Ricci flow on 4-manifolds and Seiberg-Witten equations

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Overview of the talk

- A brief review on normalized Ricci flow
    - Nonsingular solutions on compact manifolds

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  - The noncompact case
- Seiberg-Witten equations and Ricci flow

A brief review on normalized Ricci flow

A Ricci flow is a smooth family of metrics $g(t), t \in [0, T)$, on a manifold satisfying the evolution equation

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)), \tag{1}$$

where $Ric(g(t))$ denotes the Ricci curvature tensor of the metric $g(t)$.

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Ricci flow was introduced by Hamilton, and he also introduced the *volume normalized Ricci flow*

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) + \frac{2r(t)}{n} g(t), \tag{2}$$

where $r(t) = \int R(g(t)) dv_{g(t)} / \int dv_{g(t)}$ denotes the average scalar curvature of $g(t)$. The equation (2) differs from (1) by rescaling the space-time such that the volume preserves constant along the flow.

A brief review on normalized Ricci flow

A long term normalized Ricci flow $(M, g(t)), t \in [0, \infty)$, is called *nonsingular* if the curvature is uniformly bounded for all time.

A special class of nonsingular solutions is, *Ricci Solitons*. The notion of *Ricci solitons*, known as the self-similar solutions to the Ricci flow, was introduced by Hamilton in 1988. In many cases, Ricci solitons turn out to be singularity models to the Ricci flows. By definition, Ricci solitons may be considered as a natural generalizations of Einstein manifolds.

A brief review on normalized Ricci flow

The infinitesimal equation for the Ricci soliton reads

$$Ric + \frac{1}{2}\mathcal{L}_V g = \epsilon g \tag{3}$$

for some vector field V and constant $\epsilon \in \mathbb{R}$.

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If V is a gradient field of some function f, then the Ricci soliton is called *gradient Ricci soliton* with potential function f. Then the equation reads

$$Ric + \nabla^2(f) = \epsilon g. \tag{4}$$

The cases $\epsilon > 0, = 0, < 0$ correspond to shrinking, steady and expanding Ricci solitons respectively.

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The cases $\epsilon > 0, = 0, < 0$ correspond to shrinking, steady and expanding Ricci solitons respectively.

All Ricci solitons in the talk are assumed to be gradient. After a scaling of the metric, we assume $\epsilon \in \{\pm 1/2, 0\}$.

Nonsingular solutions on compact manifolds

In dimension 3, Hamilton proved in 1997:

Theorem (Hamilton, 1997)

Any nonsingular solution to the normalized Ricci flow on a closed 3-manifold satisfies one of the followings:

(1) *the solution collapses in the sense of Cheeger-Gromov;*

(2) *the solution converges subsequently to a metric of constant curvature on the manifold;*

(3) *the solution converges subsequently in the pointed Gromov-Hausdorff sense to a finite collection of complete noncompact hyperbolic pieces in the manifold; each piece is essential in the sense that the fundamental group is injective into the manifold.*

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The proof relies essentially on Hamilton and Ivey's sectional curvature pinching theorem for three dimensional Ricci flow.

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However, using Perelman's monotonicity for $\mathcal{W}$ functional, F. Q. Fang, Y.G. Zhang, Z.L. Zhang were able to prove the following generalization to higher dimensional Ricci flow:

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Theorem (Fang-Zhang-Zhang, 2006)

Any nonsingular solution to the normalized Ricci flow on a closed manifold satisfies one of the followings:

(1) *the solution collapses along a subsequence in the sense of Cheeger-Gromov;*

(2) *the solution converges subsequently to a Ricci soliton on the manifold;*

(3) *the solution converges subsequently in the pointed Gromov-Hausdorff sense to a finite collection of complete noncompact negative Einstein pieces in the manifold.*

Nonsingular solutions on compact manifolds

In the case (3), when the time is large enough, the manifold admits a thick-thin decomposition where the thick part converges to the negative Einstein pieces, while the thin part collapses. In four dimensional case, the thin part is further volume collapsed.

