Linearly dependent powers of quadratic forms
Preliminary report: 1999-2011

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11w5011 – Diophantine methods, lattices, and arithmetic theory of quadratic forms

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1. Backstory

In 1913, Ramanujan posed to the *Journal of the Indian Mathematical Society* the following question: "Shew that

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

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The next year, S. Narayanan gave the more general expression

$$(\ell x^2 - nxy + ny^2)^3 = (px^2 + mxy - my^2)^3 + (nx^2 - nxy + \ell y^2)^3 + (mx^2 - mxy - py^2)^3,$$

where

$$\ell = \lambda(\lambda^3 + 1), \quad m = 2\lambda^3 - 1, \quad n = \lambda(\lambda^3 - 2), \quad p = \lambda^3 + 1.$$

(Set $\lambda = 2$ and divide by 3 to get Ramanujan's formula.)

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First we have

$$(4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3$$
$$= (6x^2 - 4xy + 4y^2)^3 - (3x^2 + 5xy - 5y^2)^3$$

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\end{aligned}
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Furthermore, this second set of identities can be derived from the first by making the unimodular linear change of variables:

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Alas, the third transposition does not have a third representation.

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It turns out that these properties (of a third representation, and the equivalence under linear change), are not specific to Ramanujan's example. One can also write down equivalent versions for the Narayanan formulas. More to the point, up to changes of variable, these *completely* describe the solution in complex quadratic forms to $q_1^3 + q_2^3 + q_3^3 + q_4^3 = 0$, although I'll give a more symmetric formulation later.

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In 1996, C. Sándor completely solved the problem of equal sums of two cubes of quadratic forms over $\mathbb{C}$, in the sense that he gives all sets, with parameters satisfying a side-condition. He didn't present the three-fold sum of two cubes.

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- "ALL" is short for "annoying little lemma", a semi-routine bit of business which will not be proved today.

2. Introductory material – the linear case

Suppose $\{\ell_i(x,y) = \alpha_i x + \beta_i y : 1 \le i \le m\}$ is an honest set of linear forms. If $m \ge d + 2$, then $\{\ell_i^d\}$ must be a linearly dependent set, since the vector space of binary forms of degree d has basis $\{\binom{d}{j} x^{d-j} y^j : 0 \le j \le d\}$ and so has dimension $d + 1$.

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What happens if you have an honest set of quadratic forms $\{q_i : 1 \le i \le m\}$? Since $\deg q_i^d = 2d$, we again find that if $m \ge 2d+2$, then $\{q_i^d\}$ must be dependent. If $m = 2d+1$, then it is true that a *general* set $\{q_i^d\}$ is linearly independent ....

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...but there are exceptions with quadratic forms. For example, any set

$$\{(\alpha_j x^2 + \beta_j y^2)^d : 1 \le j \le d+2\}$$

must be dependent, because each element lives in the $(d+1)$-dimensional subspace $< x^{2d-2k} y^{2k} >$. (The same is true if $q_j = \alpha_j F + \beta_j G$ for any two distinct quadratic forms F, G.)

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By the two-square identity, for any linear forms ℓ_j,
$(\ell_1\ell_2 + \ell_3\ell_4)^2 + (\ell_1\ell_3 - \ell_2\ell_4)^2 = (\ell_1\ell_2 - \ell_3\ell_4)^2 + (\ell_1\ell_3 + \ell_2\ell_4)^2.$

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Not to mention the action of the orthogonal group:

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Key here is that $xy \notin < F, G >$ for $\{F, G\} = \{x^2, y^2\}$ but $(xy)^2 \in < F^2, FG, G^2 >$.

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4. Known results about $\Phi(d)$

The old results in this section have been phrased in the new notation $\mathcal{W}(r,d)$ and $\Phi(d)$.

$\boxed{\Phi(2) = 3.}$ All $\mathcal{W}(3,2)$ sets come from $\{x^2 - y^2, xy, x^2 + y^2\}$ after permutation, linear changes and scaling. The proof is similar to the one for Pythagorean triples.

