphi'(s) = s \quad \forall s \in S$

$\Rightarrow M(\mathcal{W})$ is $\underset{(D.J.S.)}{K(\pi,1)}$ and $\pi_1(M(\mathcal{W})) \cong \operatorname{Ker} \varphi'$

**example:** 3-equal arrangement: $A'_{n-1} = \text{"}S_n - (3)\text{"}$ -> triplet gr. of Khovanov

$\operatorname{Ker} \varphi' \cong$ pure triplet group

"3-equal" arrangement of type $\mathbf{W}$, $\mathbf{W}' = \mathbf{W} - (3), (4), (5)$
parabolic

⑥ Proof: a) Björner & Ziegler: if we have a simplicial decomposition $\Delta$ of the $k$-sphere and $\Delta_0$ is a subcomplex $\Rightarrow \exists$ a regular CW complex $X$ with same homotopy type as $|\Delta| \setminus |\Delta_0|$

(note: if $P$ is the face poset of $\Delta$ and $P_0$ lower oder ideal generated by $\Delta_0$ $\Rightarrow$ face poset of $X$ is dual poset of $(P \setminus P_0)$)

take the $S^{n-1}$ sphere and intersect it with $H_W$ hyp. arr. of type $W$ yields a simplicial decomposition $\Delta$ of $S^{n-1}$. Let $\Delta_0$ be the subcomplex coresp to 2-parabolic subspace arrangement $W$ i.e: subspaces fixed by rank 2-parab. subg in $P$

$$\Rightarrow \pi_1(|\Delta| \setminus |\Delta_0|) \simeq \pi_1(X)$$

what is $X$: a subcomplex of the $W$-permutahedron

$W$-permutahedron: zonotope obtained by taking the Minkowski sum of unit vector $\perp$ to $H_\alpha$

Description of $W$ permutahedron: vertices $w \in W$ $\forall w \in W$

edges $\underset{w}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{ws}{\circ}$ : edge between 2 elt of $W$ $\iff w' = ws$ for some $s \in S$

2-faces: bounded by cycles that correspond to a product of simple reflections $= e$ in $W$

4-cycle: $stst = 1$ $\rightarrow s, t$ commute
6-cycle: $(st)^3 = 1$ $\rightarrow m(s,t) = 3$
8-cycle: $(st)^4 = 1$ $\rightarrow m(s,t) = 4$ ....

$\mathbf{X}$ : is obtained from $\mathbf{W}$- permutahedron by removing the faces corresponding to $\Delta_0$

**example** : $\mathbf{W} = A_{n-1}$ 3-parabolic arr. (equal) remove the subspaces $(x, x, x, y)$

$$Gal((x,x,x,y)) = \langle (12), (23) \rangle \to \underline{6}\text{- cycles}$$

b) $\pi_1(\mathbf{X}) \simeq \pi_1(\text{2-skeleton of } \mathbf{X})$ ex: permutahedron - faces bounded by 6-cycles. So 6-cycles are holes now.

c) $\pi_1(\text{2-skeleton of } \mathbf{X}) \simeq \pi_1(\text{1-skeleton})/N \to$ where $N$ = normal subgroup generated by 4-cycles

$\simeq$ bouquet of circles $/ N$ (filled 4-cycle)

d) Let $\mathbf{W}^\infty = \mathbf{W} - (2) - (3) - \cdots$ i.e: keep only involutions (opposite of $A + Hn$)

$$\varphi : \mathbf{W}^\infty \longrightarrow \mathbf{W} \qquad \varphi(s_i) = s_i$$

$\operatorname{Ker}\varphi$ = "all cycles" = $\pi_1(\text{1-skeleton of } \mathbf{W}\text{-permutahedron}) =$

e) $\operatorname{Ker}\varphi/_N \simeq \operatorname{Ker}\varphi'$ $\quad \varphi' : \mathbf{W}^\infty + (2) = \mathbf{W}' \longrightarrow \mathbf{W}$