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In dimension four, using the Gauss-Bonnet-Chern formula and Hirzebruch signature theorem we may further prove that:

Theorem (Fang-Zhang-Zhang, 2006)

Let M be a closed four manifold admitting a nonsingular solution to the normalized Ricci flow, then one of the following three conditions hold:

(1) *M has a positive rank F-structure;*

(2) *M admits a shrinking Ricci soliton metric;*

(3) *M satisfies the Hitchin-Thorpe inequality $2\chi(M) \geq 3|\tau(M)|$.*

Sketch of the proof.

First of all, recall the evolution equation

$$\frac{\partial}{\partial t}R = \triangle R + 2|Ric^{\circ}|^2 + \frac{2}{4}R(R - r),$$

where r is the average of the scalar curature.

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where r is the average of the scalar curature.

We denote by $\breve{R}(g(t))$ the infimum of the scalar curvature. By the maximal principle, $\breve{R}(g(t))$ satisfies

$$\frac{d}{dt} \breve{R} \geq \frac{2}{4} \breve{R}(\breve{R} - r)$$

where r is the average of the scalar curvature R.

So $\breve{R}(g(t))$ increases whenever it is negative and remains nonnegative whenever $\breve{R}(g(t)) \geq 0$.

□

Continued Proof.

If $\check{R}(t) > 0$ for some time t, then $g(t)$ converges to a shrinking Ricci soliton by monotomicity of Perelman's ν functional.

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If $\check{R}(t)$ converges to zero as $t \to \infty$, then $g(t)$ converges to a Ricci flat metric.
If $\check{R}(t) < -c$ for some $c > 0$ independent of t, then

$$\frac{d}{dt}\check{R} \geq \frac{2}{4}\check{R}(\check{R} - r) \geq \frac{2c}{4}(r - \check{R})$$

implies

$$\int_0^\infty (r - \check{R})dt < \infty$$

Let us assume without loss of generality that $vol(M, g(t)) = 1$.

□

Continued Proof.

Hence

$$\int_M |R-r|dv \le \int_M (R-\check{R})dv + \int_M (r-\check{R})dv = 2\int_M (r-\check{R})dv = 2(r-\check{R}).$$

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Thus

$$\int_0^\infty \int_M |R-r|dvdt < \infty.$$

By the equation

$$\frac{\partial}{\partial t}R = \triangle R + 2|Ric^\circ|^2 + \frac{2}{4}R(R-r),$$

we obtain

□

Continued Proof.

$$\begin{aligned}
\int_0^\infty \int_M 2|Ric^\circ|^2 dvdt &= \int_0^\infty \int_M \frac{\partial}{\partial t} Rdvdt - \frac{1}{2}\int_0^\infty \int_M R(R-r)dvdt \\
&= \int_0^\infty \frac{\partial}{\partial t} rdt + \frac{1}{2}\int_0^\infty \int_M R(R-r)dvdt \\
&\leq \lim_{t\longrightarrow\infty} \sup |r(g(t)) - r_0| + \frac{C}{2}\int_0^\infty \int_M |R-r|dvdt \\
&\leq 2C + \frac{C}{2}\int_0^\infty \int_M |R-r|dvdt < \infty.
\end{aligned}$$

□

Continued Proof.

By the Chern-Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{R(g)^2}{24} + |W^+(g)|^2 + |W^-(g)|^2 - \frac{1}{2}|Ric^\circ(g)|^2\right)dv,$$

By the Hirzebruch signature theorem

$$\tau(M) = \frac{1}{12\pi^2}\int_M (|W^+(g)|^2 - |W^-(g)|^2)dv,$$

where $W^+(g)$ and $W^-(g)$ are the self-dual and anti-self-dual Weyl tensors respectively. □

Continued Proof.