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And the case of even quadratic forms implies $\boxed{\Phi(d) \leq d + 2.}$

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In view of Liouville, $\boxed{\Phi(3) = \Phi(4) = \Phi(5) = 4}$ follows from

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I'm not sure who proved the cubic one first. The quartic is a simple application of a central technique we'll talk about more later (and also goes back in some sense to Diophantus), and the quintic was found independently by Adolphe Desboves in 1880 (it's in *Dickson*) and by Noam Elkies in 1996. Noam told me he found it by replacing $\sqrt{-2}$ with a parameter and solving; there are actually several "natural" ways to derive it.

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A deep theorem of Mark Green from 1975 states that if $\{\phi_j\}$, $1 \le j \le r$, is an honest set of holomorphic functions in n complex variables and

$$\sum_{j=1}^{r} \phi_j^d = 0,$$

then $d \le (r-1)^2 - 1$. This implies that $1 + \sqrt{d+1} \le r$, and so $\boxed{\lceil\, 1 + \sqrt{d+1}\,\rceil \le \Phi(d).}$ This implies Liouville's result for $d \ge 4$. Green's approach does not lend itself to the construction of quadratic form examples.

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All known $\mathcal{W}(\Phi(d), d)$'s seem to be be very symmetric collections of quadratic forms. It's unclear whether these symmetries are inherent, a "Strong Law of Small Numbers" phenomenon, or artifacts of the techniques used. Extremal sets are often symmetric, though as we've seen this week, not necessarily as symmetric as we'd like.

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More surprisingly, for every known $\mathcal{W}(\Phi(d), d)$, there is a linear change after which $|\alpha_j| = |\gamma_j|$ in *every*

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These observations suggest a new look at an old idea of Felix Klein, introduced in his book *The Icosahedron*.

Associate to each non-zero linear form $\ell(x,y) = sx - ty$ the image of $t/s \in \mathbb{C}^*$ in the unit sphere S^2 under the Riemann map and vice-versa. (Assign $\ell(x,y) = y$ to ∞ to $(0,0,1)$. A concrete implementation of the Riemann map is:

$$p + iq \mapsto \left(\tfrac{2p}{p^2+q^2+1}, \tfrac{2q}{p^2+q^2+1}, \tfrac{p^2+q^2-1}{p^2+q^2+1}\right)$$

$$(u,v,w) \mapsto \tfrac{u+iv}{1-w}.$$

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Since $\ell(ax+by, cx+dy) = (sa-tc)x + (sb-td)y$, note that $t/s \mapsto T(t/s)$, where $T(z) = \frac{dz-b}{a-cz}$ is a Möbius transformation.

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It can be routinely checked that if $(u, v, w) \mapsto z = re^{i\theta}$, then $(-u, -v, -w) \mapsto -1/\bar{z} = -r^{-1}e^{i\theta}$. The quadratic which is the product of linear forms associated with such an antipodal pair is

$$(x - re^{i\theta}y)(x + r^{-1}e^{i\theta}y) = x^2 + \tfrac{1-r^2}{r}e^{i\theta}xy - e^{2i\theta}y^2.$$

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It follows that $x^2 + Axy + By^2$ comes from an antipodal pair if and only if $|B| = 1$ and $-A^2/B$ is a non-negative real. For example, if $B = 1$, then A has to be purely imaginary. This will happens a lot.

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If the quadratic $q(x,y)$ corresponds to points (w_1, w_2), then $p(x, e^{i\theta}y)$ corresponds to (w_1, w_2) rotated along a parallel of latitude by θ.

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Klein's original motivation was that a highly regular set of points on S^2, such as the vertices of a Platonic solid, will be invariant under a large number of rotations, hence the product of the linear forms associated to the vertices will be invariant (up to multiple) under many linear changes.

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We have repeatedly found that highly symmetric figures created by looking at the $\{q_j\}$'s in $\mathcal{W}(\Phi(d), d)$, in terms of the corresponding pairs of points on S^2.

For example, the quadratic forms from the Pythagorean parameterization $\{x^2 - y^2, x^2 + y^2, xy\}$ come from the antipodal pairs of the vertices of an octahedron:

$$(\pm 1, 0, 0) \mapsto \pm 1 \mapsto x \mp y, \quad (0, \pm 1, 0) \mapsto \pm i \mapsto x \mp iy,$$
$$(0, 0, 1) \mapsto \infty \mapsto y, \quad (0, 0, -1) \mapsto 0 \mapsto x.$$

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The antipodal pairs of the vertices of the cube $\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$ correspond to the Desboves-Elkies form: $\sum q_j^5 = 0$ and note that $\prod_j q_j = x^8 + 14x^4y^4 + y^8$ (up to multiple).

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The 10 quadratic forms corresponding to the dodecahedron only give $\sum q_j^{14} = 0$; no higher exponent.
A sum of the form

$$\sum_{k=0}^{m-1} (\zeta_m^k x^2 + axy + \zeta_m^{-k} y^2)^d = c(xy)^d$$

corresponds to two horizontal regular m-gons equally spaced with respect to the equator, plus the north and south poles.

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2. It is not hard to show that if (q_1, q_2) are distinct and relatively prime, then there is a linear change which simultaneously diagonalizes them. Thus, wlog we may assume that $q_j(x, y) = \alpha_j x^2 + \beta_j y^2$, $j = 1, 2$, so p is even!

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3. Another ALL implies that neither q_3 nor q_4 is even.
4. We now back up and study the cases in which

$$(a_3 x^2 + b_3 xy + c_3 y^2)^d + (a_4 x^2 + b_4 xy + c_4 y^2)^d$$

can be an even polynomial for $d \geq 3$.

7. The strategy for $\mathcal{W}(4,d)$ – the rabbit hole

5. There are three "obvious cases": $b_1 = b_2 = 0$,

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6. There are exceptional solutions for $d = 3, 4, 5$. For example, the family for $d = 3$ is (after scaling x, y):

$$\left(x^2 - \alpha\beta xy + y^2\right)^3 + \alpha\left(x^2 + \beta xy - y^2\right)^3$$
$$\alpha \neq \pm 1, \qquad \beta^2(1-\alpha^2) = 12.$$

Without the constraint on β, the coefficients of x^5y and xy^5 vanish; the condition comes from requiring the same for x^3y^3. In an exceptional solution, $y \mapsto -y$ gives a different solution.

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7. Once we know all the cases in which p, a sum of two d-th powers of quadratic forms, has the shape $h(x^2, y^2)$ for a form h of degree d, we use an 1851 algorithm of Sylvester to find the minimal number of linear forms ℓ_i so that $h(x, y) = \sum \ell_i(x, y)^d$ and so $p(x, y) = \sum \ell_i^d(x^2, y^2)$.

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9. The implementation of this strategy, which currently takes about 20 pages to work out completely, simultaneously establishes the uniqueness description of solutions for $\mathcal{W}(4,3)$, $\mathcal{W}(4,4)$, $\mathcal{W}(4,5)$ and the non-existence of $\mathcal{W}(4,d)$ for $d \geq 6$.

8. Characterization of $\mathcal{W}(4,3)$

Theorem: Every $\mathcal{W}(4,3)$ set is derived from the first two lines of

$$(\alpha x^2 - xy + \alpha y^2)^3 + \alpha(-x^2 + \alpha xy - y^2)^3 =$$
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This identity can be easily verified! Let $F = x^2 + y^2$ and $G = xy$. Then $F^3 - 3FG^2 = (x^2 + y^2)^3 - 3(x^2 + y^2)x^2y^2$ $= x^6 + y^6$, and

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$$= (\alpha^2 - 1)(\alpha(x^6 + y^6) + (\alpha^2 + 1)x^3y^3).$$