Thus

$$\begin{aligned} & 2\chi(M) - 3|\tau(M)| \\ \geq \quad & \liminf_{m \longrightarrow \infty} \frac{1}{4\pi^2} \int_m^{m+1} \int_M \left(\frac{1}{24} R(g(t))^2 - \frac{1}{2} |Ric^\circ(g(t))|^2\right) dvdt \\ = \quad & \liminf_{m \longrightarrow \infty} \frac{1}{4\pi^2} \int_m^{m+1} \int_M \frac{1}{24} R(g(t))^2 dvdt \geq 0. \end{aligned}$$

□

Nonsingular solutions on compact manifolds

As a corollary, we have

Corollary (Fang-Zhang-Zhang, 2006 and 2008)

If a closed four manifold admits a nonsingular solution, then its Euler characteristic is nonnegative. The Euler characteristic vanishes iff the solution collapses along a subsequence of times.

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The Hitchin-Thorpe inequality to the nonsingular solutions can be sharpened as follows:

Theorem (Fang-Zhang-Zhang, 2008; Zhang-Zhang, 2009)

Let M be a closed four manifold with Yamabe invariant $\sigma(M) \leq 0$. If M admits an eternal solution to the normalized Ricci flow with bounded scalar curvature, then

$$2\chi(M) - 3|\tau(M)| \geq \frac{1}{96\pi^2}\sigma(M)^2. \tag{5}$$

Nonsingular solutions on compact manifolds

We conjecture that the inequality may be replaced by:

Conjecture

The above inequality may be replaced by the following Hitchin-Thorpe-Gromov-Kotschick type inequality

$$2\chi(M) - 3|\tau(M)| \geq \frac{1}{1296\pi^2}\|M\|$$

where $\|M\|$ is a simplicial volume of M.

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Recently, Ishida and his coauthors studied the relationship between the existence of nonsingular solutions and smooth structures on closed four manifolds by using Seiberg-Witten invariants. For example, Ishida proved

Theorem (Ishida, 2008)

For any ℓ, there exists a simply connected four manifold M_ℓ which has at least ℓ smooth structures admitting nonsingular solutions and infinitely many other smooth structures admitting no nonsingular solutions.

The noncompact case

By Shi's work in 1989, Ricci flow with bounded curvature exists even for noncompact manifolds. Assumed the volume is bounded, then the normalized Ricci flow exists on a noncompact manifold. Y.G. Zhang, Z.L. Zhang and the author then proved:

Theorem (Fang, Zhang and Zhang, 2008)

Let $g(t)$ be a complete nonsingular solution to the normalized Ricci flow of finite volume on a noncompact manifold, then either $g(t)$ collapses along a subsequence or $g(t)$ converges along a subsequence to a collection of complete negative Einstein manifold.

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The shrinking Ricci solitons or Ricci flat manifolds never happen.

In dimension four, assumed existence of nonsingular solutions, we get

Corollary (Fang-Zhang-Zhang, 2010)

If an open complete four manifold of finite volume admits a nonsingular solution to the normalized Ricci flow, then its Euler characteristic is nonnegative.

The noncompact case

X.Z. Dai and G.F. Wei proved a noncompact version of Hithin-Thorpe inequality for Einstein four manifolds in 2006. Using their result, Y. Zhang, Z. Zhang and the author proved the following Ricci flow version for Hitchin-Thorpe inequality on noncompact manifolds:

Theorem

Let (M, g) be a Riemannian four manifold which is asymptotic to a fibred cusp at infinity. If the normalized Ricci flow starting from g is nonsingular, then M satisfies

$$2\chi(M) > 3|\tau(M) + \frac{1}{2}\mathrm{a}\lim\eta(\partial M)|, \tag{6}$$

where $\mathrm{a}\lim(\partial M)$ denotes the adiabatic limit of η invariant of the boundary.

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(M, g) is *asymptotic to a fibred cusp* if M is diffeomorphic to the interior of a compact manifold $\overline{M}$ whose boundary is a fibration $F \longrightarrow \partial M \longrightarrow B$ and the metric $g \sim dr^2 + \pi^* g_B + e^{-2r} g_F$ at infinity.