8. Characterization of $\mathcal{W}(4,3)$

The three-fold symmetry is a consequence of the sum of the cubes being a quadratic in $\{x^3, y^3\}$. Under the linear change $(x,y) \mapsto (x+iy, x-iy)$, we get a formally real version, in which the symmetry is obscured: $q_1^3 + \alpha q_2^3 = q_3^3 + \alpha q_4^3 = q_5^3 + \alpha q_6^3$, where

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Although this last expression is hard to read, notice that it's almost what Ramanujan and Narayanan were looking at. First take $y \mapsto \sqrt{3}y$, so that $\sqrt{1}2xy \mapsto 6xy$ and $y^2 \mapsto 3y^2$. Now let $\alpha = \lambda^3$, so that $q_{2j-1}^3 + \alpha q_{2j}^3 = q_{2j-1}^3 + (\lambda q_{2j})^3$. Narayanan's formula arises by taking $x \mapsto 2x - y$ and dividing by 4.

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Sándor used a different approach to this problem. He showed that all $\mathcal{W}(4,3)$ sets are given up to linear change by:

$$
\begin{aligned}
q_1 &= w_2(w_1 - w_3)x^2 + (w_1^2 - w_3^2)xy + w_4(w_4 - w_2)y^2 \\
q_2 &= -w_3(w_1 - w_3)x^2 + (w_2^2 - w_4^2)xy - w_1(w_4 - w_2)y^2 \\
q_3 &= w_4(w_1 - w_3)x^2 + (w_1^2 - w_3^2)xy + w_2(w_4 - w_2)y^2 \\
q_4 &= -w_1(w_1 - w_3)x^2 + (w_2^2 - w_4^2)xy - w_3(w_4 - w_2)y^2
\end{aligned}
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\text{where} \qquad w_1^3 + w_2^3 = w_3^3 + w_4^3, \qquad w_i \in \mathbb{C}.
$$

8. Characterization of $\mathcal{W}(4,3)$ – the rabbit hole

One other complication in figuring this out is that there is a peculiar symmetry. If we apply the unimodular transformation:

$$(x,y) \mapsto \left(\frac{x+\omega\alpha y}{\sqrt{\alpha^2-1}}, \frac{-\omega^2\alpha x - y}{\sqrt{\alpha^2-1}}\right),$$

to $q_1^3 + \alpha q_2^3 = q_3^3 + \alpha q_4^3 = q_5^3 + \alpha q_6^3$, then it turns out that $(q_1,q_2,q_3,q_4) \mapsto (q_3,-q_2,q_1,-q_4)$, so

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to $q_1^3 + \alpha q_2^3 = q_3^3 + \alpha q_4^3 = q_5^3 + \alpha q_6^3$, then it turns out that $(q_1, q_2, q_3, q_4) \mapsto (q_3, -q_2, q_1, -q_4)$, so

$$q_1^3 + \alpha q_2^3 = q_3^3 + \alpha q_4^3 \quad \mapsto \quad q_3^3 - \alpha q_2^3 = q_1^3 - \alpha q_4^3$$

And (q_5, q_6) gain the denominators we saw earlier, going to

$$\frac{1}{\alpha^2 - 1}\left(\omega^2\alpha(2+\alpha^2)x^2 + (1+5\alpha^2)xy + \omega\alpha(2+\alpha^2)y^2\right);$$

$$-\frac{1}{\alpha^2 - 1}\left(\omega^2\alpha(1+2\alpha^2)x^2 + \alpha(5+\alpha^2)xy + \omega\alpha(1+2\alpha^2)y^2\right).$$

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Put another way, suppose $\{q_1, q_2, q_3, q_4\} \in \mathcal{W}(4,3)$. Then up to a permutation,

- There exists $\{q_5, q_6\}$ so that $\{q_1, q_2, q_5, q_6\} \in \mathcal{W}(4,3)$ and $\{q_3, q_4, q_5, q_6\} \in \mathcal{W}(4,3)$.

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There is a lot of combinatorics here yet to explore.

8. Characterization of $\mathcal{W}(4,3)$ – the rabbit hole

So, suppose you are given four quadratics f_1, f_2, f_3, f_4 which satisfy $f_1^3 + f_2^3 + f_3^3 + f_4^3 = 0$, how do you determine which one of the one-parameter family does it come from, how do you find the linear change, and how do you find α? It can be done; here's the start of how you do it. Recall:

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$$q_1 = \alpha x^2 - xy + \alpha y^2, \quad q_2 = \alpha^{1/3}(-x^2 + \alpha xy - y^2),$$
$$q_3 = \omega\alpha x^2 - xy + \omega^2\alpha y^2, \quad q_4 = \alpha^{1/3}(-\omega x^2 + \alpha xy - \omega^2 y^2).$$

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There is a linear change after which the f_i's become $c_j q_j$ for some c_j's. Observe that $< f_1, f_2 >$ is a two-dimensional subspace as is $< f_3, f_4 >$ and that the intersection of these two subspaces is $< xy >$. The corresponding intersections of the other pairs of subspaces turn out to be $< (x - \omega y)(x + \omega y) >$ and $< (ax + \omega y)(x + \omega ay) >$. Now compute the same intersections for the q_i's, and try to match up factors for the linear change.

8. Characterization of $\mathcal{W}(4,3)$ – the rabbit hole

The analysis is aided by another elementary result:

Theorem

If p is a form, then there exist $f, g \in \mathbb{C}[x,y]$ such that $p = f^3 + g^3$ if and only if p is a cube, or $p = q_1 q_2 q_3$, where q_i's are distinct, but linearly dependent.

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Suppose $p = q_1^3 + q_2^3 = q_3^3 + q_4^3$ is a sum of two cubes of quadratics in more than one way. After a linear change, q_1, q_2 and p are even. Using a bunch of ALL's we can assume that

$$p(x,y) = (x^2 - r^2y^2)(x^2 - s^2y^2)(x^2 - t^2y^2)$$

where $rst \neq 0$ and $\pm r, \pm s, \pm t$ are distinct.

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More ALL's imply that each linearly dependent factorization corresponds to **one** representation of p as a sum of two cubes.

8. Characterization of $\mathcal{W}(4,3)$ – the rabbit hole

The set $\{x^2 - r^2y^2, x^2 - s^2y^2, x^2 - t^2y^2\}$ corresponds to $p = q_1^3 + q_2^3$. There are 15 ways to partition the six linear factors of p into three pairs, and we test them for linear dependence.

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$$\{x^2 - r^2y^2, (x - sy)(x + ty), (x + sy)(x - ty)\}$$

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Similarly, we have to look at dependence in sets like

$$\{(x - ry)(x + sy), (x - sy)(x + ty), (x - ty)(x + ry)\}$$

I'll skip the details. A exhaustive (exhausting?) analysis shows that everything is a linear change from the one-parameter family described earlier.

8. Characterization of $\mathcal{W}(4,3)$ – extras

There are two cases where the solutions coalesce:

$$(x^2+xy-y^2)^3+(x^2-xy-y^2)^3=2x^6-2y^6$$

is, as it stands, a sum of two cubes in two ways, and two others come from taking $y\mapsto \omega y, \omega^2 y$. (Take $\alpha=\pm i$ and $y\mapsto iy$.)

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Let $h(x,y) = xy(x^4+y^4)$ (an octahedron!) The representations are (with $\eta = \zeta_{24} = \frac{\sqrt{6}+\sqrt{2}}{4} + i \cdot \frac{\sqrt{6}-\sqrt{2}}{4}$):

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All $\mathcal{W}(4,4)$'s come from two identities: The first is

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After $(x,y) \mapsto (x+iy, x-iy)$, this becomes

$$\begin{gathered}(2x^2-2y^2)^4 + (x^2 - 2\sqrt{3}xy - y^2)^4 + (x^2 + 2\sqrt{3}xy - y^2)^4 \\ = 18(x^2+y^2)^4.\end{gathered}$$

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After $(x,y)\mapsto(x+iy,x-iy)$, this becomes

$$(2x^2-2y^2)^4+(x^2-2\sqrt{3}xy-y^2)^4+(x^2+2\sqrt{3}xy-y^2)^4$$
$$= 18(x^2+y^2)^4.$$

Setting $y\mapsto\sqrt{3}y$ makes the coefficients integral.