Let M be a compact 4-dimensional oriented Riemannian manifold. Let $P_{SO}(M) \to M$ denote its framing bundle. A spin^c-structure c is a principal Spin^c(4)-bundle $P_{Spin^c}(M) \to M$ together with a bundle map $p : P_{Spin^c}(M) \to P_{SO}(M)$ which is the standard Lie group homomorphism $\mathrm{Spin}^c(4) = Spin(4) \times_{\mathbb{Z}_2} U(1) \to SO(4)$ on every fibers. It is a classical fact that every oriented 4-manifold admits a $Spin^c$-structure, and $Spin(4) = SU(2) \times SU(2)$.

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Let $\Delta^+ : Spin^c(4) \to U(2)$ (resp $\Delta^- : Spin^c(4) \to U(2)$) be the two natural irreducible complex 2-dimensional representation, by forgetting the second (resp. the first) factor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$.

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Let S_c^+ and S_c^- be the rank 2 complex vector bundles on M associated to Δ^+ and Δ^-. We use $L = \det S_c^+ = \det S_c^-$ to denote the determinant bundle of $S_c^\pm$.

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For a connection A on L, there is a well-defined Dirac operator

$$D_A : \Gamma(S_c^+) \to \Gamma(S_c^-).$$

Ricci flow and Seiberg-Witten equations

The Clifford multiplication on the spinor bundles S_c^+ and S_c^- gives an identification of $\wedge^* T^* M$ with $\mathrm{End}(S_c^+ \oplus S_c^-)$. In particular, self dual two forms $\Lambda^{2,+} T^* M$ are identified with traceless endomorphisms in $\mathrm{End}(S_c^+, S_c^+)$.

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The Seiberg-Witten equations read

$$D_A \phi = 0 \tag{7}$$

$$F_A^+ = q(\phi) \tag{8}$$

where $\phi \in \Gamma(S_c^+)$ is a harmonic spinor, and $q(\phi) = \bar{\phi} \otimes \phi - \frac{1}{2}|\phi|^2 \mathrm{Id}$ is a section of the endomorphism bundle $\mathrm{End}(S_c^+, S_c^+)$.

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A solution (A, ϕ) of the Seiberg-Witten equation is called reducible if $\phi \equiv 0$. If there is a reducible solution, then the first Chern class $c_1(L)$ is represented by an anti-self-dual harmonic form because of the equation (8). This implies $c_1^2(L) \leq 0$, with equality iff $c_1(L)$ is a torsion class.

Ricci flow and Seiberg-Witten equations

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On the other hand, by the second equation (8) it follows that $|F_A^+| = \frac{1}{2\sqrt{2}}|\phi|^2$, and therefore,

$$\|F_A^+\|_{L^2}^2 \leq \frac{1}{8}\|R(g)\|_{L^2}^2$$

Ricci flow and Seiberg-Witten equations

Note that, by Chern-Weil theory

$$c_1^2(L) = \frac{1}{4\pi^2}(\|F_A^+\|_{L^2}^2 - \|F_A^-\|_{L^2}^2)$$

Therefore,

$$\|R(g)\|_{L^2}^2 \geq 32\pi^2 c_1^2(L)$$

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Note that the formal dimension of the Seiberg-Witten moduli space $d = \frac{1}{4}(c_1^2(L) - (2\chi(M) + 3\tau(M))$. If the Seiberg-Witten invariant of the Spin^c-structure is non-zero, then $d \geq 0$. This together with the above inequality implies that

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$$\|R(g)\|_{L^2}^2 \geq 32\pi^2(2\chi(M) + 3\tau(M))$$

By the proof of the previous Theorem and the above discussion we get immediately the following improved inequality

Theorem (Fang-Zhang-Zhang, 2006)

Let M be a closed oriented 4-manifold with nontrivial Seiberg-Witten invariant so that $\sigma_M \leq 0$. Let $g(t), t \in [0, \infty)$, be a solution to the normalized Ricci flow. If the scalar curvature $|R(g(t))| < C$ where C is a constant independent of t, then

$$\chi(M) \geq 3\tau(M)$$

where $\chi(M)$ (resp. $\tau(M)$) denote the Euler characteristic (resp. signature) of M.