9. Characterization of $\mathcal{W}(4,4)$

All $\mathcal{W}(4,4)$'s come from two identities: The first is

$$(x^2+y^2)^4 + (\omega x^2 + \omega^2 y^2)^4 + (\omega^2 x^2 + \omega y^2)^4 = 18(xy)^4.$$

After $(x,y) \mapsto (x+iy, x-iy)$, this becomes

$$(2x^2-2y^2)^4 + (x^2 - 2\sqrt{3}xy - y^2)^4 + (x^2 + 2\sqrt{3}xy - y^2)^4 \\ = 18(x^2+y^2)^4.$$

Setting $y \mapsto \sqrt{3}y$ makes the coefficients integral.
Diophantus observed that

$$u^4 + v^4 + (u+v)^4 = 2(u^2+uv+v^2)^2,$$

so any quadratic substitution making u^2+uv+v^2 a square gives a $\mathcal{W}(4,4)$. If $u = x^2+y^2$ and $v = \omega x^2 + \omega^2 y^2$, then $u+v = -(\omega^2 x^2 + \omega y^2)$ and $u^2+uv+v^2 = 3x^2y^2$.

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The other identity for fourth powers is three-fold

$$
\begin{aligned}
&\left(8\sqrt{3}\right)xy(x^6 - y^6) = \\
&(x^2 + \sqrt{3}xy - y^2)^4 - (x^2 - \sqrt{3}xy - y^2)^4 \\
&= (\omega^2 x^2 + \sqrt{3}xy - \omega y^2)^4 - (\omega^2 x^2 - \sqrt{3}xy - \omega y^2)^4 \\
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$$= (\omega x^2 + \sqrt{3}xy - \omega^2 y^2)^4 - (\omega x^2 - \sqrt{3}xy - \omega^2 y^2)^4.$$

(Note that the sum is invariant under $(x, y) \mapsto (\omega x, \omega^2 y)$, giving the other sums.) If you take a pair of the identities and flip the summands above, sometimes you get another image of the original, under a linear change, and sometimes you get

$$18x^8 - 28x^4y^4 + 18y^8$$
$$= (\sqrt{3}\, x^2 + \sqrt{2}\, xy - \sqrt{3}\, y^2)^4 + (\sqrt{3}\, x^2 - \sqrt{2}\, xy - \sqrt{3}\, y^2)^4$$
$$= (\sqrt{3}\, x^2 + i\sqrt{2}\, xy + \sqrt{3}\, y^2)^4 + (\sqrt{3}\, x^2 - i\sqrt{2}\, xy + \sqrt{3}\, y^2)^4,$$

which has no third pair.

10. Characterization of $\mathcal{W}(4,5)$

The only $\mathcal{W}(4,5)$ comes from Desboves-Elkies. Let

$$q_k(x,y) = i^k x^2 + i^{2k}\sqrt{-2}\; xy + i^{3k} y^2.$$

Then $\sum_{k=1}^4 q_k^5(x,y) = 0$, but also, by the interplay of the roots of unity,

$$\sum_{i=1}^4 q_i = \sum_{i=1}^4 q_i^2 = 0.$$

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The q_k's can be derived from these by making the substitution $q_4 = -(q_1+q_2+q_3)$ and solving $q_1^2+q_2^2+q_3^2+(q_1+q_2+q_3)^2 = 0$ in the usual Pythagorean way. But wait a minute!

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The equations $\sum_{i=1}^4 X_i = \sum_{i=1}^4 X_i^2 = 0$ define the intersection of a plane and a sphere in $\mathbb{C}^4$. This is, projectively, a curve. Unless something special is going on, this curve shouldn't contain another curve (q_1,q_2,q_3,q_4).