The case of $\chi(M) = 3\tau(M)$ is interesting, which is exactly at the borderline of the Miyaoka-Yau inequality for Kähler surfaces of general type. We will discuss shortly later.

Ricci flow and Seiberg-Witten equations

By Cheeger-Gromov's collapsing theory, a Riemannian manifold (M, g) with bounded sectional curvature admits a thick-thin decomposition $M = M^{\varepsilon} \cup M_{\varepsilon}$ for small positive number ε, where

$$M^{\varepsilon} = \{x \in M : \mathrm{Vol}(B_x(1)) \geq \varepsilon\}$$

$$M_{\varepsilon} = \{x \in M : \mathrm{Vol}(B_x(1)) < \varepsilon\}$$

For sufficiently small positive constant ε, the thin part M_{ε} admits an *F-structure* of positive rank, i.e., it admits locally fixed point free tori actions. Note that 3-manifolds with F-structures are exactly graph manifolds. By applying Cheeger-Gromov's theory we have that

Theorem (Fang-Zhang-Zhang, 2006)

Let $(M, g(t)), t \in [0, \infty)$ be a non-singular solution to the normalized Ricci flow on a closed oriented 4-manifold M with $\sigma_M < 0$. Then, for any $\delta > 0$, there is a time $T \gg 1$, and a compact 4-submanifold M^ε with boundary in M, $M^\varepsilon \subset M$, such that

(i) $\mathrm{Vol}(M - M^\varepsilon, g(T)) < \delta$, *and $M - M^\varepsilon$ admits an F-structure of positive rank.*

(ii) *The components of ∂M^ε are graph 3-manifolds.*

(iii) *M^ε admits an Einstein metric with negative scalar curvature g_∞ which is close to $g(T)|_{M^\varepsilon}$ in the C^∞-sense.*

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(i) $\mathrm{Vol}(M - M^\varepsilon, g(T)) < \delta$, *and $M - M^\varepsilon$ admits an F-structure of positive rank.*

(ii) The components of ∂M^ε are graph 3-manifolds.

(iii) M^ε admits an Einstein metric with negative scalar curvature g_∞ which is close to $g(T)|_{M^\varepsilon}$ in the C^∞-sense.

The above theorem gives an analog of the Thurston's geometrization program for 4-manifolds, however, the Einstein manifold M^ε is not well-understood.

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Theorem (Fang-Zhang-Zhang, 2006)

Let $(M, g(t))$ and $(M^{\varepsilon}, g_{\infty})$ be the same as above. If M admits a symplectic structure satisfying that $b_2^+ > 1$ and $\chi(M) = 3\tau(M)$, then g_{∞} is a complex hyperbolic metric of finite volume.

The main idea in the proof of the above theorem is as follows:

Ricci flow and Seiberg-Witten equations

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By Taubes, the Seiberg-Witten invariant of the standard Spin^c-structure induced by the almost complex structure of the symplectic manifold M is ± 1. Therefore, there are irreducible solutions (A_t, ϕ_t) for the metrics $g(t)$. The strategy is to prove, using the identity $\chi(M) = 3\tau(M)$, as $t \to \infty$, $F^+_{A_t}$ converges to a nonzero parallel self-dual form on (M_∞, g_∞). Therefore, g_∞ is a metric with special holonomy, indeed a Kähler Einstein metric.

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In the situation above, it is known that the boundary of M^ε is an infra-nil-manifold of dimension 3. It is not known whether the decomposition is "essential" or not, i.e., does the inclusion of the ∂M^ε in the thin part induce an injective homomorphism on the fundamental group? In dimension 3, by Hamilton the torus is essential, which is important for the Thurston's geometrization conjecture.

Thank you!