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More generally,

Theorem

If m cannot be written as $a(n-1)+bn, 0 \le a,b \in \mathbb{Z}$, then any symmetric form in n variables of degree m, is contained in the ideal generated by $\{\sum_{i=1}^n x_i, \dots, \sum_{i=1}^n x_i^{n-2}\}$. In particular, this is true for $m = n^2-3n+1$.

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The proof combines the Frobenius problem with Newton's theorem on symmetric forms. Unfortunately, for $n \ge 5$, the intersection $\cap_{r=1}^{n-2} \sum_{i=1}^n x_i^r$ has positive genus and so has no polynomial parameterization.

11. That synching feeling

Most of the examples in this talk involving higher degrees come from the orthogonality properties of sums of roots of unity. One simple application is:

Theorem

$$\sum_{j=0}^{k} (\zeta_{2k+2}^{-j} x^2 + \zeta_{2k+2}^{j} y^2)^{2k}$$
$$= (k+1)\binom{2k}{k} x^{2k} y^{2k} = (k+1)\binom{2k}{k} (xy)^{2k}$$

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This implies that $\Phi(2k) \le k+2$.
If $2k = 2$, $\zeta_{2k+2} = i$ and $(x^2+y^2)^2 + (ix^2 - iy^2)^2 = 2\binom{2}{1}(xy)^2$; if $2k = 4$, this is the Diophantus quartic example, in its ω-form.

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We can take $(x, y) \mapsto (x + iy, x - iy)$ and let $\theta_k = \frac{\pi}{k+1}$ to get a version with real coefficients:

$$\sum_{j=0}^{k} (2\cos(j\theta_k)(x^2 - y^2) - 4\sin(j\theta_k)xy)^{2k}$$
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Taking $k = 5$ and making a further linear change gives

$$(x^2 - 4xy + y^2)^{10} + 3^5(x^2 - y^2)^{10} + 3^5(2xy - y^2)^{10} + 3^5(2xy - x^2)^{10} + (-2x^2 + 2xy + y^2)^{10} + (x^2 + 2xy - 2y^2)^{10} = 1512(x^2 - xy + y^2)^{10}.$$

11. That synching feeling

More generally, for a parameter a,

$$\sum_{j=0}^{m-1} \zeta_m^{-rj}(x^2 + a\zeta_m^j xy + \zeta_m^{2j} y^2)^d$$

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For example,

$$\sum_{k=0}^{2}(\omega^{-k}x^2 + axy + \omega^k y^2)^2 = 3(a^2+2)x^2y^2,$$

Set $a = \sqrt{-2}$; the Klein polytope of this version of the Pythagorean formula is an octahedron resting on a face.

11. That synching feeling

What Elkies did for quintics was to observe that

$$\sum_{k=0}^{3}(i^k x^2 + i^{2k} axy + i^{3k} y^2)^5 = 40a(a^2+2)(x^7y^3 + x^3y^7),$$

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Alternatively, he might have observed that

$$\sum_{k=0}^{2} (\omega^{-k} x^2 + axy + \omega^k y^2)^5 =$$

$$(15+30a^2)(x^8y^2 + x^2y^8) + 3a(30+2a^2a^4)x^5y^5$$

$$\implies \sum_{k=0}^{2} (\omega^{-k} x^2 + \tfrac{i}{\sqrt{2}} xy + \omega^k y^2)^5 = \left(\tfrac{3i}{\sqrt{2}} xy\right)^5.$$

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The Klein polytope rotates from a cube from its xyz orientation to one in which vertices are at the north and south poles.

11. That synching feeling

One can do this for higher degrees, at the cost of either more terms or more complicated equations for a. For example,

$$\sum_{k=0}^{3}(i^{-k}x^2 + axy + i^k y^2)^6 = 12(2+5a^2)(x^{10}y^2 + x^2y^{10}) + p(a)x^6y^6.$$

$$\implies$$

$$\sum_{k=0}^{3}\left(i^{-k}x^2 + \sqrt{-\tfrac{2}{5}}xy + i^k y^2\right)^6 = -\frac{5632}{125}x^6y^6 = 11\cdot\left(\sqrt{\tfrac{-8}{5}}xy\right)^6,$$

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showing that $\Phi(6) = 5$. Three other $\mathcal{W}(5,6)$'s have the shape

$$(x^2 + cxy + y^2)^6 + (x^2 - cxy + y^2)^6 = \sum_{k=1}^{3}(\alpha_k x^2 + \beta_k y^2)^6,$$

where c^2 is a root of $t^3 + 80t^2 + 1360 + 4480$; c is purely imaginary. There may be other $\mathcal{W}(5,6)$'s as well.

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Similarly, but more uglily,

$$\sum_{k=0}^{3} \left(i^{-k}x^2 + \sqrt{-\tfrac{6}{5}}xy + i^k y^2 \right)^7 = -\frac{2^{23/2}3^{1/2} \cdot 13}{5^{7/2}} i(xy)^7.$$

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which comes from zapping the coefficients of $x^{11}y^3$ and x^3y^{11}. More generally, if $d=2k+1$, then

$$\sum_{j=0}^{k}\left(\zeta_{k+1}^{j}x^2+axy+\zeta_{k+1}^{-j}y^2\right)^{2k+1}=$$

$$f(a)(x^{3k+2}y^k+x^ky^{3k+2})+g(a)x^{2k+1}y^{2k+1}.$$

Choose $a\neq 0$ so that $f(a)=0$ (possible when $k\geq 2$ since $\deg f=k+1$), and it follows that $\Phi(2k+1)\leq k+2$.

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Finally and miraculously,

$$\sum_{k=0}^{4}(\zeta_5^k x^2 + a\ x\ y + \zeta_5^{-k} y^2)^{14} =$$

$$f(a)(x^{24}y^4 + x^4y^{24}) + g(a)(x^{19}y^9 + x^9y^{19}) + h(a)x^{14}y^{14},$$

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$$\text{where} \quad f(a) = 455(1+a^2)(1+11a^2) \quad \text{and}$$

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Take $a = i$. Let $q_k(x,y) = \zeta_5^k x^2 + i\ x\ y + \zeta_5^{-k} y^2$, $0 \le k \le 4$ and $q_5(x,y) = \sqrt{-5}\ x\ y$ (another miracle in the constant). Then

$$\sum_{j=0}^{5} q_j^{14}(x,y) = 0.$$

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I don't know *why* the 14th degree identity is true. Possible hint:

$$\sum_{j=0}^{5} q_j^{2k}(x,y) = 0 \quad \text{for} \quad k = 1,2,4,7$$

But *why* do the quartic q_j^2's lie on $\cap \sum_{i-1}^{6} X_i^k$ for $k = 1,2,4,7$?

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Open questions:

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- What can be said about $\mathcal{W}(r, d_1) \cap \mathcal{W}(r, d_2)$? The examples at $d = 5, 14$ suggest that the champions can fight in several different weight divisions.

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12. What's next?

- Given a family, is there an easy way to determine whether there is a linear change making it real, or rational?
- It is provable that no linear change makes the Desboves-Elkies example real, but it's not hard to give a $\mathcal{W}(5,5) \subset \mathbb{Z}[x,y]$. It may be sensible to define $\Phi_{\mathbb{R}}(d)$ and $\Phi_{\mathbb{Q}}(d)$.
- Euler gave a famous example of binary septics over $\mathbb{Q}$ which satisfy $f_1^4 + f_2^4 = f_3^4 + f_4^4$. What happens if you replace "quadratic forms" with "degree k forms"?
- Many algebraic geometers in the audience have been internally screaming during this talk that all I'm doing is looking at curves parameterized by quadratics which lie on the Fermat surface:

$$X_1^d + \cdots + X_r^d = 0$$

Granted. Aside from Green's theorem, how does this help?

13. Oh, look, I have some more time